

Pion mass dependence of the HVP contribution to muon $g - 2$

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Abstract. One of the systematic errors in some of the current lattice computations of the HVP contribution to the muon anomalous magnetic moment $g - 2$ is that associated with the extrapolation to the physical pion mass. We investigate this extrapolation assuming lattice pion masses in the range of 220 to 440 MeV with the help of two-loop chiral perturbation theory, and find that such an extrapolation is unlikely to lead to control of this systematic error at the 1% level. This remains true even if various proposed tricks to improve the chiral extrapolation are taken into account.

1 Introduction

The leading-order (LO), hadronic vacuum polarization (HVP) contribution to the muon anomalous magnetic moment $a_\mu = (g - 2)/2$ is given by the integral

$$a_\mu^{\text{HLO}} = -4\alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w(Q^2) (\Pi(Q^2) - \Pi(0)), \quad (1)$$

where $\Pi(Q^2)$ is obtained from the electromagnetic (EM) hadronic vacuum polarization

$$\Pi_{\mu\nu}(Q) = (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu) \Pi(Q^2), \quad (2)$$

α is the fine-structure constant, and $w(Q^2)$ is a known weight function [1, 2]. The integrand in Eq. (1) is strongly peaked at $Q^2 \sim m_\mu^2/4 \sim (50 \text{ MeV})^2$ because of the presence of the factor $w(Q^2)/Q^2$. With the smallest momentum on a periodic volume at present being at least 100 MeV or larger, all the usual systematic errors afflicting a precision determination of $\Pi(Q^2)$ are “magnified,” and it is thus important to investigate each of the sources of systematic error in detail, especially if the aim is a computation of a_μ^{HLO} with an accuracy better than 1%. Here, we consider the extrapolation to the physical pion mass from a lattice computation using pion masses in the range 220 to 440 MeV, assuming that all other errors (in particular, lattice spacing and finite volume effects) would be under control. In particular, we investigate the use of the so-called “ETMC” trick [3], and variants thereof, which were designed to improve the chiral behavior of a_μ^{HLO} , *i.e.*, to “flatten” the dependence of a_μ^{HLO} on m_π .

For the published version of this work, see Ref. [4].

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2 The ETMC trick

In Ref. [3], it was proposed to use, instead of Eq. (1), the expression¹

$$a_\mu^{\text{HLO}} = -4\alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w(Q^2) \left(\Pi \left(\frac{m_{\rho,\text{latt}}^2}{m_\rho^2} Q^2 \right) - \Pi(0) \right), \quad (3)$$

where $m_\rho = 775$ MeV is the physical rho mass, while $m_{\rho,\text{latt}}$ is the rho mass extracted from the lattice ensemble also used to compute $\Pi(Q^2)$. Of course, at the physical pion mass (assuming extrapolation to the continuum limit and infinite volume), we have that $m_{\rho,\text{latt}} = m_\rho$, and Eq. (3) is identical to Eq. (1).

The observation that the extrapolation to the physical pion mass using Eq. (3) may be flatter than that using Eq. (1) rests on the observation that vector-meson dominance (VMD) gives a fairly good approximation to the HVP. Inspired by VMD, let us assume a model for the HVP of the form

$$\Pi^{\text{VMD}}(Q^2) = \frac{f_\rho^2 m_\rho^2}{m_\rho^2 + Q^2} + \Delta\Pi(Q^2), \quad (4)$$

where $\Delta\Pi(Q^2)$ estimates the difference between perturbation theory and VMD for $Q^2 \gg m_\rho^2$. It follows that, with $\Pi(Q^2)$ computed on the lattice

$$\Pi^{\text{VMD}} \left(\frac{m_{\rho,\text{latt}}^2}{m_\rho^2} Q^2 \right) = \frac{f_\rho^2 m_{\rho,\text{latt}}^2}{m_{\rho,\text{latt}}^2 + \frac{m_{\rho,\text{latt}}^2}{m_\rho^2} Q^2} + \Delta\Pi \left(\frac{m_{\rho,\text{latt}}^2}{m_\rho^2} Q^2 \right) = \frac{f_\rho^2 m_\rho^2}{m_\rho^2 + Q^2} + \Delta\Pi \left(\frac{m_{\rho,\text{latt}}^2}{m_\rho^2} Q^2 \right), \quad (5)$$

where we assumed that the dependence of $f_{\rho,\text{latt}} \approx f_\rho$, for which there is some evidence. As the perturbative tail for large Q^2 , $\Delta\Pi(Q^2)$, only contributes a few percent to the total a_μ^{HLO} , we see that, to the extent that VMD gives a good estimate, indeed Eq. (3) is less dependent on m_π than Eq. (1).

This simple argument suggests that indeed using Eq. (3) instead of Eq. (1) should improve the extrapolation to the physical pion mass. However, it is well known that the two-pion cut also contributes significantly (of order 10%) to a_μ^{HLO} , and a detailed investigation is necessary to check the improvement in accuracy that can be gained with this trick. In fact, in Ref. [3], the lowest pion mass in the lattice computations was about 300 MeV, and the results of that work suggest that the extrapolation from that value to the physical value is rather long. A variant of this trick was also used by HPQCD in Ref. [5], where the lowest-order pion loop contribution (*cf.* Sec. 3 below) at the lattice pion mass was subtracted first, before applying the ETMC trick to the modified data, after which the lowest-order pion loop was added back in at the physical pion mass. We will also consider this variant in the investigation we report on below. We will do so using next-to-next-to-leading-order (NNLO) Chiral Perturbation Theory (ChPT) for the vacuum polarization.

3 Chiral perturbation theory

To simplify matters, we will first consider the $I = 1$ component of the HVP, returning later to the full EM case. The results we will need from ChPT were calculated in Ref. [6]. For $I = 1$, schematically, it looks like

$$\Pi^{I=1}(Q^2) - \Pi^{I=1}(0) = -4(F(Q^2) - F(0)) - \frac{4Q^2}{f_\pi^2} F^2(Q^2) + \frac{16Q^2}{f_\pi^2} L_9 F(Q^2) + 8C_{93} Q^2 + C(Q^2)^2, \quad (6)$$

¹We discuss a specific example of the ETMC trick, the one which has been most used in practice.

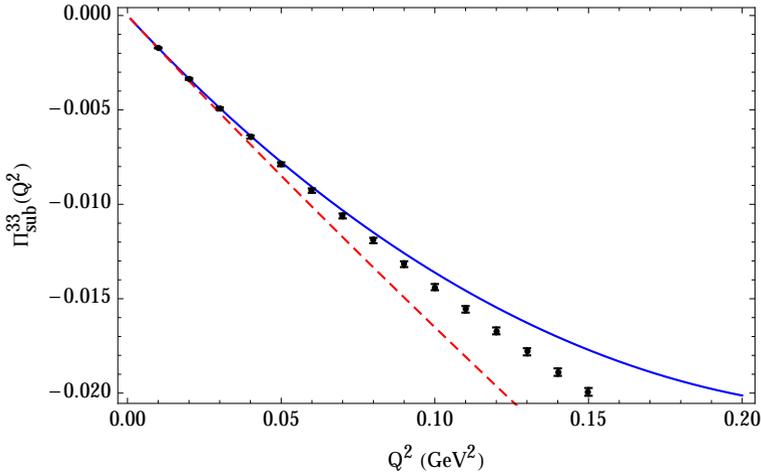


Figure 1. Comparison of the HVP between data from hadronic tau decays (black points) and ChPT. The blue solid curve includes the C term in Eq. (6) with value $C' = 0.289 \text{ GeV}^{-4}$; the red dashed curve has $C = 0$.

where $F(Q^2)$ is a known function (representing two-pion and two-kaon cuts), and $L_9 = 0.00593(43)$ [7], $C_{93} = -0.01536(44) \text{ GeV}^{-2}$ [8] and $C = 0.289 \text{ GeV}^{-4}$ [8] are NLO, NNLO, and NNNLO low-energy constants (LECs), respectively. C represents the analytic $(Q^2)^2$ dependence at NNNLO, but only models the Q^2 dependence at that order, because the non-analytic terms at that order are not known.

Figure 1 shows the comparison between ChPT and data for $\Pi^{I=1}(Q^2) - \Pi^{I=1}(0)$, where the data were extracted from the ALEPH data for non-strange, vector hadronic tau decays [9]. The construction of these data points from the data, using fits to extend the measured spectral function beyond m_τ^2 [10], was described in detail in Ref. [4]. The ChPT curves in the figure include the C term in Eq. (6) (blue solid curve), or have it set equal to zero (red dashed curve). The figure shows that ChPT agrees well with the data to about $Q^2 \sim 0.1 \text{ GeV}^2$, if the phenomenological C term is included. In order to make this quantitative, we consider a cut-off version of a_μ^{HLO} :

$$a_\mu^{\text{HLO},I=1}(Q_{\text{max}}^2) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{Q_{\text{max}}^2} \frac{dQ^2}{Q^2} w(Q^2) (\Pi^{I=1}(Q^2) - \Pi^{I=1}(0)), \quad (7)$$

so that for Q_{max}^2 small enough, we can use the ChPT representation for the integrand. Taking $Q_{\text{max}}^2 = 0.1 \text{ GeV}^2$, we find that

$$\begin{aligned} a_\mu^{\text{HLO},I=1}(0.1) &= 9.81 \times 10^{-8}, & (\text{data}), \\ &= 9.73 \times 10^{-8}, & (\text{ChPT}), \\ a_\mu^{\text{HLO},I=1}(\infty) &= 11.95 \times 10^{-8}, & (\text{data}). \end{aligned} \quad (8)$$

This shows that for $Q_{\text{max}}^2 = 0.1 \text{ GeV}^2$ ChPT agrees with the data to better than 1%, and that the interval between 0 and 0.1 GeV^2 yields 82% of the value of $a_\mu^{\text{HLO},I=1}$.

We will use $a_\mu^{\text{HLO},I=1}(0.1)$ in order to investigate the dependence on the pion mass, using ChPT in order to do so. As already discussed in Sec. 2, also the rho mass depends on the pion mass, and it is this dependence which lies at the heart of the trick employed in Refs. [3, 5] to extrapolate to the

physical pion mass. In ChPT, dependence on the rho mass will show up through the mass dependence of effective versions, $C_{93,\text{latt}}^{\text{eff}}$ and $C_{\text{latt}}^{\text{eff}}$, of the (mass-independent) LECs C_{93} and C obtained in fits employing the form (6) above while also incorporating higher-order mass-dependent terms in the chiral expansion. Assuming VMD, one expects these effective LECs to scale with m_ρ as

$$C_{93,\text{latt}}^{\text{eff}} = \hat{C}_{93}^{\text{eff}} \frac{m_\rho^2}{m_{\rho,\text{latt}}^2}, \quad C_{\text{latt}}^{\text{eff}} = \hat{C} \frac{m_\rho^4}{m_{\rho,\text{latt}}^4}, \quad (9)$$

where $\hat{C}_{93}^{\text{eff}}$ and \hat{C} are the effective LECs at the physical pion mass. We will employ Eq. (6) with C_{93} and C replaced by $C_{93,\text{latt}}^{\text{eff}}$ and $C_{\text{latt}}^{\text{eff}}$ to construct a model for use in calculating $a_\mu^{\text{HLO},I=1}(Q_{\text{max}}^2)$ for non-physical values of the pion mass and exploring the impact of the ETMC and HPQCD tricks.

4 Pion mass dependence of $a_\mu^{\text{HLO}}(0.1)$

We will first consider the pion mass dependence of $a_\mu^{\text{HLO},I=1}$ by considering the contribution up to $Q_{\text{max}}^2 = 0.1 \text{ GeV}^2$, *i.e.*, $a_\mu^{\text{HLO},I=1}(0.1)$. Of course, the “missing” part (the contribution from $Q^2 > 0.1 \text{ GeV}^2$) also depends on the pion mass. However, the pion mass dependence of that part is inherently different from that of $a_\mu^{\text{HLO},I=1}(0.1)$, and it would be a fluke if this would change the lessons we will learn from considering $a_\mu^{\text{HLO},I=1}(0.1)$. Moreover, we find that these lessons do not change when we take $Q_{\text{max}}^2 = 0.2 \text{ GeV}^2$ [4].

In order to study the pion mass dependence of $a_\mu^{\text{HLO},I=1}(0.1)$, we need the pion mass dependence of m_K and f_π , both of which appear in Eq. (6), as well as of m_ρ , which appears “indirectly,” through Eq. (9). As an example, we take a range of pion masses between 220 MeV and 440 MeV, with corresponding values for m_K , f_π and m_ρ from MILC [12]. They are shown in Table 1. We use these

Table 1. Masses and f_π values used in our tests. All numbers in MeV.

m_π	m_K	f_π	m_ρ
223	514	98	826
262	523	101	836
313	537	104	859
382	558	109	894
440	581	114	929

values to calculate, using ChPT, three values of $a_\mu^{\text{HLO},I=1}(0.1)$, and show these in Fig. 2, for each of the lines in Table 1. The black dot shows the value at the physical point; while the points at the larger pion masses show the values at the points of Table 1, with the lower set corresponding to the unmodified values for $a_\mu^{\text{HLO},I=1}(0.1)$, the middle set the values modified by the use of the ETMC trick, and the upper set of points those modified by the HPQCD version of the ETMC trick.

The three different panels all show the same points, but the fits are different. The curves correspond to fits to the functional forms

$$\begin{aligned} a_\mu^{\text{HLO},I=1,\text{quad}}(0.1) &= Am_{\pi,\text{latt}}^4 + Bm_{\pi,\text{latt}}^2 + C, & (\text{quadratic}), & (10) \\ a_\mu^{\text{HLO},I=1,\text{log}}(0.1) &= A \log(m_{\pi,\text{latt}}^2/m_\pi^2) + Bm_{\pi,\text{latt}}^2 + C, & (\text{logarithmic}), & \\ a_\mu^{\text{HLO},I=1,\text{in}}(0.1) &= A/m_{\pi,\text{latt}}^2 + Bm_{\pi,\text{latt}}^2 + C, & (\text{inverse}), & \end{aligned}$$

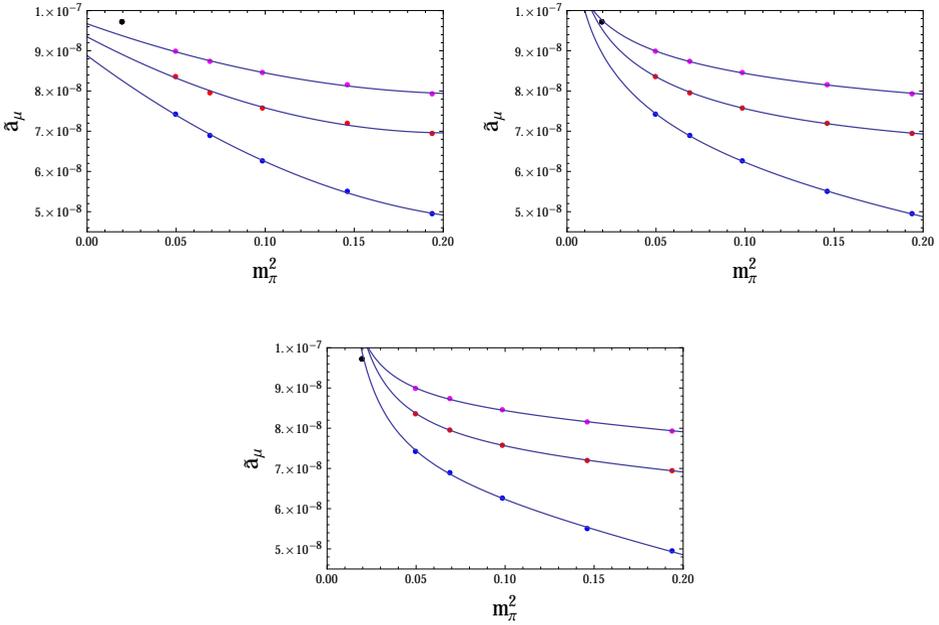


Figure 2. The unmodified and ETMC- and HPQCD-improved versions of $\tilde{a}_\mu \equiv a_\mu^{\text{HLO},J=1}(0.1)$ as a function of m_π^2 . In each plot, the upper (magenta) data points are HPQCD-improved, the middle (red) data points ETMC-improved and the lower (blue) data points unimproved. The (black) point in the upper left corner of each plot is the “physical” point, $a_\mu^{\text{HLO},J=1}(0.1) = 9.73 \times 10^{-8}$ (cf. Eq. (8)). Fits are “quadratic” (upper left panel), “log” (upper right panel) and “inverse” (lower panel). For further explanation, see text.

with the upper left panel showing the quadratic fit, the upper right panel showing the logarithmic fit, and the lower panel the inverse fit. The logarithmic form is inspired by ChPT, even though the values of m_π for which the chiral logarithm should be visible are far smaller than the physical pion mass (see below). The inverse fit was employed in Ref. [5].

The figures show that all fits lead to extrapolations that miss the physical point, with possible exceptions for the logarithmic fit if one uses either version of the ETMC-modified points, and the inverse fit if one uses the unmodified points. We quantify these deviations in Table 2. In the case of

Table 2. Values for $a_\mu^{\text{HLO},J=1}(0.1) \times 10^8$ for the three types of fit (cf. Eq. (10)) and the three data sets. For reference, the correct model value is $a_\mu^{\text{HLO},J=1}(0.1) \times 10^8 = 9.73$ (cf. Eq. (8)).

	unimproved data	ETMC-improved data	HPQCD-improved data
quadratic fit	8.26	8.91	9.38
logarithmic fit	8.96	9.55	9.77
inverse fit	9.93	10.46	10.33

real QCD, we will not have access to the “exact” value of $a_\mu^{\text{HLO},J=1}(0.1)$, as we have here in this test, of course. Therefore, one will have to resort to a different method for estimating the systematic error associated with the extrapolation to the physical pion mass. With no solid theoretical guidance on what the fit form should be, the best option is to take the difference between two “reasonable” fits. Taking,

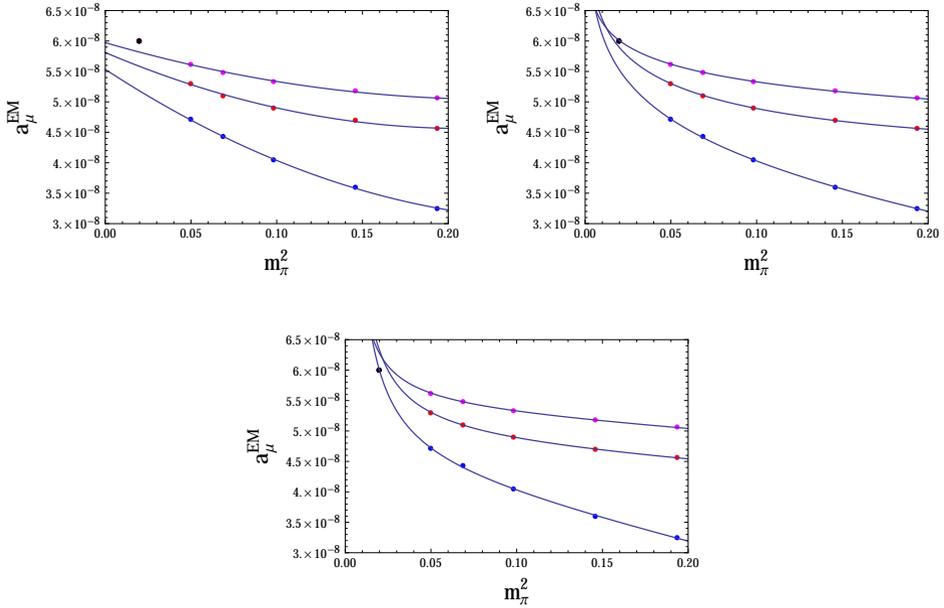


Figure 3. The unmodified and ETMC- and HPQCD-improved versions of $a_\mu^{\text{HLO,EM}}(0.1)$ as a function of m_π^2 . In each plot, the upper (magenta) data points are HPQCD-improved, the middle (red) data points ETMC-improved and the lower (blue) data points unimproved. The (black) point in the upper left corner of each plot is the “physical” point, $a_\mu^{\text{HLO,EM}}(0.1) = 6.00 \times 10^{-8}$. Fits are “quadratic” (upper left panel), “log” (upper right panel) and “inverse” (lower panel). For further explanation, see text.

for instance, the relative difference between extrapolations using the quadratic and logarithmic fits, one finds differences of 8%, 7% and 4% for the unimproved, ETMC-improved and HPQCD-improved data. We see that indeed the two variants of the ETMC trick do improve the chiral extrapolation, but not sufficiently to reach the desired accuracy of below 1%. We have also redone this exercise omitting the largest pion mass (440 MeV), but found that this makes no significant difference. We emphasize that the logarithmic fit is *not* theoretically preferred. While ChPT does predict a logarithm as in Eq. (10) [4], its coefficient is far smaller than the values we find in unconstrained fits. The reason is that the scale for the chiral logarithm as predicted in ChPT is set by the muon mass, while for the data we consider here, the pion mass is much larger than the muon mass. For more discussion of this point, we refer to Ref. [4].

We repeated this test for the EM case, and the corresponding plots are shown in Fig. 3. Here we lack the direct comparison with experimental data, but otherwise, the exercise can be carried out in the same manner as for the $I = 1$ case. We thus have to assume that also in this case the representation of the EM HVP by ChPT to NNLO (augmented with the C term) is good enough up to Q_{max}^2 equal to $0.1 - 0.2 \text{ GeV}^2$. The numerical results analogous to those of Table 2 are shown in Table 3. In this case, systematic error estimates obtained by taking the relative difference between the quadratic and logarithmic fits amounts to 6%, 5% and 3% for the unimproved, ETMC-improved, and HPQCD-improved data. While these numbers are slightly better than those for the $I = 1$ case, the conclusion is the same: both variants of the ETMC trick do not allow chiral extrapolations with an accuracy of less than 1%. Again, omitting the largest pion mass does not make a significant difference.

Table 3. Values for $a_\mu^{\text{HLO,EM}}(0.1) \times 10^8$ for the three types of fit (*cf.* Eq. (10)) and the three data sets. For reference, the correct model value is $a_\mu^{\text{HLO,EM}}(0.1) \times 10^8 = 6.00$.

	unimproved data	ETMC-improved data	HPQCD-improved data
quadratic fit	5.19	5.59	5.82
logarithmic fit	5.55	5.91	6.02
inverse fit	6.04	6.37	6.30

5 Conclusion

We used a ChPT-inspired model to investigate the extrapolation of the leading-order hadronic contribution to the muon anomalous magnetic moment, a_μ^{HLO} , from lattice pion masses of order 200 to 400 MeV to the physical pion mass. We found that such pion masses are too large to allow for a reliable extrapolation, if the aim is an extrapolation error of less than 1%. This is true even if various tricks to improve the extrapolation are employed [3, 5].

In order to perform our study, we had to make certain assumptions. First, we assumed that useful insight into the pion mass dependence could be obtained by focussing on the contribution to a_μ^{HLO} up to $Q_{max}^2 = 0.1 \text{ GeV}^2$. This restriction is necessary if we want to take advantage of information on the mass dependence from ChPT, since only in this range ChPT provides a reasonable representation of the HVP; we believe this is not a severe restriction. Changing Q_{max}^2 to 0.2 GeV^2 makes no qualitative difference to our conclusions.

Second, we assumed Eq. (9) for the dependence of the effective LECs $C_{93,\text{latt}}^{\text{eff}}$ and $C_{\text{latt}}^{\text{eff}}$ on the pion mass. While this is a phenomenological assumption, we note that this assumption is in accordance with the ideas underlying the ETMC and HPQCD tricks, so that those tricks should work well if indeed this assumption is correct. There are two reasons that the modified extrapolations nevertheless do not work well enough to achieve the desired better than 1% accuracy. One is the fact that in addition to the physics of the ρ , the two-pion intermediate state contributing to the non-analytic terms in Eq. (6), also beyond leading order, plays a significant role as well. The second reason is that, although ChPT provides a simple functional form for the chiral extrapolation of $a_\mu^{\text{HLO}}(Q_{max}^2 = 0.1 \text{ GeV}^2)$ for pion masses much smaller than the muon mass, this is not useful in practice, so that one needs to rely on phenomenological fit forms, such as those of Eq. (10).

In order to eliminate the systematic error from the chiral extrapolation, which we showed to be very difficult to estimate reliably, one needs to compute a_μ^{HLO} at, or close to, the physical pion mass. In contrast to the experience with simpler quantities such as meson masses and decay constants, even an extrapolation from approximately 200 MeV pions turns out to be a long extrapolation.

It would be interesting to consider the case in which extrapolation from larger than physical pion masses is combined with direct computation near the physical pion mass in order to reduce the total error on the final result. This case falls outside the scope of the study presented here, because the trade-off between extrapolation and computation at the physical point is expected to depend on the statistical errors associated with the ensembles used for each pion mass. However, our results imply that also in this case a careful study should be performed. The methodology developed in this paper can be easily adapted to different pion masses and extended to take into account lattice statistics, and we thus recommend such a study for any computation of the HVP contribution to the muon anomalous magnetic moment that relies on extrapolation from larger than physics pion masses.

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