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Generating Function Approach to the Derivation of Higher-Order Iterative Methods for Solving Nonlinear Equations

Tugal Zhanlav[1,⋆], *Ochbadrakh* Chuluunbaatar[1,2,⋆⋆], and *Vandandoo* Ulziibayar[1,3,⋆⋆⋆]

[1] *Institute of Mathematics, National University of Mongolia, Mongolia*
[2] *Joint Institute for Nuclear Research, Dubna, 141980 Moscow region, Russia*
[3] *School of Applied Sciences, Mongolian University of Science and Technology, Mongolia*

Abstract. In this paper we propose a generating function method for constructing new two and three-point iterations with p ($p = 4, 8$) order of convergence. This approach allows us to derive a new family of optimal order iterative methods that include well known methods as special cases. Necessary and sufficient conditions for p-th ($p = 4, 8$) order convergence of the proposed iterations are given in terms of parameters τ_n and α_n. We also propose some generating functions for τ_n and α_n. We develop a unified representation of all optimal eighth-order methods. The order of convergence of the proposed methods is confirmed by numerical experiments.

1 Introduction

Solving nonlinear equations is important in many applied mathematics and theoretical physics problems. In recent years, a number of higher-order iterative methods have been developed and analyzed on this issue, see [1–11] and references therein. Motivated by the recent results in [11], in this paper we introduce a generating function method for the construction of new two and three-point iterations with p-th order of convergence. This paper is organized as follows. Section 2 is devoted to the construction of a generating function for the optimal fourth-order method. We then present some choices for the parameters τ_n and α_n. Some iterations are proposed among which some are already well known. In Section 3 we propose a family of optimal eighth-order methods, that include many well-known methods as particular cases. In our previous paper [11] we have considered two and three-point iterative methods of solving nonlinear equation $f(x) = 0$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = y_n - \bar{\tau}_n \frac{f(y_n)}{f'(x_n)}, \tag{1}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad z_n = y_n - \bar{\tau}_n \frac{f(y_n)}{f'(x_n)}, \qquad x_{n+1} = z_n - \alpha_n \frac{f(z_n)}{f'(x_n)}. \tag{2}$$

We have proved in [11] the following theorems:

⋆e-mail: tzhanlav@yahoo.com
⋆⋆e-mail: chuka@jinr.ru
⋆⋆⋆e-mail: ulzii@jinr.ru

Theorem 1. *Assume that $f(x)$ is smooth enough function with a simple root $x^* \in I$ and the initial approximation x_0 is close enough to x^*. Then the iterative method (1) has fourth-order of convergence if and only if the parameter τ_n is given by*

$$\tau_n = 1 + \theta_n + 2\theta_n^2 + O(\theta_n^3), \qquad \theta_n = \frac{f(y_n)}{f(x_n)}. \tag{3}$$

Theorem 2. *Assume that all assumptions of Theorem 1 are fulfilled. Then the three-point iterative methods (2) has an eighth-order of convergence if and only if the parameters $\bar{\tau}_n$ and α_n are given by*

$$\bar{\tau}_n = 1 + 2\theta_n + \beta\theta_n^2 + \gamma\theta_n^3 + \cdots, \qquad \left(\bar{\tau}_n = \frac{\tau_n - 1}{\theta_n}\right), \tag{4}$$

and

$$\alpha_n = 1 + 2\theta_n + (\beta + 1)\theta_n^2 + (2\beta + \gamma - 4)\theta_n^3 + (1 + 4\theta_n)\frac{f(z_n)}{f(y_n)} + O(\theta_n^4). \tag{5}$$

Our approach in [11] is constructive in the sense that it proposes a new way to obtain optimal order iterations (see [11] for details). An extended version of the present paper will be published elsewhere.

2 Construction of optimal fourth-order methods

The Theorems 1 and 2 not only give sufficient conditions for iterations of p-th ($p = 4, 8$) order of convergence, but they also allow us to construct new iterations with p order of convergence. Obtaining new optimal methods of order four is still important, because they combine higher-order of convergence and low computational cost. We consider the following choice of the parameter τ_n

$$\tau_n = H(\theta_n), \tag{6}$$

where $H(\theta)$ is a real function to be determined properly. Obviously τ_n will satisfy the condition (3) if

$$H(0) = 1, \qquad H'(0) = 1, \qquad H''(0) = 4. \tag{7}$$

We call the function $H(\theta)$ satisfying conditions (7) a generating function for the iteration (1). The construction of the generating function allows us to derive a new optimal order family of iterations. The following theorem is a consequence of Theorem 1.

Theorem 3. *Assume that all assumptions of Theorem 1 are fulfilled. Then the optimal fourth-order two-point iterations (1) are obtained by the generating function (6) satisfying the conditions (7).*

Many different variants of the generating function $H(x)$, satisfying condition (7) are possible. We cite here one simple form, namely

$$H(x) = \frac{1 + (1 - m\alpha)x + (2 - m\alpha + \frac{m(m-1)}{2}\alpha^2)x^2 + \omega x^3}{(1 - \alpha x)^m}, \qquad \alpha, m, \omega \in \mathbb{R}. \tag{8}$$

The optimal two-point iterations (1) with $\tau_n = H(\theta_n)$ given by (8) include many well-known iterations as special cases. If $\omega = 0$, $m = 1$ and $\alpha = 2 - b$, $b \in \mathbb{R}$, then (1) leads to King's method [5]. If $\alpha = 0$, $m = 1$ and $\omega = 1$ in (8), then (1) yields a modification of Potra-Ptak's method [4]. If $\alpha = m = 1$ and $\omega = -1$ in (8), then (1) leads to Maheshwari's method [7]. If $\alpha = 1$, $m = 2$ and $\omega = 0$ in (8), then (1) leads to Chun and Lee's method [2]. Recently, Behl et al [12] proposed a general class of fourth-order optimal methods that includes the well-known Ostrowski's and King's family as special cases. We note that this general class of optimal fourth-order iterations is also included in our methods with

$\tau_n = H(\theta_n)$ given by (8) as a special case. Namely, if $m = 3$, α replaced by $-\alpha$ and $\omega = \alpha^2 + \frac{5}{3}\alpha + \frac{4}{3}$ or $\omega = \left(1 - \frac{\beta}{6}\right)\alpha^3 + \alpha^2 - 2\alpha$, then the iterations (1) with $\tau_n = H(\theta_n)$ given by (8) reduce to (3.8) and (3.10) in [12], respectively. This shows that our class of optimal fourth-order methods is wider than that of [12]. So, we have obtained an optimal fourth-order convergence family of iterative methods with three degrees of freedom based on the generating function method.

3 Proper representation of the optimal order three-point iterative methods

Recently, based on optimal fourth-order methods some higher-order, in particular eighth order three-point methods have been proposed for solving nonlinear equations. It is easy to show that $\tau_n = H(\theta_n)$ given by (8) satisfies the condition (4) provided

$$\beta = \omega + 2m\alpha - \frac{m(m-1)}{2}\alpha^2 + \frac{m(m-1)(m-2)}{6}\alpha^3, \tag{9a}$$

$$\gamma = \omega m\alpha + m(m+1)\alpha^2 - \frac{(m-1)m(m+1)}{3}\alpha^3 + \frac{(m-2)(m-1)m(m+1)}{8}\alpha^4. \tag{9b}$$

Table 1. Optimal order three-point iterative methods recovered at special parameter values

	m	$\alpha_n - (1+4\theta_n)[f(z_n)/f(y_n)]$	Methods
1	0	$1 + 2\theta_n + (\beta+1)\theta_n^2 + (2\beta+\gamma-4)\theta_n^3$	$\beta = \gamma = 1$, [14] Maheshwari-based $\beta = 4,\ \gamma = 8$, Method 1 in [14] see [11] $\beta = 3,\ \gamma = 4$, Chun Lee [2]
2	1	$(2-\theta_n)/(6\theta_n^2 - 5\theta_n + 2)$	$p = -3,\ d = 0,\ q = 5/2$, Maheshwari-based optimal methods [16]
3	1	$[2\beta - 1 + 2\beta(\beta-2)\theta_n]/$ $[2\beta - 1 + 2(\beta^2 - 4\beta + 1)\theta_n + (1+4\beta)\theta_n^2]$	$d = 0,\ q = 2(\beta^2 - 4\beta + 1)/(1-2\beta)$, $p = (1+4\beta)/(1-2\beta)$, King-based optimal methods [17]
4	1	$1/(1 - 2\theta_n - \theta_n^2)$	$\beta = 4,\ \gamma = 8,\ q = 2,\ p = 1,\ d = \omega = 0$, method 4 in [14]

The following theorem is a consequence of Theorem 2.

Theorem 4. *Assume that all assumptions of Theorem 1 are fulfilled. Then the family of three-point iterative methods (2) has an eighth-order of convergence if and only if the parameters τ_n and α_n are given by (6), (8) and*

$$\alpha_n = \left(H(\theta_n) + \theta_n + (\beta-1)\theta_n^2 + (\beta+\gamma-4)\theta_n^3\right) + (1+4\theta_n)\frac{f(z_n)}{f(y_n)}. \tag{10}$$

Thus, we propose the families of three-point iterative methods (2) with generating function $\tau_n = H(\theta_n)$. They include many well-known eighth-order methods, as particular cases (see Table 1). The expression in brackets in (10) can be approximated by a simple rational function without loss of generality. Then α_n can be represented as

$$\alpha_n = \frac{1 - (2 - mq)\theta_n + c\theta_n^2 + \omega\theta_n^3}{\left(1 - \theta_n(d\theta_n^2 + p\theta_n + q)\right)^m} + (1+4\theta_n)\frac{f(z_n)}{f(y_n)}, \qquad q, p, d, m \in \mathbb{R}, \tag{11}$$

where

$$c = \beta + 1 - m\left(p + 2q + \frac{1}{2}(m-1)q^2\right),$$

$$\omega = (2\beta + \gamma - 4) - m\left(d + 2p + \left(\beta + 1 + (1-m)p\right)q - (m-1)q^2 + \frac{(m-1)(m-2)}{6}q^3\right). \tag{12}$$

We call the optimal order three-point iterative methods (2), with parameters τ_n and α_n given by (6), (8) and (11) respectively, proper representations. It is easy to show that all the well-known optimal order three-point iterative methods can be represented uniquely in the proper form (see [1–3, 6, 8–11, 13–20] and references therein). It should be mentioned that Wu and Lee in [10] first used a proper representation of (2). Thus, by means of (6), (8) and (11) we find a unified representation of all optimal order three-point iterations. It should be mentioned that the order of convergence of the proposed methods was confirmed by numerical experiments.

Conclusions

The construction of the generating function for τ_n and α_n allows us to derive new optimal order family of iterations. This family includes many known iterations as special cases. We develop a unified and proper representation of optimal eighth-order three-point methods. The sufficient and necessary conditions for iterations (2) to be p ($p = 4, 8$) order of convergence are also given in term of parameters τ_n and α_n.

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