

Supermultiplets in $\mathcal{N}=1$ SUSY SU(2) Yang-Mills Theory

Sajid Ali¹, Georg Bergner², Henning Gerber^{1,*}, Pietro Giudice¹, Simon Kuberski¹, István Montvay³, Gernot Münster¹, Stefano Piemonte⁴, and Philipp Scior¹

¹*Institut für Theoretische Physik, Universität Münster, Wilhelm-Klemm-Str. 9, D-48149 Münster*

²*Theoretisch-Physikalisches Institut, Universität Jena, Max-Wien-Platz 1, D-07743 Jena*

³*Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, D-22603 Hamburg*

⁴*Fakultät für Physik, Universität Regensburg, Universitätsstr. 31, D-93053 Regensburg*

Abstract. We study $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (SYM) on the lattice. The non-perturbative nature of supersymmetric field theories is still largely unknown. Similarly to QCD, SYM is confining and contains strongly bound states. Applying the variational method together with different smearing techniques we extract masses of the lightest bound states such as gluino-gluon, glueball and mesonic states. As these states should form supermultiplets, this study allows to check whether SYM remains supersymmetric also on the quantum level.

1 Introduction

The Standard Model of particle physics accurately describes current particle physics experiments. However, it is neither theoretically plausible as a fundamental theory (e.g. hierarchy problem), nor does it explain the observation of dark matter. It should therefore be viewed as a low energy effective theory of some more complete theory of physics. Deviations from the Standard Model are expected to be observed when measuring with more precision or at higher energies.

One possible extension of the Standard Model which cures the hierarchy problem and also provides candidates for dark matter is supersymmetry. Supersymmetry relates bosonic and fermionic fields to one another and implies that for every particle there exists a supersymmetric partner particle of the same mass and a spin different by $1/2$. Since this is not observed by current experiments, supersymmetry must be broken if it exists.

Supersymmetric models have been analyzed extensively within the framework of perturbation theory. On the other hand, it is desirable to gain more insight in their non-perturbative properties. We study $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (SYM) on the lattice. It is the simplest supersymmetric theory containing non-abelian gauge fields. Important topics to be studied are, besides others, the question whether a supersymmetric continuum limit exists and what the particle spectrum of such a theory looks like.

When treated within the framework of lattice field theory, supersymmetry is broken due to the breaking of Poincaré invariance by the lattice. In Ref. [1] it has been suggested that $\mathcal{N} = 1$ SYM can be studied on the lattice by extrapolating to the chiral and continuum limit, see [2] for a detailed

*Speaker, E-mail: h.gerber@uni-muenster.de

review. An important check of supersymmetry restoration in this limit is to study the multiplet formation which was predicted in Ref. [3] and Ref. [4]. The authors argue that if SYM is unbroken, chiral supermultiplets are formed from glueball states with quantum numbers 0^{++} and 0^{-+} (in J^{PC} notation), $a-f_0$ and $a-\eta'$ mesons¹ and additional gluino-gluon states.

The groundstates of these particles have been investigated in Ref. [5] for SYM with the gauge group $SU(2)$. Indeed, a mass degeneracy of these states (except for the 0^{-+} -glueball) was found in the chiral and continuum limit. The goal of this work is to analyze also the masses of the corresponding first excited states.

1.1 The Theory

The Lagrangian of SYM in the continuum reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu,a} + \frac{i}{2}\bar{\lambda}^a (\not{D}\lambda)^a - \frac{m_g}{2}\bar{\lambda}^a \lambda^a, \quad (1)$$

where $F^{\mu\nu,a}$ is the field strength tensor built from the gluon field A_μ^a . The corresponding superpartner is the gluino field $\lambda^{a,\alpha}$. Supersymmetry requires the gluino to be in the adjoint representation of the gauge group and to be of Majorana type, i.e. fulfill the Majorana condition

$$\bar{\lambda} = \lambda^T C, \quad (2)$$

where C is the charge conjugation matrix. The covariant derivative for the gauge group $SU(2)$ is given by

$$(D_\mu \lambda)^a = \partial_\mu \lambda^a + g\epsilon_{abc}A_\mu^b \lambda^c. \quad (3)$$

The gluino mass term breaks supersymmetry softly. In order to approach the chiral limit, the bare gluino mass m_g needs to be fine-tuned such that the renormalized gluino mass vanishes, see also [6].

For the lattice discretization we use Wilson fermions and a tree level Symanzik improved gauge action. Additionally, we use one level of stout smearing [7] for the links in the fermionic part of the action in order to improve the signal-to-noise ratio of the measurements.

2 Techniques

2.1 Variational Method

For the extraction of the excited state masses we use the variational method as proposed in Ref. [8]. It requires solving a generalized eigenvalue problem (GEVP) of the form

$$C(t)\vec{v}^{(n)} = \omega^{(n)}(t, t_0)C(t_0)\vec{v}^{(n)}. \quad (4)$$

Here $C(t)$ is the correlation matrix, $\omega^{(n)}(t, t_0)$ and $\vec{v}^{(n)}$ are the corresponding generalized eigenvalues and eigenvectors. The entries of the correlation matrix are given by correlation functions of different interpolating fields O_i of the respective state:

$$C_{ij}(t) = \langle O_i(t)O_j^\dagger(0) \rangle. \quad (5)$$

¹The names stem from the similarity of these states with the corresponding particles in QCD. The fermionic constituents are, however, in the adjoint and not in the fundamental representation of the group, see section 1.1.

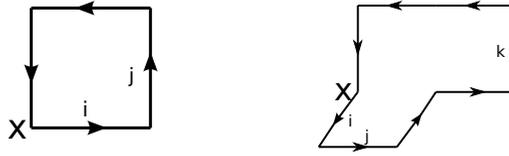


Figure 1. Plaquette $P_{ij}(x)$ and gauge-loop $C_{ijk}(x)$ used in the interpolating field of 0^{+-} -glueball.

The large- t behavior of the generalized eigenvalues allows to extract the mass m_n of the excited states by fitting the eigenvalues $\omega^{(n)}(t, t_0)$ to exponentials:

$$\lim_{t \rightarrow \infty} \omega^{(n)}(t, t_0) \propto e^{-m_n(t-t_0)} \left(1 + \mathcal{O}\left(e^{-\Delta m_n(t-t_0)}\right) \right), \quad \Delta m_n = \min_{l \neq n} |m_l - m_n|. \quad (6)$$

For a good signal it is preferable to choose a set of interpolating fields which on the one hand have good overlaps with the physical states and on the other hand have only small mutual overlaps.

2.2 Interpolating fields

The interpolating fields O_i used in the correlation matrix (5) have to match the quantum numbers of the respective state to be analyzed. The basic interpolating fields that we use are described below. In order to improve the signal-to-noise ratio and to create different interpolating fields for each state we apply different smearing methods to these operators (see Sect. 2.3).

The basic interpolating fields for glueballs are built from gauge link loops that respect the parity quantum number of the respective state. For the 0^{++} -glueballs we use a sum of gauge plaquettes

$$O_{\text{gb}^{++}}(x) = \text{Tr} [P_{12}(x) + P_{23}(x) + P_{31}(x)], \quad (7)$$

where P_{ij} denotes a plaquette in the i - j direction (see Fig. 1). For the pseudo-scalar glueball 0^{-+} we use

$$O_{\text{gb}^{-+}}(x) = \sum_{R \in \mathbf{O}_h} [\text{Tr} [C(x)] - \text{Tr} [PC(x)]], \quad (8)$$

where the sum is over all rotations of the cubic group \mathbf{O}_h and PC is the parity conjugate of loop C ; it is depicted in Fig. 1.

The basic interpolating fields for the mesons are

$$O_{\text{a-f}_0}(x) = \bar{\lambda}(x)\lambda(x), \quad O_{\text{a-}\eta'}(x) = \bar{\lambda}(x)\gamma_5\lambda(x). \quad (9)$$

When inserted into the correlation matrix, Wick contractions of these fields lead to connected and disconnected pieces

$$\langle \bar{\lambda}(x)\Gamma\lambda(x)\bar{\lambda}(y)\Gamma\lambda(y) \rangle = \text{Tr} [\Gamma\Delta(x, x)] \text{Tr} [\Gamma\Delta(y, y)] - 2\text{Tr} [\Gamma\Delta(x, y)\Gamma\Delta(y, x)], \quad (10)$$

where $\Delta(x, y)$ denotes the propagator from x to y (spin and group indices suppressed) and Γ represents $\mathbb{1}$ or γ_5 .

The interpolating field of the gluino-gluon state is given by

$$O_{\text{g-g}}^\alpha(x) = \sum_{i < j=1}^3 \sigma_{ij}^{\alpha\beta} \text{Tr} [P_{ij}(x)\lambda^\beta(x)], \quad (11)$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

2.3 Smearing Methods

In order to improve the signal-to-noise ratio and to obtain different interpolating fields O_i for the variational method, we use APE-smearing [9] for the gauge links $U_\mu(x)$ and gauge invariant Jacobi smearing [10, 11] for the gluino field $\lambda(x)$.

An APE-smearing step is given by

$$U'_\mu(x) = U_\mu(x) + \epsilon_{\text{APE}} \sum_{v=\pm 1, v \neq \mu}^{\pm 3} U_v^\dagger(x + \hat{\mu}) U_\mu(x + \hat{v}) U_v(x), \quad (12)$$

where ϵ_{APE} is a parameter of the smearing method that needs to be optimized to maximize the signal-to-noise ratio. The APE-smearing step can be iteratively applied to obtain higher smearing levels. The gluino field is smeared by applying the Jacobi smearing function:

$$\lambda'^{bb}(\vec{x}, t) = \sum_{\vec{y}} F_{\beta b, \alpha a}(\vec{x}, \vec{y}, t) \lambda^{\alpha a}(\vec{y}, t), \quad (13)$$

where the smearing function $F_{\beta b, \alpha a}(\vec{x}, \vec{y}, t)$ is given by

$$F_{\beta b, \alpha a}(\vec{x}, \vec{y}, t) = \delta_{\vec{x}, \vec{y}} \delta_{\beta \alpha} + \delta_{\beta \alpha} \sum_{i=1}^{N_J} \left(\kappa_J \sum_{\mu=1}^3 \left[\delta_{\vec{y}, \vec{x} + \hat{\mu}} U_\mu(\vec{x}, t) + \delta_{\vec{y} + \hat{\mu}, \vec{x}} U_\mu^\dagger(\vec{x}, t) \right] \right)_{ba}^i, \quad (14)$$

with the smearing parameter κ_J and the smearing level N_J .

2.4 Identifying the eigenvalues

The errors of the generalized eigenvalues that are used to fit the masses and the final errors of the masses are estimated by resampling the data using the Jackknife method.

Fitting an exponential to the n -th generalized eigenvalue $\omega^{(n)}(t, t_0)$ requires identifying it out of all the generalized eigenvalues at t on each Jackknife sample. Originally it was proposed to identify the eigenvalues at each t by their magnitude, $\omega^{(0)} > \omega^{(1)} > \omega^{(2)}, \dots$. However, due to limited statistics the eigenvalues fluctuate on the Jackknife samples. If the fluctuations of two or more eigenvalues overlap, it is not suitable any longer to identify them by their magnitude. In this situation we use a different approach, which is to identify the generalized eigenvalues by their corresponding generalized eigenvectors $\vec{v}^{(n)}$ in the following way:

1. Solve the GEVP for all t , find a t_1 where the fluctuations of the $\omega^{(n)}(t_1, t_0)$ are small enough such that they do not overlap. t_1 should be chosen as large as possible, so that the eigenvectors have sufficiently stabilized. Sort the eigenvalues by their magnitude as originally proposed. The corresponding eigenvectors are called $\vec{w}^{(n)}$.
2. Solve the GEVP at all other t , call the eigenvectors $\vec{v}^{(n)}(t, t_0)$.
3. For each t calculate a matrix of scalar products $M_{mn} = |\vec{w}^{(m)} \cdot \vec{v}^{(n)}(t, t_0)|$.
4. (a) Scan for largest entry $M_{n_{\max} m_{\max}}$ in M .
 (b) Identify $\omega^{(m_{\max})}(t, t_0)$ to be the eigenvalue corresponding to $\vec{v}^{(n_{\max})}(t, t_0)$.
 (c) Delete m_{\max} -th row and n_{\max} -th column from M .
 (d) Repeat 3. until all eigenvalues are identified.

A comparison of this method with the earlier one is shown in Fig. 2.

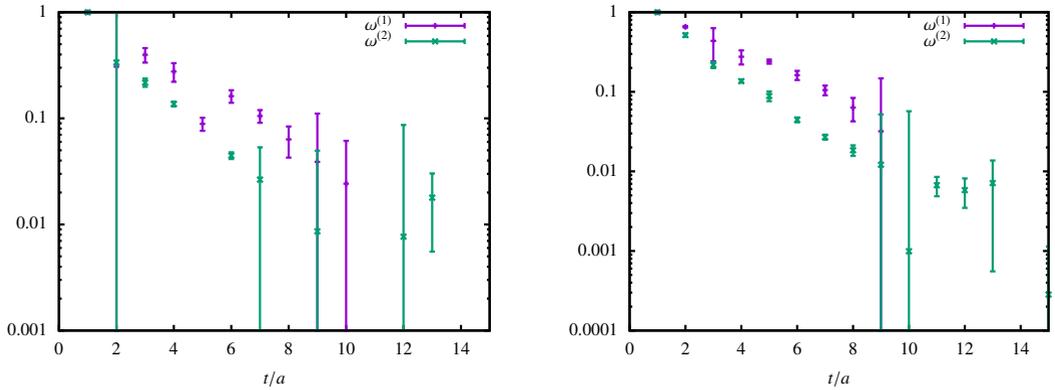


Figure 2. Left: Generalized eigenvalues in the 0^{++} -channel at $\beta = 1.9$, $\kappa = 0.1433$ sorted by magnitude, right: sorted by scalar products of eigenvectors. Six interpolating fields were used in the GEVP; for better visibility only two of the six eigenvalues are displayed. Note that the eigenvalues at $t = 5$ have been misidentified when sorted by magnitude. The identification using scalar products allows to use the data up to $t = 8$.

2.5 Optimization of the smearing parameters

In order to determine the optimal values for the Jacobi smearing parameters κ_J and N_J , we measured the smearing radius defined as

$$R_J^2 = \frac{\sum_{\vec{x}} |\vec{x}|^2 |F(\vec{x}, 0)|^2}{\sum_{\vec{x}} |F(\vec{x}, 0)|^2}.$$

The measurement shows that there is a critical parameter $0.15 < \kappa_J^c < 0.2$ (see Fig. 3). For values above κ_J^c the smearing function (14) diverges in the limit $N_J \rightarrow \infty$, while for smaller values it converges. Since we use Jacobi smearing to create new interpolating fields from the basic ones, it is more suitable to choose a value for κ in the diverging regime. In order to keep numerical errors small, we chose $\kappa_J^c = 0.2$ which is just above the critical κ_J^c . Optimizing the signal-to-noise ratio, we chose to use the smearing levels 0, 40 and 80. In principle more smearing levels could be used, but from our experience it makes the identification of the eigenvalues (see Sect. 2.4) unstable and therefore does not improve the results.

A similar analysis was also done for the APE-smearing. In the gluino-gluon measurements we chose $\epsilon_{APE} = 0.4$ and the smearing levels 0, 5, 15, 25, ..., 95. In the glueball measurements we chose the smearing levels 10, 50 and 80.

2.6 Exploiting mixing

Operators with the same quantum numbers are expected to mix. We expect the a - f_0 meson to mix with the 0^{++} -glueball since they share the same quantum numbers. The same is true for the a - η' meson and the 0^{-+} -glueball. This mixing allows both, mesonic and gluonic, interpolators to be used in the correlation matrix (5) for the extraction of a mixed state. The full correlation matrix then has the following form:

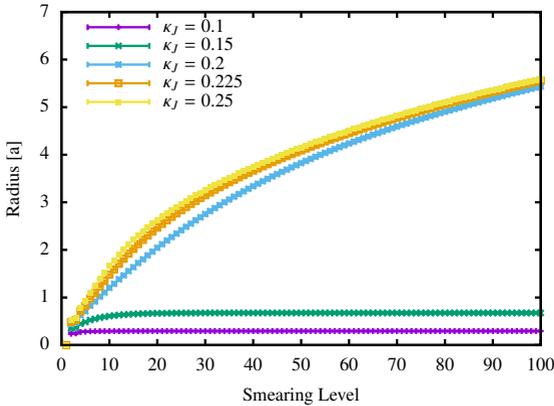


Figure 3. Measurement of the Jacobi smearing radius for different values of κ_J at $\beta = 1.9$, and $\kappa = 0.14415$. For $\kappa_J \geq 0.2$ the smearing function diverges for $N_J \rightarrow \infty$. For $\kappa_J \leq 0.15$ it converges.

$$C(t) = \begin{pmatrix} \left\langle O_{\text{gb}^{++}}(t) O_{\text{gb}^{++}}^\dagger(0) \right\rangle & \left\langle O_{\text{gb}^{++}}(t) O_{a-f_0}^\dagger(0) \right\rangle \\ \left\langle O_{a-f_0}(t) O_{\text{gb}^{++}}^\dagger(0) \right\rangle & \left\langle O_{a-f_0}(t) O_{a-f_0}^\dagger(0) \right\rangle \end{pmatrix}. \quad (15)$$

Here each entry is a submatrix consisting of the different interpolating fields for each operator. Since the mesonic and the gluonic operators are very different from one another, the mutual overlap is expected to be small and using this larger correlation matrix should improve the signal drastically.

3 Preliminary Results

We have used lattices for $\beta = 1.9$ at four different hopping parameters $\kappa \in \{0.1433, 0.14387, 0.14415, 0.14435\}$. For each value of κ there are more than 10.000 lattice gauge configurations available of which only every 8th configurations has been measured so far. Our results are therefore only preliminary.

The results (see Fig. 4) indicate that the variational method in combination with different smearing levels and different interpolating fields can indeed be used to extract the masses of the excited states. As expected, we observe a much cleaner signal in the 0^{++} -channel when using both, mesonic and gluonic, interpolators than when using them separately. Curiously, in the 0^{-+} -channel the correlation matrix is block-diagonal within errors, i.e. there seems to be no mixing between the $a\text{-}\eta'$ meson and the 0^{-+} -glueball. The groundstate mass of the 0^{-+} -glueball seems to be in the range of the excited states. These two findings might be an indication that the 0^{-+} -component of the lowest chiral multiplet only consists of $a\text{-}\eta'$ and that the 0^{-+} -glueball actually belongs to a different, heavier multiplet as was already proposed in Ref. [5].

For the final results the statistics will be increased and a detailed analysis of autocorrelations will be done. A chiral extrapolation to the critical κ_c will be performed. Previous results [5] indicate that the ensemble at $\beta = 1.9$ is already close to the continuum limit. To provide an estimate of lattice discretization errors, we will also analyze a coarser lattice spacing at $\beta = 1.75$.

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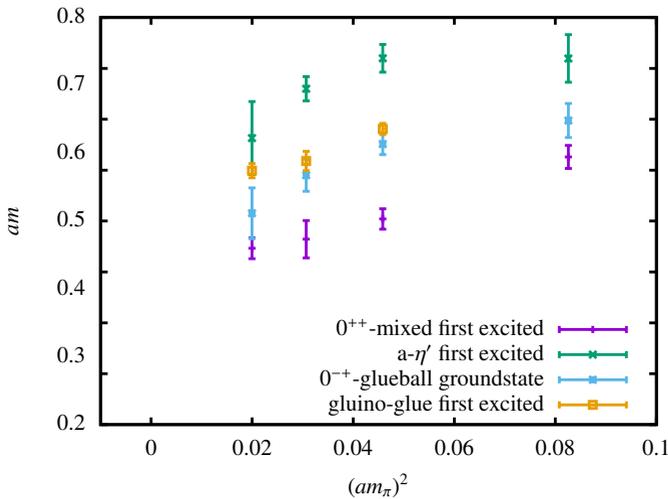


Figure 4. Preliminary results for $\beta = 1.9$. The extracted masses are plotted against the measured squared mass of the unphysical adjoint pion m_π . The gluino mass scales proportionally to m_π^2 , thus the chiral limit is at $m_\pi^2 = 0$ [12].

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