Generating Function Approach to the Derivation of Higher-Order Iterative Methods for Solving Nonlinear Equations

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Abstract. In this paper we propose a generating function method for constructing new two and three-point iterations with \( p (p = 4, 8) \) order of convergence. This approach allows us to derive a new family of optimal order iterative methods that include well known methods as special cases. Necessary and sufficient conditions for \( p \)-th \( (p = 4, 8) \) order convergence of the proposed iterations are given in terms of parameters \( \tau_n \) and \( \alpha_n \). We also propose some generating functions for \( \tau_n \) and \( \alpha_n \). We develop a unified representation of all optimal eighth-order methods. The order of convergence of the proposed methods is confirmed by numerical experiments.

1 Introduction

Solving nonlinear equations is important in many applied mathematics and theoretical physics problems. In recent years, a number of higher-order iterative methods have been developed and analyzed on this issue, see [1–11] and references therein. Motivated by the recent results in [11], in this paper we introduce a generating function method for the construction of new two and three-point iterations with \( p \)-th order of convergence. This paper is organized as follows. Section 2 is devoted to the construction of a generating function for the optimal fourth-order method. We then present some choices for the parameters \( \tau_n \) and \( \alpha_n \). Some iterations are proposed among which some are already well known. In Section 3 we propose a family of optimal eighth-order methods, that include many well-known methods as particular cases. In our previous paper [11] we have considered two and three-point iterative methods of solving nonlinear equation \( f(x) = 0 \)

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = y_n - \overline{\tau}_n \frac{f(y_n)}{f'(x_n)},
\]

(1)

and

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \overline{\tau}_n \frac{f(y_n)}{f'(x_n)}, \quad x_{n+1} = z_n - \alpha_n \frac{f(z_n)}{f'(x_n)}.
\]  

(2)

We have proved in [11] the following theorems:

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**Theorem 1.** Assume that $f(x)$ is smooth enough function with a simple root $x^* \in I$ and the initial approximation $x_0$ is close enough to $x^*$. Then the iterative method (1) has fourth-order of convergence if and only if the parameter $\tau_n$ is given by

$$
\tau_n = 1 + \theta_n + 2\theta_n^2 + O(\theta_n^3), \quad \theta_n = \frac{f(y_n)}{f(x_n)}.
$$

**Theorem 2.** Assume that all assumptions of Theorem 1 are fulfilled. Then the three-point iterative methods (2) has an eighth-order of convergence if and only if the parameters $\tau_n$ and $\alpha_n$ are given by

$$
\tau_n = 1 + 2\theta_n + \beta \theta_n^2 + \gamma \theta_n^3 + \cdots, \quad \left(\tau_n = \frac{\tau_n - 1}{\theta_n}\right),
$$

and

$$
\alpha_n = 1 + 2\theta_n + (\beta + 1)\theta_n^2 + (2\beta + \gamma - 4)\theta_n^3 + (1 + 4\theta_n) \frac{f(z_n)}{f(y_n)} + O(\theta_n^6).
$$

Our approach in [11] is constructive in the sense that it proposes a new way to obtain optimal order iterations (see [11] for details). An extended version of the present paper will be published elsewhere.

### 2 Construction of optimal fourth-order methods

The Theorems 1 and 2 not only give sufficient conditions for iterations of $p$-th $(p = 4, 8)$ order of convergence, but they also allow us to construct new iterations with $p$ order of convergence. Obtaining new optimal methods of order four is still important, because they combine higher-order of convergence and low computational cost. We consider the following choice of the parameter $\tau_n$

$$
\tau_n = H(\theta_n),
$$

where $H(\theta)$ is a real function to be determined properly. Obviously $\tau_n$ will satisfy the condition (3) if

$$
H(0) = 1, \quad H'(0) = 1, \quad H''(0) = 4.
$$

We call the function $H(\theta)$ satisfying conditions (7) a generating function for the iteration (1). The construction of the generating function allows us to derive a new optimal order family of iterations. The following theorem is a consequence of Theorem 1.

**Theorem 3.** Assume that all assumptions of Theorem 1 are fulfilled. Then the optimal fourth-order two-point iterations (1) are obtained by the generating function (6) satisfying the conditions (7). Many different variants of the generating function $H(x)$, satisfying condition (7) are possible. We cite here one simple form, namely

$$
H(x) = \frac{1 + (1 - ma)x + (2 - ma + \frac{m(m - 1)}{2}a^2)x^2 + \omega x^3}{(1 - ax)^m}, \quad \alpha, m, \omega \in \mathbb{R}.
$$

The optimal two-point iterations (1) with $\tau_n = H(\theta_n)$ given by (8) include many well-known iterations as special cases. If $\omega = 0$, $m = 1$ and $\alpha = 2 - b$, $b \in \mathbb{R}$, then (1) leads to King’s method [5]. If $\alpha = 0$, $m = 1$ and $\omega = 1$ in (8), then (1) yields a modification of Potra-Ptak’s method [4]. If $\alpha = m = 1$ and $\omega = -1$ in (8), then (1) leads to Maheshwari’s method [7]. If $\alpha = 1$, $m = 2$ and $\omega = 0$ in (8), then (1) leads to Chun and Lee’s method [2]. Recently, Behl et al [12] proposed a general class of fourth-order optimal methods that includes the well-known Ostrowski’s and King’s family as special cases. We note that this general class of optimal fourth-order iterations is also included in our methods with
\( \tau_n = H(\theta_n) \) given by (8) as a special case. Namely, if \( m = 3, \alpha \) replaced by \(-\alpha\) and \( \omega = \alpha^2 + \frac{3}{2} \alpha + \frac{4}{3} \) or \( \omega = (1 - \frac{3}{2}) \alpha^3 + \alpha^2 - 2\alpha \), then the iterations (1) with \( \tau_n = H(\theta_n) \) given by (8) reduce to (3.8) and (3.10) in [12], respectively. This shows that our class of optimal fourth-order methods is wider than that of [12]. So, we have obtained an optimal fourth-order convergence family of iterative methods with three degrees of freedom based on the generating function method.

### 3 Proper representation of the optimal order three-point iterative methods

Recently, based on optimal fourth-order methods some higher-order, in particular eighth order three-point methods have been proposed for solving nonlinear equations. It is easy to show that \( \tau_n = H(\theta_n) \) given by (8) satisfies the condition (4) provided

\[
\begin{align*}
\beta &= \omega + 2m\alpha - \frac{m(m-1)}{2}\alpha^2 + \frac{m(m-1)(m-2)}{6}\alpha^3, \\
\gamma &= \omega m\alpha + m(m+1)\alpha^2 - \frac{(m-1)m(m+1)}{3}\alpha^3 + \frac{(m-2)(m-1)m+1}{8}\alpha^4.
\end{align*}
\]

\[\text{(9a)}\]
\[\text{(9b)}\]

<p>| Table 1. Optimal order three-point iterative methods recovered at special parameter values |
|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>( m )</th>
<th>( \alpha_n - (1 + 4\theta_n)[f(z_n)/f(y_n)] )</th>
<th>Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 1 + 2\theta_n + (\beta + 1)\theta_n^2 + (2\beta + \gamma - 4)\theta_n^3 )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( 1 )</td>
<td>( (2 - \theta_n)/(6\theta_n^2 - 5\theta_n + 2) )</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( 1 )</td>
<td>( [2\beta - 1 + 2\beta(\beta - 2)\theta_n]/(2\beta - 1 + 2(\beta^2 - 4\beta + 1)\theta_n + (1 + 4\beta)\theta_n^2) )</td>
</tr>
<tr>
<td>( 4 )</td>
<td>( 1 )</td>
<td>( 1/(1 - 2\beta_n - \theta_n^3) )</td>
</tr>
</tbody>
</table>

The following theorem is a consequence of Theorem 2.

**Theorem 4.** Assume that all assumptions of Theorem 1 are fulfilled. Then the family of three-point iterative methods (2) has an eighth-order of convergence if and only if the parameters \( \tau_n \) and \( \alpha_n \) are given by (6), (8) and

\[
\alpha_n = \left( H(\theta_n) + \theta_n + (\beta - 1)\theta_n^2 + (\beta + \gamma - 4)\theta_n^3 \right) + (1 + 4\theta_n) \frac{f(z_n)}{f(y_n)}.
\]

\[\text{(10)}\]

Thus, we propose the families of three-point iterative methods (2) with generating function \( \tau_n = H(\theta_n) \). They include many well-known eighth-order methods, as particular cases (see Table 1). The expression in brackets in (10) can be approximated by a simple rational function without loss of generality. Then \( \alpha_n \) can be represented as

\[
\alpha_n = \frac{1 - (2 - m\phi)\theta_n + c\theta_n^3 + \omega\theta_n^3}{(1 - \theta_n(d\theta_n^2 + p\theta_n + q))^{m+1}} + (1 + 4\theta_n) \frac{f(z_n)}{f(y_n)}, \quad q, p, d, m \in \mathbb{R},
\]

\[\text{(11)}\]

where

\[
\begin{align*}
c &= \beta + 1 - m \left( p + 2q + \frac{1}{2}(m - 1)q^2 \right), \\
\omega &= (2\beta + \gamma - 4) - m \left( d + 2p + (\beta + 1 + (1 - m)p)q - (m - 1)q^2 + \frac{(m - 1)(m - 2)}{6}q^3 \right).
\end{align*}
\]

\[\text{(12)}\]
We call the optimal order three-point iterative methods (2), with parameters $\tau_n$ and $\alpha_n$ given by (6), (8) and (11) respectively, proper representations. It is easy to show that all the well-known optimal order three-point iterative methods can be represented uniquely in the proper form (see [1–3, 6, 8–11, 13–20] and references therein). It should be mentioned that Wu and Lee in [10] first used a proper representation of (2). Thus, by means of (6), (8) and (11) we find a unified representation of all optimal order three-point iterations. It should be mentioned that the order of convergence of the proposed methods was confirmed by numerical experiments.

Conclusions

The construction of the generating function for $\tau_n$ and $\alpha_n$ allows us to derive new optimal order family of iterations. This family includes many known iterations as special cases. We develop a unified and proper representation of optimal eighth-order three-point methods. The sufficient and necessary conditions for iterations (2) to be $p$ ($p = 4, 8$) order of convergence are also given in term of parameters $\tau_n$ and $\alpha_n$.

Acknowledgments

The work was supported partially by the Foundation of Science and Technology of Mongolia under grant SST_007/2017. O. Ch. acknowledges support from the Hulubei-Meshcheryakov program JINR-Romania.

References