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Generalized System of Riccati-Type Equations

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Abstract. A new system of generalized Riccati-type equations is derived. An interconnection between the solutions of n-th order differential equations and the solutions of a generalized system of Riccati-type equations is established. Inverse mapping from the solutions of generalized Riccati-type equations onto the linearly independent solutions of the n-th order differential equation is constructed.

1 Introduction

The structure of the standard Riccati equation is defined in terms of a first order derivative and a second order polynomial. The Riccati equation is associated foremost with differential equation and the Möbius transformation [1]. Analogously, the generalized system of n-th order Riccati-type equations is also associated with the n-th order differential equations. In the present paper this problem is formulated as follows.

Consider the evolution equation with respect to a parameter t generated by the finite dimensional operator H

$$\frac{d}{dt}\Psi(t) = H\Psi(t), \quad \Psi(0) = \Psi_0, \tag{1}$$

the direct closed-form solution of which is given by the formula

$$\Psi(t) = \exp(tH)\,\Psi_0.$$

The finite dimensional operator H is represented by an $n \times n$ matrix which obeys the characteristic polynomial equation

$$f(H) = 0. \tag{2}$$

As a matter of convenience let us suppose that *the characteristic polynomial* coincides with the *minimal polynomial*

$$f(X) = X^n + \sum_{k=1}^{n}(-1)^k a_k X^{n-k}, \qquad a_k \in C. \tag{3}$$

Let E be a *companion matrix* of the polynomial $f(X)$ associated with operator H. The companion matrix obeys the same characteristic equation, namely,

$$f(E) = 0. \tag{4}$$

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Besides the evolution equation generated by the operator H one may consider an evolution generated by the polynomial $f(X)$. This evolution is described by the n-th order Riccati equation

$$\frac{d}{dt}U = f(U). \tag{5}$$

The aim of the present contribution is to establish a mapping between solutions of the equations (1) and (5).

2 Generalized trigonometric functions as solutions of high-order Riccati equation

In the same way as the usual complex algebra induces the trigonometry, the *general complex algebra* GC_n induces representations of the set of generalized trigonometric functions [2], [3]. A matrix representation of the GC_n algebra is given by the companion matrix. The *companion matrix* E is the representation of the equivalent class of all $n \times n$ matrices with trace a_1, determinant a_n and the sum of corresponding minors a_i, $2, \ldots, n-1$. The explicit form of this matrix is defined as follows

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & (-1)^{n+1}a_n \\ 1 & 0 & 0 & 0 & 0 & (-1)^n a_{n-1} \\ 0 & 1 & 0 & 0 & 0 & (-1)^{n-1}a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}. \tag{6}$$

An analogy of the Euler formula is defined by the series

$$\exp\left(\sum_{k=1}^{n-1} E^k \phi_k\right) = Q(E), \tag{7}$$

where the polynomial $Q(U)$ denotes an $(n-1)$-degree polynomial of the form

$$Q(U) = g_0(\phi) + Ug_1(\phi) + U^2 g_2(\phi) + \cdots + U^{n-1}g_{n-1}(\phi). \tag{8}$$

The parameter ϕ stands for the set of $(n-1)$ parameters $\phi := (\phi_1, \phi_2, \phi_3, \ldots, \phi_{n-1})$.

The structure of the set of differential equations for generalized trigonometric functions $g_0(\phi)$, $g_1(\phi)$, $g_2(\phi)$, …, $g_{n-1}(\phi)$ is governed by the matrix E and its degrees $E^k, k = 1, \ldots, n-1$ formulated in the standard way:

$$\frac{\partial}{\partial \phi_k}\mathbf{v}^g(\phi) = E^k \mathbf{v}^g(\phi), \qquad \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}), \quad k = 1, \ldots, n-1; \tag{9}$$

where $\mathbf{v}^g(\phi)$ means a vector of components

$$\mathbf{v}^g = [g_0, g_1, g_2, \ldots, g_{n-1}]^T. \tag{10}$$

As proved in [4], the differential equations (2.9) are reduced to an n-th order Riccati equation

$$\frac{d}{d\phi_{n-1}}U = f(U), \tag{11}$$

under the set of constraints

$$g_k(\phi) = 0, \qquad k = 2, 3, \ldots, n-1. \tag{12}$$

In this approach the solution of the n-th order Riccati equation is defined as a fraction of two trigonometric functions

$$U(\phi_{n-1}) = -\frac{g_0(\phi_{n-1})}{g_1(\phi_{n-1})}, \tag{13}$$

where ϕ_{n-1} depends of $(n-2)$ parameters $\phi_{n-1}(\phi_1, \phi_2, \ldots, \phi_{n-2})$, this dependence in a implicit way is defined by the constraints (12). The transformation of the linear system of evolution equations into the canonical form of the n-th order Riccati equation requires the $n-2$ constraints (12). Under these constraints the polynomial $Q(U)$ of order $(n-1)$ is reduced to a linear function of the form

$$Q(U) = g_0 + U g_1. \tag{14}$$

Then, the solution of the equation $Q(U) = 0$ turns out to the solution to the n-th order Riccati equation (11). This observation prompts us the idea to seek differential equations for the roots of the polynomial $Q(U)$ of order $(n-1)$. As a result, we get a system of Riccati-type equations for the functions

$$u_k = u_k(\phi), \quad k = 1, 2, 3, \ldots, n-1, \qquad \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}), \tag{15}$$

where u_k are roots of the polynomial $Q(u_k)$.

3 System of Riccati-type equations

Consider the polynomial of the $(n-1)$-th order,

$$Q(U) = \sum_{j=0}^{n-1} U^j g_j(\phi), \tag{16}$$

with roots $u_k(\phi)$, $k = 1, 2, 3, \ldots, n-1$; $\phi = (\phi_1, \phi_2, \ldots, \phi_{n-1})$. The coefficients of the polynomial $g_j(\phi)$, $j = 0, 1, 2, \ldots, n-1$ are solutions of the system of evolution equations

$$\partial_i \, g_j = \sum_{m=1}^{n} (E^i)^m_j g_{m-1}, \qquad i = 1, \ldots, n-1. \tag{17}$$

The main result is given by the following

Theorem. *The functions $u_k(\phi)$, $k = 1, \ldots, n-1$ obey the following system of nonlinear equations*

$$F(u_m) \sum_{k=1}^{n-p} a_{n-k-p} \partial_k \, u_m = A_p \, f(u_m), \qquad m = 1, \ldots, n-1. \tag{18}$$

where $F(u_m)$ is the $(n-2)$-degree truncated polynomial

$$F(u_m) = \left.\frac{dQ(U)}{dU}\right|_{U=u_m} = u_m^{n-2} + \sum_{k=0}^{n-3} u_m^k A_k(m) = \prod_{k=1,\, k\neq m}^{n-1} (u_m - u_k), \tag{19}$$

and $A_p(m)$ is the p-th coefficient of the polynomial $F(u_m)$. □

The explicit form of the system of equations (19) is presented as follows

$$F(U)\,(\partial_{n-1} - a_1\partial_{n-2} + a_2\partial_{n-3} - a_3\partial_{n-4} + \cdots + (-1)^{n-1}a_{n-2}\partial_1)U = A_{n-1}\, f(U),$$

$$\ldots$$

$$F(U)\,(\partial_k - a_1\partial_{k-1} + a_2\partial_{k-2} - a_3\partial_{k-3} + \cdots + (-1)^k a_{k-1}\partial_1)U = A_k\, f(U),$$

$$\ldots$$

$$F(U)\,(\partial_5 - a_1\partial_4 + a_2\partial_3 - a_3\partial_2 + a_4\partial_1)U = A_5\, f(U),$$

$$F(U)\,(\partial_4 - a_1\partial_3 + a_2\partial_2 - a_3\partial_1)U = A_4\, f(U),$$

$$F(U)\,(\partial_3 - a_1\partial_2 + a_2\partial_1)U = A_3\, f(U),$$

$$F(U)(\partial_2 - a_1\partial_1)U = A_2\, f(U),$$

$$F(U)\partial_1 U = A_1\, f(U). \tag{20}$$

If the basic polynomial $f(X)$ is a polynomial of the n-th order then the functions u_k, $k = 1, \ldots, n-1$ are roots of the following polynomial of the order $(n-1)$:

$$Q(U) = g_{n-1}U^n + \cdots + g_3U^3 + g_2U^2 + g_1U + g_0. \tag{21}$$

Each of the roots u_k obeys the system of differential equations (20), where

$$F(U) = A_1U^{n-2} + A_2U^{n-3} + \cdots + A_{n-2}U + A_{n-1}, \tag{22}$$

and

$$A_1 = 1, \qquad -A_2 = V + W + Y + Z, \qquad A_3 = VW + VY + VZ + WY + WZ + YZ + \cdots\,. \tag{23}$$

4 Conclusion

We have derived the system of Riccati-type equations from the linear system of evolution equations. The method can be applied in the problem of transformation of the n-th order differential equation into a Riccati-type equation. It is expected, the present method will be useful in the theory of finite dimensional quantum mechanics.

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