

A lattice calculation of the hadronic vacuum polarization contribution to $(g - 2)_\mu$

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Abstract. We present results of calculations of the hadronic vacuum polarisation contribution to the muon anomalous magnetic moment. Specifically, we focus on controlling the infrared regime of the vacuum polarisation function. Our results are corrected for finite-size effects by combining the Gounaris-Sakurai parameterisation of the timelike pion form factor with the Lüscher formalism. The impact of quark-disconnected diagrams and the precision of the scale determination is discussed and included in our final result in two-flavour QCD, which carries an overall uncertainty of 6%. We present preliminary results computed on ensembles with $N_f = 2 + 1$ dynamical flavours and discuss how the long-distance contribution can be accurately constrained by a dedicated spectrum calculation in the iso-vector channel.

1 Introduction

The persistent deviation of 3.5 – 4 standard deviations between the direct measurement of the muon’s anomalous magnetic moment $a_\mu = \frac{1}{2}(g - 2)_\mu$ and the value predicted by theory [1] may signal a limitation in the ability of the Standard Model (SM) to provide an accurate and precise description of particle properties. Two new experiments, E989 at Fermilab and E34 at J-PARC, will increase the precision of the direct experimental determination of a_μ by up to a factor of four, which calls for a similar reduction in the uncertainty of the SM prediction. Since the latter is dominated by the contributions from low-energy QCD, it is particularly timely to revisit the determinations of the hadronic vacuum polarisation and hadronic light-by-light scattering contributions, a_μ^{hvp} and a_μ^{hlbl} , respectively. Since the current estimates for these quantities either rely on experimental data or on model assumptions, it is highly desirable to compute them via a first-principles approach such as lattice QCD. However, in order to be competitive with the dispersive approach, any lattice QCD calculation of a_μ^{hvp} must be able to control all sources of error at the level of 0.5% or better, if such studies are to have an impact on testing the limits of the SM. By contrast, a determination of a_μ^{hlbl} with an overall error of about 15% will constitute a major step forward.

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Here we report on our ongoing effort in determining a_μ^{hvp} in lattice QCD with fully controlled errors. In particular, we describe our calculation in QCD with two flavours, which is focussed on methodology, and which has been published in [2]. We also discuss preliminary results obtained on gauge ensembles with 2 + 1 dynamical quark flavours.

2 Methodology

The leading hadronic vacuum polarisation (HVP) is accessible in lattice QCD via two types of integral representations. The first is formulated in terms of a convolution integral over Euclidean momenta [3, 4] and reads

$$a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 K(Q^2) \hat{\Pi}(Q^2), \quad (1)$$

where $K(Q^2)$ is a known kernel function, and $\hat{\Pi}(Q^2) = 4\pi^2 (\Pi(Q^2) - \Pi(0))$ denotes the subtracted vacuum polarisation which is obtained from the correlator of the electromagnetic current $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \dots$ via

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0) \rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2). \quad (2)$$

Alternatively, one can express a_μ^{hvp} via the ‘‘time-momentum representation’’ (TMR) [5], by integrating the spatially summed correlator $G(x_0)$ multiplied by a kernel $\tilde{K}(x_0)$ over Euclidean time according to

$$a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{K}(x_0) G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle, \quad (3)$$

$$\tilde{K}(x_0) = 4\pi^2 \int_0^\infty dQ^2 K(Q^2) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{1}{2}Qx_0\right) \right]. \quad (4)$$

A major difficulty, which is encountered in both approaches, is associated with controlling the infrared regime of the vacuum polarisation. In the first approach, one must determine the additive renormalisation $\Pi(0)$ and a sufficiently precise representation of $\Pi(Q^2)$ for values of Q^2 that do not exceed the muon mass by much. While the use of partially twisted boundary conditions is helpful [6] in this regard, it does not remove the need for simulating large lattice volumes in order to reliably constrain $\Pi(Q^2)$ in the low-momentum regime [7].

If one employs the TMR, one is confronted with the problem of controlling the long-distance regime of $G(x_0)$, which is difficult owing to its rapidly rising noise-to-signal ratio. Therefore one has to resort to some kind of model for the large- x_0 behaviour. Furthermore, in the light quark sector, $G(x_0)$ is dominated by a two-pion state in a p -wave as $x_0 \rightarrow \infty$. Possible choices for the description of the iso-vector component of $G(x_0)$ at large distances include a naive single exponential function that falls off with the mass of the ρ -resonance or an *ansatz* that supplements the single exponential with the contribution from the two-pion state. A third possibility is based on the Gounaris-Sakurai parameterisation [8] of the timelike pion form factor that enters the iso-vector spectral function [9, 10]. In ref. [2] we presented a detailed discussion of the different strategies to control the low-energy regime.

3 Lattice calculation, systematic effects and results

Our results for the hadronic vacuum polarisation have been obtained using gauge ensembles generated as part of the CLS effort. An estimate for a_μ^{hvp} in two-flavour QCD, including a detailed error budget

has been published in [2]. This study was performed using two degenerate flavours of $O(a)$ improved Wilson quarks and the Wilson plaquette action, using the parameterisation of the clover coefficient c_{sw} from [11]. Extrapolations to the physical point have been performed using data at three different values of the lattice spacing (i.e. $a = 0.049, 0.066$ and 0.079 fm) and pion masses ranging from 185 – 495 MeV, always keeping $m_\pi L \geq 4$. In addition, we present preliminary results for QCD with $N_f = 2 + 1$ dynamical flavours [12], based on the tree-level Symanzik improved gauge action with c_{sw} tuned according to ref. [13]. The current range of lattice spacings and pion masses comprises $a = 0.050, 0.065, 0.085$ fm and $m_\pi = 200 - 420$ MeV, which is similar to the two-flavour case. We postpone the inclusion of isospin-breaking effects to future work (see [14, 15] for pilot studies).

In the following we focus on the discussion of systematic effects arising from the long-distance behaviour of the vector correlator, which is also connected with the issue of finite-volume effects. The accuracy of the scale determination is also an important factor for the overall precision of the lattice prediction of a_μ^{hvp} . For this purpose we restrict the discussion to the TMR. It is then convenient to define the quark-connected contribution of flavour $f = ud, s, c$ to the vector correlator via

$$G^f(x_0) = -\frac{a^3}{3} \sum_k \sum_{\vec{x}} q_f^2 Z_V \left\langle V_{k,f}^{\text{con}}(x_0, \vec{x}) V_{k,f}^{\text{loc}}(0) \right\rangle, \quad (5)$$

where q_f is the quark electric charge¹, Z_V denotes the renormalisation factor of the local vector current $V_{\mu,f}^{\text{loc}}(x) = \bar{\psi}_f(x) \gamma_\mu \psi_f(x)$, and $V_{\mu,f}^{\text{con}}(x)$ represents the conserved (point-split) current. The corresponding contribution to the hadronic vacuum polarisation of flavour f is then given by

$$(a_\mu^{\text{hvp}})^f = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{K}(x_0) G^f(x_0), \quad f = ud, s, c. \quad (6)$$

The statistical precision of the light quark contribution $(a_\mu^{\text{hvp}})^{ud}$, which dominates the total HVP, is limited by the exponentially growing noise-to-signal ratio of the integrand in eq. (6). This is shown in Figure 1 where we have plotted the product $\tilde{K}(x_0) G^{ud}(x_0)$ in units of m_μ versus the Euclidean time separation x_0 [fm]. Above a certain value, i.e. $x_0^{\text{cut}} \approx 1.3$ fm, the signal has deteriorated to such an extent that one has to resort to a model for $G^{ud}(x_0)$. The various coloured bands compare the extension based on the single exponential and the Gounaris-Sakurai (GS) parameterisation. Overall, one finds that both types of extension yield consistent results for $(a_\mu^{\text{hvp}})^{ud}$ within errors.

It is also possible to determine finite-volume corrections based on the GS model. To be more explicit, it is useful to start with the isospin decomposition of the vector correlator, i.e.

$$G(x_0) = G^{\rho\rho}(x_0) + G^{I=0}(x_0), \quad G^{\rho\rho}(x_0) = \frac{9}{10} G^{ud}(x_0), \quad (7)$$

where $G^{\rho\rho}(x_0)$ denotes the iso-vector contribution, which is proportional to the connected light-quark contribution $G^{ud}(x_0)$. In infinite volume the iso-vector part is given by

$$G^{\rho\rho}(x_0, \infty) = \int_{2m_\pi}^\infty d\omega \omega^2 \rho(\omega^2) e^{-\omega|x_0|} = \frac{1}{48\pi^2} \int_{2m_\pi}^\infty d\omega \omega^2 \left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2 e^{-\omega|x_0|}, \quad (8)$$

where $\rho(\omega^2)$ is the (continuous) spectral function, and $F_\pi(\omega)$ denotes the pion form factor in the timelike regime. In a finite volume, characterised by a box size L , one encounters a discrete spectrum, i.e.

$$G^{\rho\rho}(x_0, L) \stackrel{x_0 \rightarrow \infty}{=} \sum_n |A_n|^2 e^{-\omega_n x_0}, \quad \omega_n = 2 \sqrt{m_\pi^2 + k^2}. \quad (9)$$

¹For $f = ud$, the combined iso-symmetric light quark contribution yields $q_{ud}^2 = 5/9$.

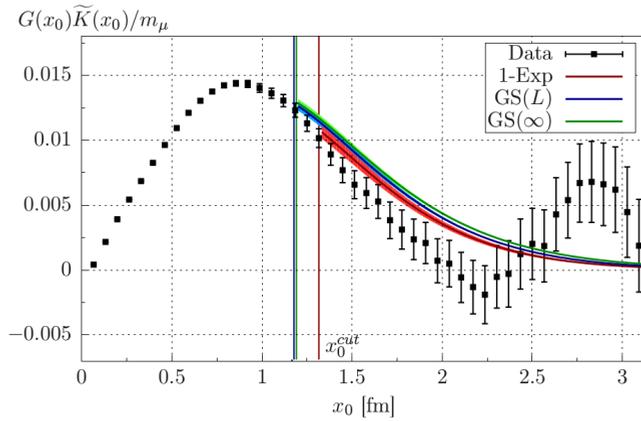


Figure 1. Light quark contribution to the integrand, $\tilde{K}(x_0)G^{ud}(x_0)$, in units of the muon mass, computed in two-flavour QCD at a pion mass of 185 MeV [2]. Data are shown by black filled squares, while extensions of the vector correlator for $x_0 > 1.2$ fm based on a single exponential and the GS parameterisation (with and without finite-volume correction) are represented by coloured bands.

The energies ω_n are related to the scattering momentum via the Lüscher condition [16]:

$$\delta_1(k) + \phi(q) = 0 \text{ mod } \pi, \quad q = \frac{kL}{2\pi}. \quad (10)$$

In the inelastic region the amplitudes A_n can be expressed in terms of the timelike pion form factor [9]

$$|A_n|^2 = \frac{2k^2}{3\pi\omega_n^2} \frac{|F_\pi(\omega_n)|^2}{k\phi'(k) + k\delta_1'(k)}. \quad (11)$$

Given input data for $F_\pi(\omega) \equiv |F_\pi(\omega)|e^{i\delta_1(k)}$, one may then determine the finite-volume shift by forming the difference $G^{\rho\rho}(x_0, \infty) - G^{\rho\rho}(x_0, L)$ and inserting it into eq. (3). In the absence of any direct calculation of ω_n and $|A_n|$, one can resort to the GS parameterisation of $F_\pi(\omega)$ in terms of the resonance mass m_ρ and the width Γ_ρ . Both parameters can be determined from lattice data via the following procedure: In a first step, the mass of the ground state is identified with the ρ -meson mass extracted from a smeared vector correlation function computed in an auxiliary lattice calculation via a two-state fit with the lowest level set to $2\sqrt{m_\pi^2 + (2\pi/L)^2}$. Then, by inserting the GS model for $|F_\pi|$ and δ_1 into eq. (11) one obtains an expression for the correlator $G^{\rho\rho}(x_0)$ in terms of the GS parameters (m_ρ, Γ_ρ). Fitting $G^{\rho\rho}(x_0)$ to the form in eq. (9) with m_ρ fixed to the estimate extracted from the smeared correlator yields the width parameter Γ_ρ . During this fit one inserts the GS parameterisation for δ_1 into the Lüscher condition and determines the scattering momenta k of excited states in the iso-vector channel, which, in turn, yield the corresponding energy levels and matrix elements ω_n and $|A_n|^2$. In this way one can evaluate $G^{\rho\rho}(x_0, L)$ in eq. (9) for, say, a handful of states. The procedure is completed by inserting the GS parameterisation into eq. (8), which yields the correlator $G^{\rho\rho}(x_0, \infty)$. Since the iso-scalar part in eq. (7) is sub-dominant, one may approximate it by a single exponential whose fall-off is given by $m_\omega \approx m_\rho$. We have determined the finite-volume shift in $(a_\mu^{\text{hvp}})^{ud}$ according to the above procedure, for all of our two-flavour ensembles with pion masses $m_\pi \leq 340$ MeV. Unsurprisingly, the biggest correction of +3 % was encountered at our smallest pion mass of 185 MeV.

Another source of uncertainty that has received relatively little attention in most of the earlier calculations of a_μ^{hvp} arises from the uncertainty in the lattice scale. Although a_μ^{hvp} is dimensionless, there are two ways in which its determination in lattice QCD introduces a scale dependence.

Firstly, the muon mass m_μ enters the kernel function $\tilde{K}(x_0)$ via the dimensionless combination $x_0 m_\mu$. Secondly, the masses of the dynamical quarks enter implicitly via the lattice evaluation of the vector correlator. Therefore, a_μ^{hvp} can be thought of as a function in the dimensionless combinations $M_\mu \equiv m_\mu/\Lambda$, $M_u \equiv m_u/\Lambda$, $M_d \equiv m_d/\Lambda$, \dots , where Λ is the quantity that sets the lattice scale. The scale setting error $\Delta\Lambda$ then induces a corresponding uncertainty in a_μ^{hvp} , i.e.

$$\Delta a_\mu^{\text{hvp}} = \left| \Lambda \frac{da_\mu^{\text{hvp}}}{d\Lambda} \right| \frac{\Delta\Lambda}{\Lambda} = \left| M_\mu \frac{\partial a_\mu^{\text{hvp}}}{\partial M_\mu} + \sum_{f=1}^{N_f} M_f \frac{\partial a_\mu^{\text{hvp}}}{\partial M_f} \right| \frac{\Delta\Lambda}{\Lambda}. \quad (12)$$

It is useful to replace $M_u, M_d \dots$ and their derivatives by suitable meson masses in units of Λ . In the isospin limit the above expression can then be rewritten as

$$\Delta a_\mu^{\text{hvp}} = \left| M_\mu \frac{\partial a_\mu^{\text{hvp}}}{\partial M_\mu} + M_\pi \frac{\partial a_\mu^{\text{hvp}}}{\partial M_\pi} + M_K \frac{\partial a_\mu^{\text{hvp}}}{\partial M_K} + \dots \right| \frac{\Delta\Lambda}{\Lambda}. \quad (13)$$

When working with the TMR one can determine the derivative term involving the muon mass via [2]

$$M_\mu \frac{\partial a_\mu^{\text{hvp}}}{\partial M_\mu} = -a_\mu^{\text{hvp}} + \left(\frac{\alpha}{\pi} \right)^2 \int_0^\infty dx_0 G(x_0) J(x_0), \quad J(x_0) = x_0 \tilde{K}'(x_0) - \tilde{K}(x_0), \quad (14)$$

where the kernel $J(x_0)$ can be easily computed using the series expansion of $\tilde{K}(x_0)$ from appendix B in [2]. The derivative w.r.t. the pion mass M_π can be estimated from the slope of the chiral extrapolation of $(a_\mu^{\text{hvp}})^{ud}$ at $m_\pi = m_\pi^{\text{phys}}$ (see left panel of Figure 2). With these results at hand one finds

$$\frac{\Delta a_\mu^{\text{hvp}}}{a_\mu^{\text{hvp}}} = \left| \underbrace{M_\mu \frac{\partial a_\mu^{\text{hvp}}}{\partial M_\mu}}_{1.8} + \underbrace{M_\pi \frac{\partial a_\mu^{\text{hvp}}}{\partial M_\pi}}_{-0.18(6)} \right| \frac{\Delta\Lambda}{\Lambda}. \quad (15)$$

The factor multiplying the scale setting uncertainty is thus dominated by the contribution from the muon mass, with only a 10% reduction coming from the light quarks. Heavier quark flavours are likely to have an even smaller effect. The lesson one can draw from this analysis is the fact that the proportionality between the relative uncertainties of a_μ^{hvp} and the lattice scale Λ is a number of order 1. Therefore, the lattice scale must be known to within a fraction of a percent, if one is to reach the precision goal in the determination of a_μ^{hvp} .

We have subjected our data for the various flavour contributions to a_μ^{hvp} to combined chiral and continuum extrapolations, using a variety of different *ansätze*, such as

$$\begin{aligned} \text{Fit A: } & \alpha_1 + \alpha_2 m_\pi^2 + \alpha_3 m_\pi^2 \ln m_\pi^2 + \alpha_4 a, & \text{Fit B: } & \beta_1 + \beta_2 m_\pi^2 + \beta_3 m_\pi^4 + \beta_4 a, \\ \text{Fit C: } & \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 a, & \text{Fit D: } & \delta_1 + \delta_2 a. \end{aligned} \quad (16)$$

Since we did not include the $O(a)$ improvement term in the vector currents, we expect that the leading cutoff effects are linear in the lattice spacing, which accounts for the term of order a in the above expressions. Fits A and B each contain a term allowing for a curvature in the chiral behaviour of a_μ^{hvp} . Other models describing the pion mass dependence contain terms that diverge in the chiral limit, such as $1/m_\pi^2$ or $\ln m_\pi^2$. While the latter is only justified in the region where $m_\pi < m_\mu$, inverse powers of m_π^2 may over-amplify the pion mass dependence around the physical pion mass, as was noted in [17]. We have therefore excluded singular *ansätze* in our final analysis. After adding the contributions from the light, strange and charm quarks, we arrive at our final estimate for a_μ^{hvp} [2]:

$$a_\mu^{\text{hvp}} = (654 \pm 32_{\text{stat}} \pm 17_{\text{syst}} \pm 10_{\text{scale}} \pm 7_{\text{FV}} \pm 0_{-10}^{\text{disc}}) \cdot 10^{-10}, \quad (17)$$

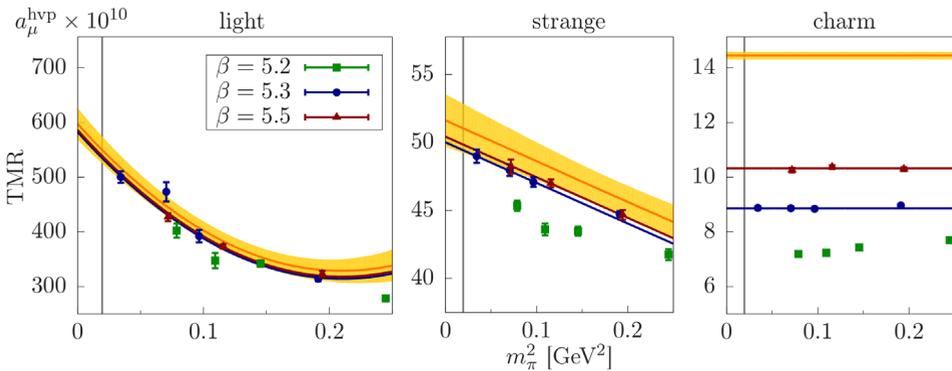


Figure 2. Combined chiral and continuum extrapolations of the individual quark flavour contributions to a_μ^{hvp} computed using the TMR in two-flavour QCD [2]. The yellow band represents the pion mass dependence in the continuum limit. The vertical line indicates the physical value of the pion mass.

where the statistical and systematic errors have been estimated via the “extended frequentist’s method” which combines the standard bootstrap technique with a number of procedural variations in the analysis. The remaining systematic errors refer to the scale uncertainty, the error assigned to the estimation of the finite-volume shift and the upper bound on the effect of including quark-disconnected diagrams. The calculation of the latter on a subset of gauge ensembles is described in appendix D of ref. [2]. A crucial ingredient for obtaining accurate results is the cancellation of stochastic noise between the contributions of the light and strange quark flavours, discussed in [18].

A compilation of results for a_μ^{hvp} from different collaborations is shown in Figure 3. Overall there is good agreement among different groups, with the largest observed deviations between individual calculations amounting to about 2.5 standard deviations. It is clear, though, that the overall errors have to be further reduced in order to be competitive with the dispersive analysis.

We have started to compute a_μ^{hvp} on a set of CLS ensembles generated with $N_f = 2 + 1$ flavours of dynamical quarks. In order to combat the problem of topology freezing, a large fraction of the ensembles have open boundary conditions [26, 27], which precludes the determination of the vacuum polarisation via the four-dimensional Fourier transform of eq. (2). Since the TMR is independent of the type of boundary conditions, we exclusively focus on this approach in all our calculations for $N_f = 2 + 1$. One important additional ingredient relative to the earlier two-flavour calculation is the use of the $O(a)$ improved versions of the local and point-split vector currents [28].

In Figure 4 we show once more the integrand of eq. (3), computed at a pion mass of 200 MeV at $a = 0.065$ fm. The data confirm that a single exponential provides a good approximation of the long-distance behaviour of the integrand. We also display the results of a state-of-the-art calculation aimed at determining the energies ω_n and corresponding matrix elements A_n of the vector current for the n^{th} energy eigenstate in the iso-vector channel. This not only allows for a precise direct calculation of the long-distance behaviour of $G^{\text{pp}}(x_0, L)$ but also provides the determination of the timelike pion form factor that is necessary to compute the finite-volume shift. An account of our method has been provided in [29]. The results from the spectrum calculation demonstrates that the iso-vector correlator $G^{\text{pp}}(x_0)$ is saturated by the first four states in that channel, for $x_0 \gtrsim 1.7$ fm. It is also interesting to note that the two-pion state starts to dominate the correlator for $x_0 \gtrsim 3$ fm. The explicit spectrum calculation provides a highly accurate and precise determination of the vector correlator at long distances, especially since the error grows only linearly with x_0 , in contrast to the exponential error growth of G^{pp} itself.

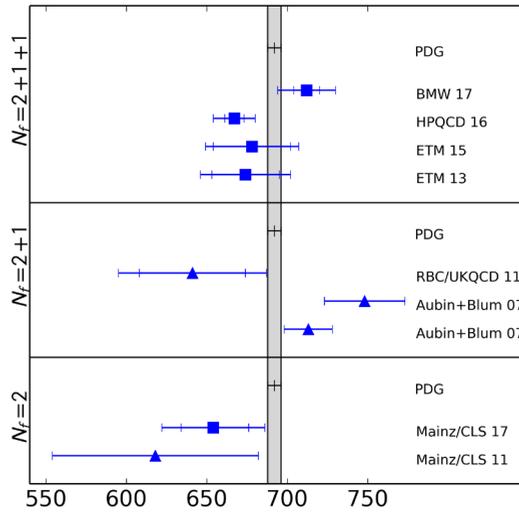


Figure 3. Compilation of results for the hadronic vacuum polarisation contribution a_μ^{hvp} in units of 10^{-10} . The three panels represent calculations with different numbers of quarks in the sea. Squares denote estimates including the contributions from u, d, s, c quarks in the valence sector, while triangles represent results for u, d, s quarks only. The meaning of the labels is: Mainz/CLS 11 [6], Mainz/CLS 17 [2], Aubin+Blum 07 [19], RBC/UKQCD 11 [20], ETM 13 [21], ETM 15 [22], HPQCD 16 [23] and BMW 17 [24]. See [25] for another recent calculation. The vertical band represents the result from dispersion theory.

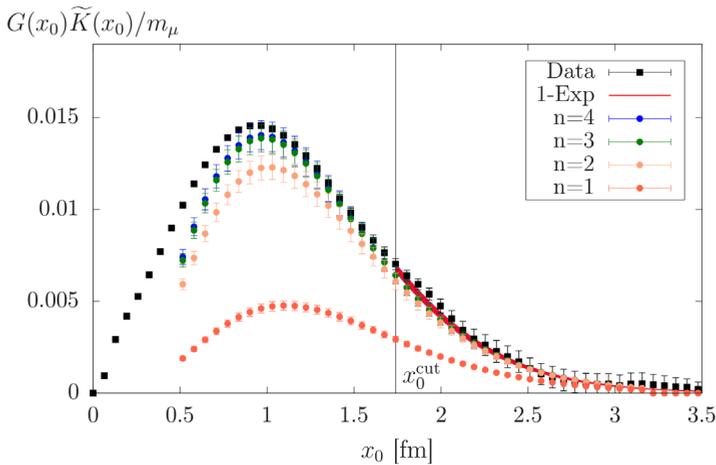


Figure 4. Preliminary results for the light quark contribution to the integrand, $\tilde{K}(x_0)G^{ud}(x_0)$, in units of m_μ , computed for $N_f = 2 + 1$ at $m_\pi = 200$ MeV. Data points are shown by black filled squares. The red circles denote the two-pion contribution to the iso-vector correlator $G^{\rho\rho}$, with the remaining coloured points showing the accumulated contributions from the higher excited states.

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