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On the Generation of Random Ensembles of Qubits and Qutrits

Computing Separability Probabilities for Fixed Rank States

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Abstract. The generation of random mixed states is discussed, aiming for the computation of probabilistic characteristics of composite finite dimensional quantum systems. In particular, we consider the generation of random Hilbert-Schmidt and Bures ensembles of qubit and qutrit pairs and compute the corresponding probabilities to find a separable state among the states of a fixed rank.

1 Introduction

The easily stated problem of finding the probability of a binary composite quantum system being in a separable state turns out to be a challenging computational issue even for simple systems, such as qubit-qubit or qubit-qutrit pairs. Although, the issue of the "separability probability" was formulated almost twenty years ago, nowadays it still remains a subject of interest in the theory of quantum information and quantum communications. For references on relevant studies, rigorous notions and definitions we cite a comprehensive review [1] and a recent paper by P. Slater [2]. Based on the classical theorems of the geometric probability theory [3, 4] attributed to J. Sylvester, the computation of the quantum "separability probability" can be reduced to the evaluation of the relative volume of the separable states with respect to the volume of the whole state space. In order to avoid these subtle and cumbersome numerical calculations of multidimensional integrals (for a generic 2-qubit case the integrals are over the semi-algebraic domain of a 15-dimensional Euclidean space), the Monte-Carlo ideology with a specific algorithm for the generation of random variables has been used (cf. [5]–[8] and references therein). Our report aims to present several results on the numeric studies of the separability probabilities for random qubit-qubit and qubit-qutrit Hilbert-Schmidt and Bures ensembles. In particular, we discuss the dependence of the "separability probability" on the Bloch vectors of each subsystem for the random states of all the possible ranks.

2 The separability probability

There are two equivalent ways to define and compute the separability probability of quantum states. The first is to follow the geometric probability theory, considering the state space of a composite

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quantum system as the Riemannian space and defining the separability probability as the two volumes ratio:

$$\mathcal{P}_{\text{sep}} = \frac{\text{Vol (Separable states)}}{\text{Vol (All states)}} \,. \tag{1}$$

The second way consists in generating the ensemble of random states distributed in correspondence with the volume measure in (1) and evaluating the separability probability:

$$\mathcal{P}_{\text{sep}} = \frac{\text{Number of separable states in ensemble}}{\text{Total number of states in ensemble}} \,. \tag{2}$$

Representation (1) is well appropriate for theoretical studies, but attempting to use it for a practical evaluation leads to big computational difficulties (see e.g., [2] and references therein). In its turn, the equation (2) allows to use highly effective numerical computational techniques. Below, based on representation (2), the results of the studies on the separability probability of the qubit-qubit and qubit-qutrit systems will be stated.

3 Generating the Hilbert-Schmidt and Bures random states

Keeping in mind the calculations of the separability probability of states with different ranks we briefly describe the algorithm for the generation of density matrices from the Hilbert-Schmidt and the Bures ensembles of a n-level quantum system.

- The first step in the construction of both ensembles is the generation of $n \times n$ matrices Z from the complex Ginibre ensemble, i.e. the family of matrices in which each entry is an independent complex Gaussian random variable of mean zero and variance one;
- Using Z and its complex conjugate $Z^\dagger$ one can represent the density matrices ϱ_{HS} describing states from the Hilbert-Schmidt ensemble in the following form:

$$\varrho_{\text{HS}} = \frac{ZZ^\dagger}{\operatorname{tr}\left(ZZ^\dagger\right)} \,; \tag{3}$$

- The density matrices ϱ_{B} from the Bures ensemble are generated using the Ginibre matrices Z and matrices $U \in SU(n)$ distributed over the $SU(n)$ group manifold according to the Haar measure,

$$\varrho_{\text{B}} = \frac{(I+U)\,ZZ^\dagger\left(I+U^\dagger\right)}{\operatorname{tr}\left((I+U)\,ZZ^\dagger\,(I+U^\dagger)\right)} \,. \tag{4}$$

Since we are interested in the study of the entanglement of states with a fixed rank of density matrices, the above algorithm requires further specification. For states of a rank lower than the maximal one we proceed as follows.

•**Rank-3 states**• We start with the generation of complex rank-3 Ginibre matrices. It is known that any rank-3 complex 4×4 matrix admits the following representation:

$$Z = P_Z \left(\begin{array}{ccc|c} & & & x_1 \\ & A & & x_2 \\ & & & x_3 \\ \hline y_1 & y_2 & y_3 & D \end{array}\right) Q_Z \,, \tag{5}$$

where P_Z and Q_Z denote the permutation matrices and A is a regular element of the Ginibre ensemble of 3×3 matrices. Additionally assuming that the 3-tuples $Y = (y_1, y_2, y_3)$ and $X = (x_1, x_2, x_3)$ are

Table 1. The separability probability for the maximal rank-4 states in qubit-qubit and qubit-qutrit systems

States	System	Probability
HS ensemble	$2 \otimes 2$	0.2424
	$2 \otimes 3$	0.0270
Bures ensemble	$2 \otimes 2$	0.0733
	$2 \otimes 3$	0.0014

Table 2. The separability probability for non-maximal rank qubit-qubit states

States	Rank	Probability
HS ensemble	3	0.1652
	2	0
	1	0
Bures ensemble	3	0.0494
	2	0
	1	0

composed of standard normal random complex variables, while the entry D is $D = YA^{-1}X$, we arrive at the presentation of the rank-3 elements from the Ginibre ensemble of complex 4×4 matrices.
•**Rank-2 states**• Similarly, the 4×4 Ginibre matrix of rank-2 can be written as,

$$Z = P_Z \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) Q_Z \,, \tag{6}$$

where A, B and C are 2×2 complex Ginibre matrices, while the 2×2 matrix D reads, $D = CA^{-1}B$.
•**Rank-1 states**• Finally, the 4×4 complex Ginibre matrix Z of rank-1 admits the representation:

$$Z = P_Z \left(\begin{array}{c|ccc} a & y_1 & y_2 & y_3 \\ \hline x_1 & & & \\ x_2 & & D & \\ x_3 & & & \end{array}\right) Q_Z \,, \tag{7}$$

where a, x_1, x_2, x_3 and y_1, y_2, y_3 are standard normal random complex variables, while D is:

$$D = \frac{1}{a} \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}.$$

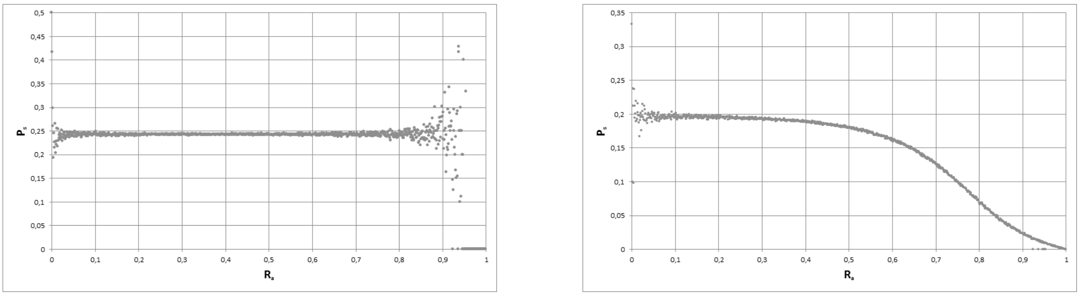
Now, using the generic Ginibre matrix and the representations (5)–(7) for non-maximal rank matrices we can build both the Hilbert-Schmidt and the Bures states of the required rank following either equation (3) or equation (4).

4 Computing the separability probability

The computation of the sought-for separability probability (2) requires testing the generated states on their separability. With this goal in mind, the Peres-Horodecki criterion [1] has been used in our calculations.

4.1 Results of the computations

In table 1 the results of the calculations of the separability probability of the generic (rank-4) states from the Hilbert-Schmidt and Bures ensembles of qubit-qubit and qubit-qutrit pairs are collected. The results of the computations of the separability probabilities of non-maximal rank states of a qubit-qubit system are given for both ensembles in table 2. It is worth noticing the zero separability probability for the rank-2 and rank-1 states. This result is consistent with the known statement, that if a density matrix of 2-qubits ϱ is such that $\operatorname{rank}(\varrho) < d_A = \operatorname{rank}(\varrho_A)$,[1] then ϱ is not separable [9]. Indeed, the above mentioned inequality holds true, since during the generation of rank-2 and rank-1 almost all reduced matrices are not singular.



(a) Maximal rank states

(b) Rank-3 states

Figure 1. The separability probability for the Hilbert-Schmidt qubit-qubit states as a function of the Bloch radius of the first qubit

5 Concluding remarks

Using the above described algorithms as tools for the generation of ensembles of random states of different ranks, various characteristics of entanglement in composite quantum systems can be studied. As an example, in figures 1a and 1b, we illustrate the separability probability of 2-qubit systems as a function of the Bloch radius of the constituent qubit for the states of maximal and sub-maximal ranks. Our analysis shows that the distribution of the separability probability with respect to the Bloch radius of the qubit is uniform for maximal rank states (see figure 1a), while for the rank-3 states the deviation from the total separability probability is depicted in figure 1b.

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[1] Here, ϱ_A denotes the reduced density matrix obtained by taking the partial trace with respect to the second qubit.