Topology in the $SU(N_f)$ chiral symmetry restored phase of unquenched QCD and axion cosmology

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Abstract. The axion is one of the more interesting candidates to make the dark matter of the universe, and the axion potential plays a fundamental role in the determination of the dynamics of the axion field. Moreover, the way in which the $U(1)_A$ anomaly manifests itself in the chiral symmetry restored phase of QCD at high temperature could be tested when probing the QCD phase transition in relativistic heavy ion collisions. With these motivations, we investigate the physical consequences of the survival of the effects of the $U(1)_A$ anomaly in the chiral symmetric phase of QCD, and show that the free energy density is a singular function of the quark mass m, in the chiral limit, and that the σ and $\bar{\pi}$ susceptibilities diverge in this limit at any $T \geq T_c$. We also show that the difference between the $\bar{\pi}$ and $\bar{\delta}$ susceptibilities diverges in the chiral limit at any $T \geq T_c$, a result that can be contrasted with the existing lattice calculations; and discuss on the generalization of these results to the $N_f \geq 3$ model.

1 Introduction

We report the results of an investigation on the physical consequences of the survival of the topological effects of the axial anomaly in the high temperature phase of QCD. This contribution to the Lattice 2017 Symposium is based on references [1], [2] and we refer the interested reader to these papers. To summarize the main results, our starting hypothesis in [1] was to assume that the perturbative expansion of the free energy density in powers of the quark mass, m, has a non-vanishing convergence radius in the high temperature chiral symmetric phase of QCD. This is just what we expect on physical grounds if all correlation lengths remain finite in the chiral limit, and the spectrum of the model shows therefore a mass gap also in this limit. The main conclusion obtained from this hypothesis was that all the topological effects of the axial anomaly should disappear in this phase, the topological susceptibility and all θ-derivatives of the free energy density vanish, and the theory becomes θ-independent at any $T > Tc$ in the infinite-volume limit. Accordingly, the free energy density should be a singular function of the quark mass, in the chiral limit, if the topological effects of the $U(1)_A$ anomaly survive in the chiral symmetry restored phase of QCD at finite temperature, and the main purpose of reference [2] was to investigate this issue. The starting hypothesis in [2] was to assume that the topological effects of the anomaly survive in the high temperature phase of QCD,

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and the model shows therefore a non-trivial θ-dependence in this phase. Under this assumption we showed that indeed the free energy density is a singular function of the quark mass, m, in the chiral limit, at any $T > T_c$, and that the correlation length and the σ and $\bar{\pi}$ susceptibilities diverge in this limit, as well as the difference between the $\bar{\pi}$ and $\bar{\delta}$ susceptibilities.

The relevance of these results is due to the fact that the topological effects of the $U(1)_A$ axial anomaly in the high temperature phase of QCD play a fundamental role in the determination of the dynamics of the axion field, which is one of the more interesting candidates to make the dark matter of the universe. Moreover these results could be tested when probing the QCD phase transition in relativistic heavy ion collisions.

The first investigations on this subject started long time ago. The idea that the chirally restored phase of two-flavor QCD is symmetric under $U(2) \times U(2)$ rather than $SU(2) \times SU(2)$ was raised by Shuryak in 1994 [3] based on an instanton liquid-model study. In 1996 Cohen [4] also got this result formally from the QCD functional integral under some assumptions. However immediately after several calculations questioning this result appeared [5]-[8]. Since then much work has been done, and references [1]-[2], [9]-[13] contain recent work concerning theoretical developments, the computation of the topological susceptibility (with discrepant results, as shown in this conference), and the computation of correlation functions, spectral density of the Dirac operator and susceptibilities.

2 σ and η susceptibilities

Our starting point is the Euclidean continuum Lagrangian of N_f flavors QCD with a θ-term

$$L = \sum_f L_F^f + \frac{1}{4} F_{\mu\nu}^a(x) F_{\mu\nu}^a(x) + i\theta \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) \tag{1}$$

where L_F^f is the fermion Lagrangian for the f-flavor, and

$$Q = \frac{g^2}{64\pi^2} \int d^4x\, \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) \tag{2}$$

is the topological charge of the gauge configuration.

To avoid ultra-violet divergences we will assume Ginsparg-Wilson fermions, which show an $U(1)_A$ anomalous symmetry, good chiral properties, a quantized topological charge and an exact index theorem on the lattice, so they share all essential ingredients with the continuum formulation. We will also assume in what follows that the topological effects of the $U(1)_A$ axial anomaly survive in the high temperature chiral symmetric phase of QCD, and that the partition function and the free energy density

$$Z(\theta) = e^{-V_x L_t E(\beta,m,\theta)} \tag{3}$$

show a non-trivial dependence on the θ parameter. $E(\beta, m, \theta)$ in (3) is the free energy density, β the inverse gauge coupling, m the quark mass, and L_t the lattice temporal extent or inverse physical temperature T. Moreover the mean value of any intensive operator O, as for instance the scalar and pseudoscalar condensates, or any correlation function, in the $Q = 0$ topological sector, can be computed as

$$\langle O \rangle_{Q=0} = \frac{\int d\theta \langle O \rangle_\theta Z(\theta, m)}{\int d\theta Z(\theta, m)} \tag{4}$$

with $\langle O \rangle_\theta$ the mean value of O computed with the integration measure (1). Because the free energy density, as a function of θ, has its absolute minimum at $\theta = 0$ for non-vanishing quark masses, the following relation holds in the infinite lattice volume limit

$$\langle O \rangle_{Q=0} = \langle O \rangle_{\theta=0} \tag{5}$$

and we want to remark that, as discussed in [1], in spite of the fact that the $Q = 0$ topological sector is free from the global $U(1)_A$ anomaly, and spontaneously breaks the $U(N_f)_A$ axial symmetry at $T = 0$, equation (5) is compatible with a massive flavor-singlet pseudoscalar meson in the chiral limit.

Let us consider, for simplicity, the two-flavor model with degenerate up and down quark masses. In the high temperature phase the $SU(2)_A$ symmetry is fulfilled in the ground state for massless quarks, and therefore the mean value of the flavor singlet scalar condensate $\langle S \rangle$, as well as of any order parameter for this symmetry, vanishes in the chiral limit. Moreover the infinite lattice volume limit and the chiral limit should commute, provided the order parameter remains bounded. In addition equation (5) implies that the $SU(2)_A$ symmetry is also fulfilled in the the $Q = 0$ topological sector. However, the $U(1)_A$ symmetry should be spontaneously broken in this sector, giving account in this way for the $U(1)_A$ anomaly[1].

Let us assume that the flavor singlet scalar susceptibility, the σ-susceptibility, remains finite when we approach the chiral limit. Then the flavor singlet scalar condensate at $\theta = 0$, which is equal to the mean value of the same operator in the $Q = 0$ sector, shows this linear behavior

$$\langle S \rangle_{\theta=0} = \langle S \rangle_{Q=0} \underset{m\to 0}{\approx} \chi_\sigma(0)\, m \tag{6}$$

with the quark mass for small quark masses. Because the $Q = 0$ sector is free from the global $U(1)_A$ anomaly, we can write the following Ward identities

$$\chi_{\bar{\pi}}(m)_{Q=0} = \chi_\eta(m)_{Q=0} = \frac{\langle S \rangle_{Q=0}}{m} \to \chi_\sigma(0) \tag{7}$$

which tell us that the pion, eta and sigma susceptibilities in the chiral limit in this sector are equal to the sigma susceptibility in full QCD at $\theta = 0$. Moreover we can demonstrate, with the help of an anomalous $U(1)_A$ transformation, the following equation

$$\langle S(x) S(0) \rangle^{m=0}_{Q=0} = \frac{1}{2} \langle S(x) S(0) \rangle^{m=0}_{\theta=0} + \frac{1}{2} \langle P(x) P(0) \rangle^{m=0}_{\theta=0} \tag{8}$$

which relates the flavor singlet scalar correlation function in the $Q = 0$ sector in the chiral limit, with the same quantity, and the eta-correlation function in full *QCD*.

Equation (8) implies the following relation

$$\chi_\sigma(0) = \chi_\sigma(0)_{Q=0} = \frac{\chi_\sigma(0) + \chi_\eta(0)}{2} \tag{9}$$

between the σ and η susceptibilities in full *QCD*. The fulfillment of this equation requires the equality of the sigma and eta susceptibilities in the chiral limit, in contradiction with the assumption that the topological effects of the $U(1)_A$ axial anomaly survive in the high temperature phase of QCD. We conclude therefore that the assumption on the finitude of the correlation length and σ-susceptibility in the chiral limit is not compatible with the survival of the topological effects of the $U(1)_A$ anomaly in the high temperature phase of *QCD*. Hence the σ-susceptibility should diverge in the chiral limit, and the free energy density should be singular at vanishing quark masses.

[1]The Goldstone theorem however can be fulfilled without a Nambu-Goldstone boson [1].

Following the standard wisdom, the critical behavior of the model should be well described by a power law behavior for the flavor singlet scalar condensate

$$\langle S \rangle_{\theta=0} {}_{m\to 0} \approx C(T)\, m^{\frac{1}{\delta}} \tag{10}$$

with $\delta > 1$. But equation (10) implies that the flavor singlet scalar susceptibility diverges at any $T \geq T_c$ in the chiral limit

$$\chi_{\sigma}(m) \approx C(T)\, \frac{1}{\delta} m^{\frac{1-\delta}{\delta}} \tag{11}$$

and because the $SU(2)_A$ symmetry is not anomalous, the pion susceptibility verifies this Ward identity

$$\chi_{\bar{\pi}}(m) = \frac{\langle S \rangle}{m} \tag{12}$$

and also diverges in the chiral limit this way

$$\chi_{\bar{\pi}}(m) \approx C(T)\, m^{\frac{1-\delta}{\delta}}. \tag{13}$$

Moreover the vector meson $\bar{\delta}$ susceptibility, $\chi_{\bar{\delta}}$, which is bounded by the scalar susceptibility, χ_{σ} , verifies the following inequality

$$\chi_{\bar{\pi}}(m) - \chi_{\bar{\delta}}(m) \geq \chi_{\bar{\pi}}(m) - \chi_{\sigma}(m) \approx C(T)\, \frac{\delta - 1}{\delta} m^{\frac{1-\delta}{\delta}} \tag{14}$$

which shows that this quantity, which is an order parameter for the $U(1)_A$ axial symmetry, also diverges in the chiral limit.

The zero-temperature two-flavor Schwinger model is a good toy model for testing these results. The $SU(2)_A$ chiral symmetry is fulfilled in the vacuum in this model because of the Coleman-Mermin-Wagner theorem. Moreover it is well known that the topological effects of the $U(1)_A$ anomaly are relevant in this model since it shows a non-trivial dependence on the θ-parameter [14]. It has been shown that the free energy density is singular in the chiral limit [15] in this model, and that the pions are massless in this limit [16]; in agreement with the discussion developed in this section.

3 Phase diagram of QCD in the Q=0 topological sector

These results can also be shown by performing a qualitative analysis of the phase diagram of QCD in the $Q = 0$ sector. The $SU(2)_A$ symmetry is fulfilled in QCD at any $T > T_c$, and therefore the up and down scalar condensates $\langle S_u \rangle$, $\langle S_d \rangle$ vanish in the chiral limit $m_u = m_d = 0$. However if we consider QCD with two non degenerate quark flavors, and take the limit $m_u \to 0$ keeping m_d fixed, or vice versa, the condensate $\langle S_u \rangle$, or $\langle S_d \rangle$, takes a non-vanishing mean value due to the fact that the $U(1)_u$ symmetry at $m_u = 0$, or the $U(1)_d$ symmetry at $m_d = 0$, which would enforce the condensate to be zero, is anomalous. But since equation (5) can be applied to these condensates, this result tell us that the $Q = 0$ topological sector, which is free from the global axial anomaly, spontaneously breaks the $U(1)_u$ axial symmetry at $m_u = 0, m_d \neq 0$, and the $U(1)_d$ symmetry at $m_d = 0, m_u \neq 0$. The phase diagram of QCD in the $Q = 0$ topological sector, in the (m_u, m_d) plane, shows therefore two first order phase transition lines, which coincide with the coordinate axes, finishing at the end point $m_u = m_d = 0$, which is a critical point for any $T > T_c$.

Equation (5) tell us that the critical chiral equation of state of QCD at $\theta = 0$ should be the same as the one of the $Q = 0$ topological sector, and should show therefore a divergent correlation length at any $T > T_c$ in the chiral limit. We expect therefore a continuous finite temperature chiral transition, and a divergent correlation length for any $T \geq T_c$, and because the symmetry breaking pattern is, in the two flavor model, $SU(2)_L \times SU(2)_R \to SU(2)_V$, the critical equation of state should be that of the three-dimensional $O(4)$ vector universality class [17], which shows a critical exponent $\delta = 4.789(6)$ [18] ($\delta = 3$ in the mean field or Landau approach).

For $N_f \geq 3$ a similar argument on the phase diagram of the $Q = 0$ sector applies, but the scenario that emerges in this case is not plausible because no stable fixed points are expected in the corresponding Landau-Ginzburg-Wilson Φ^4 theory compatible with the given symmetry-breaking pattern [19].

4 Conclusions and comments

We started recently an investigation of the topological properties of QCD in the high temperature chiral symmetric phase in reference [1]. The starting hypothesis in [1] was to assume that the perturbative expansion of the free energy density in powers of the quark mass, m, has a non-vanishing convergence radius in the high temperature chiral symmetric phase of QCD, which is just what we expect if all correlation lengths remain finite in the chiral limit, and the spectrum of the model shows therefore a mass gap also in this limit. The main conclusion in [1] was that all the topological effects of the axial anomaly should disappear in this phase, the topological susceptibility and all θ-derivatives of the free energy density vanish, and the theory becomes θ independent at any $T > T_c$ in the infinite-volume limit. Accordingly, the free energy density should be a singular function of the quark mass, in the chiral limit, if the topological effects of the $U(1)_A$ anomaly survive in the chiral symmetry restored phase of QCD at finite temperature.

Ongoing with this research line, the main purpose in reference [2] was to further investigate this issue. To this end our starting hypothesis was to assume that the topological effects of the anomaly survive in the high temperature phase of QCD, and the model shows therefore a non-trivial θ-dependence in this phase. Under this assumption we have shown that indeed, the free energy density is a singular function of the quark mass, m, in the chiral limit at any $T > T_c$, and that the correlation length and the σ and $\bar{\pi}$ susceptibilities diverge in this limit. Under the same assumption we have also shown that the difference between the $\bar{\pi}$ and $\bar{\delta}$ susceptibilities diverges in the chiral limit at any $T \geq T_c$.

This result seems to be excluded by recent results of Tomiya et al. [20] from numerical simulations of two-flavor QCD, thus suggesting the effects of the $U(1)_A$ anomaly are absent in the chiral symmetric phase of two-flavor QCD. However, previous results by Dick et al. [21] on larger lattices, but using overlap fermions only in the valence sector, seem to predict a divergent $\chi_{\bar{\pi}}(m) - \chi_{\bar{\delta}}(m)$ in the chiral limit, in agreement with equation (14). Hence the numerical results of [20] and [21] are in disagreement and do not allow to get a definite answer. Any further clarification of the numerical results for $\chi_{\bar{\pi}}(m) - \chi_{\bar{\delta}}(m)$ would be therefore very welcome.

We have also discussed that the previous results for the two-flavor model apply also to $N_f \geq 3$. However, universality and renormalization-group arguments, based on the most general Landau-Ginzburg-Wilson Φ^4 theory compatible with the given symmetry-breaking pattern, make this scenario not plausible because no stable fixed points are expected in the corresponding Landau-Ginzburg-Wilson Φ^4 theory for $N_f \geq 3$ [19].

References

[1] V. Azcoiti, *Phys. Rev. D* **94**, 094505 (2016).
[2] V. Azcoiti, *Phys. Rev. D* **96**, 014505 (2017).
[3] E.V. Shuryak, *Comments Nucl. Part. Phys.* **21**, 235 (1994).
[4] T. D. Cohen, *Phys. Rev. D* **54**, R1867 (1996).
[5] C. Bernard, T. Blum, C. DeTar, S. Gottlieb, U. M. Heller, . E. Hetrick, K. Rummukainen, R. Sugar, D. Toussaint, and M. Wingate, *Phys. Rev. Lett.* **78**, 598 (1997).
[6] S. Chandrasekharan and N. H. Christ, *Nucl.Phys.Proc.Suppl.* **47**, 527 (1996).
[7] J.B. Kout, J.F. Lagae and D.K. Sinclair, *Nucl.Phys.Proc.Suppl.* **53**, 269 (1997).
[8] G. Boyd, F. Karsch, E. Laermann and M. Oevers, *hep-lat/9607046* (1996).
[9] S. Aoki, H. Fukaya and Y. Taniguchi, *Phys. Rev. D* **86**, 114512 (2012).
[10] T. Kanazawa and N. Yamamoto, *J. High Energy Phys.* **01**, (2016) 141.
[11] C. Bonati, M. D'Elia, M. Mariti, G. Martinelli, M. Mesiti, F. Negro, F. Sanfilippo, G. Villadoro, *J. High Energy Phys.* **03**, (2016) 155.
[12] P. Petreczky, H-P. Schadler and S. Sharma, *Phys. Lett. B* **762**, 498 (2016).
[13] Sz. Borsanyi et al., *Nature (London)* **539** , 69 (2016).
[14] S. Coleman, *Ann. Phys.* **101**, 239 (1976).
[15] A.V. Smilga, *Phys. Lett. B* **278**, 371 (1992).
[16] C. Gattringer, I. Hip and C.B. Lang, *Phys. Lett. B* **466**, 287 (1999).
[17] R.D. Pisarski and F. Wilczek, *Phys. Rev. D* **29**, 338 (1984).
[18] M. Hasenbusch, *J. Phys. A* **34** 8221 (2001).
[19] E. Vicari, *Proc.Sci., LATTICE2007* (2007) 023.
[20] A. Tomiya, G. Cossu, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko and J. Noaki, *arXiv:1612.01908 [hep-lat].*
[21] V. Dick, F. Karsch, E. Laermann, S. Mukherjee, and S. Sharma, *Phys. Rev. D* **91**, 094504 (2015).