

Weakly coupled conformal gauge theories on the lattice

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Abstract. Results are reported for the β -function of weakly coupled conformal gauge theories on the lattice, SU(3) with $N_f = 14$ fundamental and $N_f = 3$ sextet fermions. The models are chosen to be close to the upper end of the conformal window where perturbation theory is reliable hence a fixed point is expected. The study serves as a test of how well lattice methods perform in the weakly coupled conformal cases. We also comment on the 5-loop β -function of two models close to the lower end of the conformal window, SU(3) with $N_f = 12$ fundamental and $N_f = 2$ sextet fermions.

1 Introduction

We study gauge theories inside the conformal window close to its upper end. In this region the gauge group, number of massless fermion flavors and the representation they carry are such that the perturbative β -function possesses a fixed point at a small value of the renormalized coupling. Hence it is expected that the loop expansion is reliable and the existence of the infrared fixed point will not be spoiled by any non-perturbative effect. The corresponding models are genuine interacting conformal field theories with non-trivial (small) anomalous dimensions [1, 2].

Our main motivation is that there has been extensive lattice study of models close to the lower end of the conformal window because of their relevance for BSM model building in recent years [3]. Close to the lower end of the conformal window non-perturbative effects are relevant because in the conformal case the fixed point coupling is large and in the chirally broken case the entire low energy dynamics is dictated by non-perturbative effects similarly to QCD. In principle lattice simulations are an ideal tool to determine whether a given model is inside or outside the conformal window exactly because the lattice setup can capture all non-perturbative effects. Nevertheless systematic effects can be large sometimes leading to controversies for models close to the lower end of the conformal window. We study here the weakly coupled conformal case which can be tested on the lattice with predictable and controlled results in sufficient orders of perturbation theory. We are interested in how the lattice tool set is able to identify conformality, if there are any unexpected systematic effects and how ambiguous or unambiguous the lattice results are.

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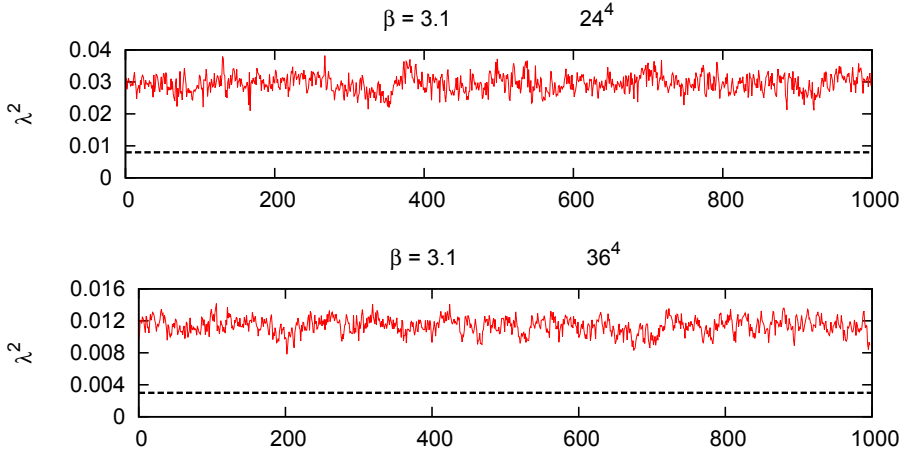


Figure 1. Monte Carlo history of the lowest Dirac operator, $D^\dagger D$, eigenvalues for the $N_f = 14$ fundamental model at the lowest bare coupling β on two selected volumes. Every 10^{th} configuration was measured, the runs correspond to 10000 trajectories in total. The dashed lines show the chosen lower bound for the Remez algorithm. The histories for different bare coupling β and the $N_f = 3$ sextet model are similar.

In this work the finite volume gradient flow [4] running coupling scheme [5, 6] is used to calculate the β -function and to probe the infrared dynamics.

2 Two weakly coupled conformal theories

The models we study are both $SU(3)$ gauge theories. One of them has $N_f = 14$ fundamental, the other $N_f = 3$ sextet (two-index-symmetric) fermions. In what follows we will sometimes refer to the first simply as $N_f = 14$ and the latter as $N_f = 3$, without specifying the representation. The β -functions in $\overline{\text{MS}}$ are known to 5-loops [7, 8],

$$\mu^2 \frac{dg^2}{d\mu^2} = \sum_{i=1}^5 b_i \frac{g^{2i+2}}{(16\pi^2)^i} \tag{1}$$

where the coefficients b_i for the two models are

$$\begin{aligned} N_f = 14 : \quad & b_1 = -\frac{5}{3}, \quad b_2 = \frac{226}{3}, \quad b_3 = \frac{70547}{54}, \quad b_4 = -15506.48, \quad b_5 = -668754.5 \\ N_f = 3 : \quad & b_1 = -1, \quad b_2 = 148, \quad b_3 = \frac{3493}{2}, \quad b_4 = -22834.07, \quad b_5 = -2365262.5 \end{aligned}$$

The corresponding fixed points at increasing loop order are then simply

$$\begin{aligned} N_f = 14 : \quad & g_{*2}^2 = 3.494, \quad g_{*3}^2 = 2.696, \quad g_{*4}^2 = 2.810, \quad g_{*5}^2 = 2.926 \\ N_f = 3 : \quad & g_{*2}^2 = 1.067, \quad g_{*3}^2 = 0.993, \quad g_{*4}^2 = 0.999, \quad g_{*5}^2 = 1.002 \end{aligned} \tag{2}$$

Clearly, these fixed points are all rather small and do not change much beyond 3-loops so it is reasonable to expect that the loop expansion is already a good approximation. This is especially the case for

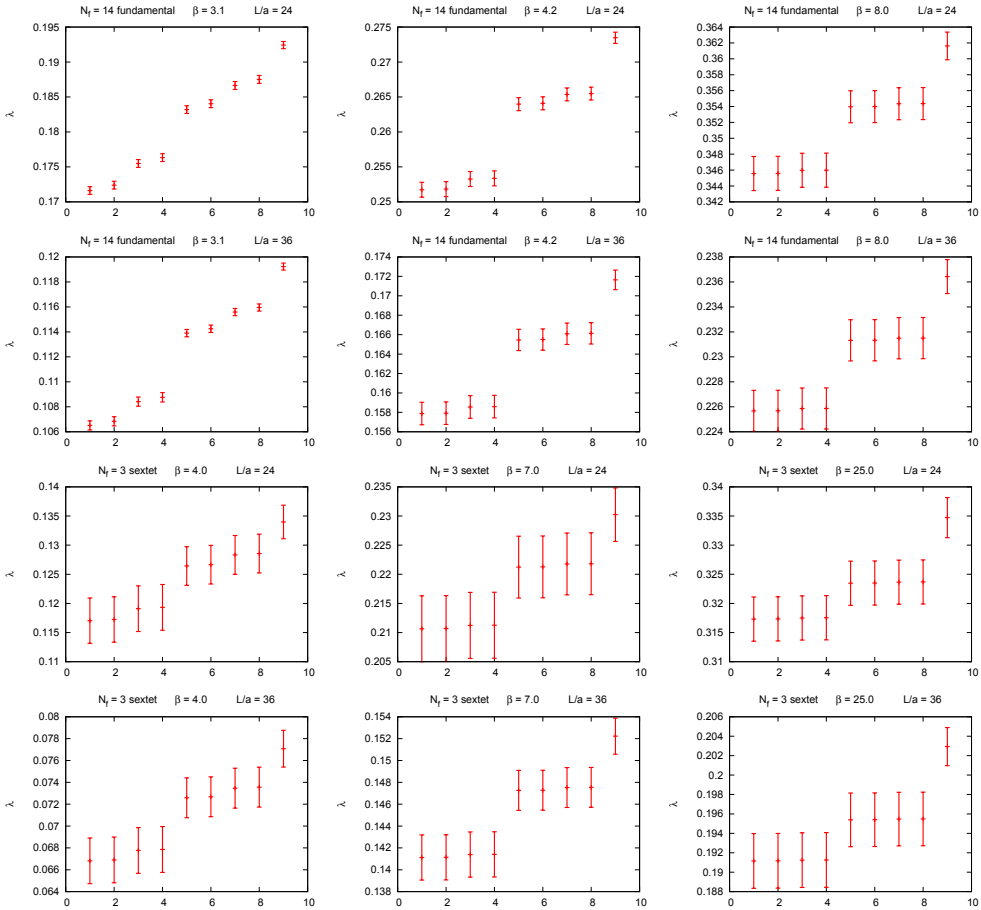


Figure 2. The lowest 9 eigenvalues for both models at the lowest, medium and largest bare coupling β on two selected volumes. Top two rows: $N_f = 14$ fundamental, bottom two rows: $N_f = 3$ sextet. Bare coupling β is increasing from left to right.

the $N_f = 3$ sextet model, which is closer to the upper end of its conformal window than the $N_f = 14$ fundamental model. For the sextet model asymptotic freedom is lost at $N_f = 3.3$ while for the fundamental model at $N_f = 16.5$. Actually $N_f = 14$ was chosen so that it is further from the upper end but still such that the perturbative calculation is presumably trustworthy. Not being very close to the upper end of the conformal window will have practical consequences as explained in the next section.

The fact that the fixed points in both models are expected to be at small couplings in $\overline{\text{MS}}$ indicates that scheme dependence is also expected to be small. Hence a comparison with our scheme (which is different from $\overline{\text{MS}}$) is meaningful.

3 Numerical simulation

The simulations are carried out via the staggered discretization using stout improvement. The running coupling is defined in the finite volume gradient flow scheme with periodic gauge fields and fermions

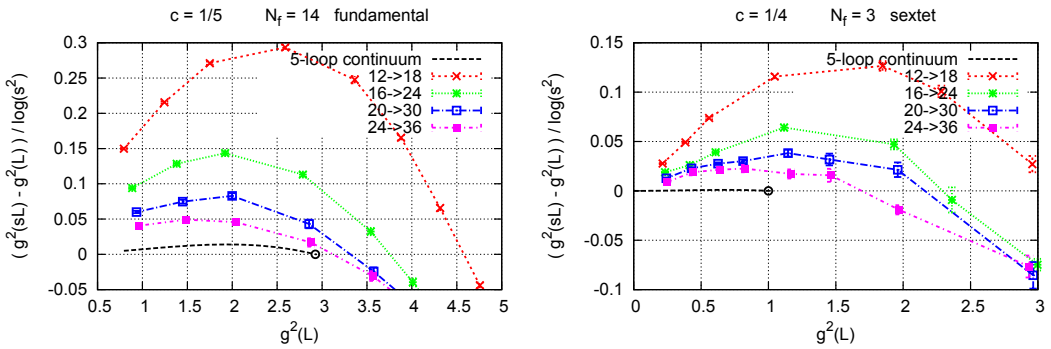


Figure 3. Measured discrete β -function as a function of $g^2(L)$ for the $N_f = 14$ fundamental (left) and the $N_f = 3$ sextet model (right). The 5-loop $\overline{\text{MS}}$ continuum result is also shown. Note that the maximum of the latter is quite small, ≈ 0.014 (left) and ≈ 0.001 (right) hence hardly visible on the plot.

which are anti-periodic in all four directions. The coupling in this scheme is given by

$$g^2(L) = \frac{128\pi^2}{3(N^2 - 1)(1 + \delta(c))} \langle t^2 E(t) \rangle \quad (3)$$

where $N = 3$, $c = \sqrt{8}t/L$ is a constant, $\delta(c)$ is a known factor and we set $c = 1/5$ for definiteness; see [5, 6] for more details. Due to finite volume and the remnant chiral symmetry of staggered fermions the bare mass can be set to zero, $m = 0$, and the only parameters are the lattice volume and the bare coupling β .

Since the flavor content in neither model is divisible by 4 the rooting procedure is required, implemented by the RHMC algorithm. A necessary ingredient is the Remez algorithm which requires preset values for the interval on which the fourth root will be approximated by a rational function. Since the mass is zero, a good choice for the lower end of this interval needs to be measured first. This is shown in figure 1 for two examples. Once an appropriate lower bound is found the simulation is stable. The validity of the rooting procedure at zero fermion mass and finite volume was described in detail in [16]. Key is the finite gap in the Dirac spectrum due to the anti-periodic boundary conditions of the fermions. As the continuum is approached taste breaking is reduced and already at the lowest bare coupling β the quartets in the Dirac spectrum are clearly visible. We measured the lowest 9 eigenvalues and show examples of the restoration of quartet degeneracy in figure 2.

The discretization of the observable E in equation (3) is done by the symmetric clover definition while the Symanzik tree level improved gauge action is used both for the dynamical gauge action in the simulations and for the evolution of the gradient flow. In the terminology of [9, 10] this setup corresponds to the SSC discretization.

Once the renormalized couplings are measured the discrete β -function, $(g^2(sL) - g^2(L))/\log(s^2)$ for some finite ratio s is straightforward to obtain. We choose $s = 3/2$. The continuum will be approached by four pairs of lattice volumes, $12 \rightarrow 18$, $16 \rightarrow 24$, $20 \rightarrow 30$ and $24 \rightarrow 36$. The renormalized coupling is measured at various bare couplings on all volume pairs and the results are shown in figure 3. Also shown in the plot is the 5-loop $\overline{\text{MS}}$ result for comparison. Note that the maximum of the β -function in the $\overline{\text{MS}}$ 5-loop case is rather small, around 0.014 and 0.001 for the $N_f = 14$ and $N_f = 3$ models, respectively. The maximum being very small in the latter is a direct consequence of its closeness to the upper end of the conformal window. Also note that the 5-loop

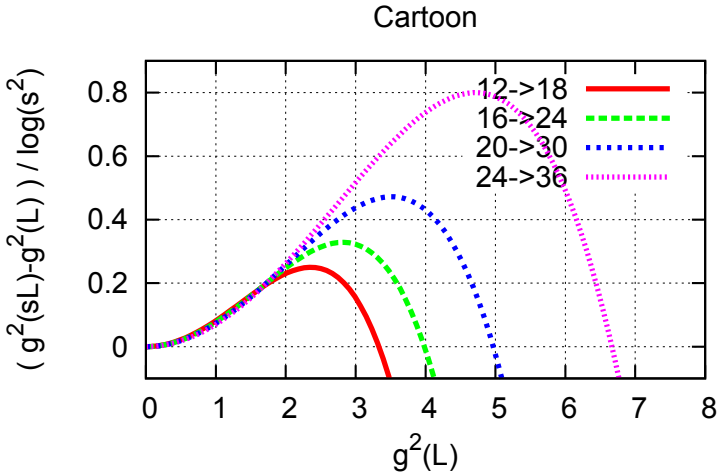


Figure 4. A sketch (not real data) illustrating cases where the zero of the β -function at finite lattice volume keeps increasing towards the continuum. In these cases it is a very delicate issue whether the zeros converge to a finite value or run off to infinity. In the latter case the continuum theory is not conformal.

beta-function possesses a second zero too, but at relatively large coupling for both models, $g^2 \simeq 7.32$ for $N_f = 14$ and $g^2 \simeq 6.39$ for $N_f = 3$. More importantly the location of the second zero at 4-loops is at $g^2 \simeq 18.6$ and 19.8 , respectively, hence convergence is not reached at the 5-loop level for these second zeros, unlike for the first zero. Non-perturbative lattice calculations are needed to rule in or rule out the second zero in the β -function as hinted from 5-loops at strong coupling.

A number of observations are in order. Both models exhibit the following property: the discrete β -function possesses a zero on finite lattice volumes. One needs to be very careful about its interpretation though because it is entirely possible that towards the continuum limit these zeros disappear. In particular if the location of the zeros is *increasing* as the lattice volume is increasing, i.e. towards the continuum limit, then it is not at all clear whether the zeros converge somewhere finite or run off to infinity. Hence a fully controlled continuum extrapolation is mandatory in these cases before definite conclusions can be drawn. A sketch is shown in figure 4 illustrating the potential problem.

Note further though that in both models the zeros of the β -function at finite lattice volume are such that they are *decreasing* as the lattice volume is increasing. In other words, closer and closer to the continuum the detected zero becomes smaller and smaller. In this case, contrary to the situation sketched in figure 4, it is rather easy to conclude assuming that the observed trend does not change. This is because in any case we know that both models are asymptotically free hence a positive β -function is guaranteed at some small positive g^2 . Hence the convergence of the zeros to a positive and finite g_*^2 is guaranteed as $L/a \rightarrow \infty$, i.e. the continuum model is conformal.

Finally, it must be emphasized that an important assumption is absolutely necessary for the above argument, namely that the observed trend of decreasing zeros with increasing lattice volume does not change as the continuum is approached further. If this assumption turns out to be wrong then the conclusion about conformality was premature.

The actual continuum limit of the β -function at fixed $g^2(L)$ as performed usually is extremely demanding given our data. This is because as noted above the expected maximum of the β -function

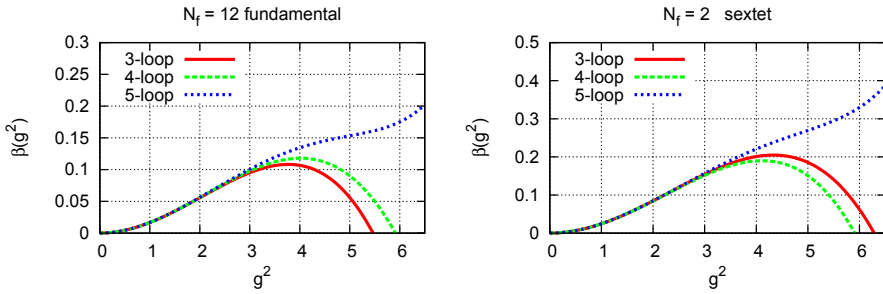


Figure 5. The β -function of the $N_f = 12$ fundamental model (left) and the $N_f = 2$ sextet model (right) in $\overline{\text{MS}}$ for increasing loop orders.

is tiny and one needs to resolve this tiny value from zero, in order to establish a positive continuum result. This requires very small absolute errors on the renormalized couplings. Our current errors even though rather small in relative terms, are far too large in absolute value for this. Hence we are not able to perform a controlled continuum extrapolation that is precise enough.

For instance, using our data we can interpolate the discrete β -function from nearby data points to $g^2(L) = 0.6$ on all lattice volume pairs for the $N_f = 3$ sextet model using various polynomial orders. The variation of the interpolated values with the polynomial order is taken into account as a systematic error and is added to the statistical error in quadrature. The obtained discrete β -function values are then extrapolated to the continuum linearly in a^2/L^2 . We may take the continuum limit using all 4 data points or by dropping the roughest one and using only 3. Both have very good χ^2/dof and hence the difference in the final result is again taken as a systematic error which is added to the statistical one; see figure 6. We obtain 0.002(2) in this particular example for the continuum value which is clearly consistent with the expected 0.001 from the 5-loop $\overline{\text{MS}}$ result but is also consistent with zero hence is not very predictive. At other values of $g^2(L)$ our observations are similar.

Unfortunately this property, namely that the maximum of the β -function is very small, seems to be a common feature of all weakly coupled gauge theories as it is a direct consequence of being close to the upper end of the conformal window.

4 Moving away from the weakly coupled regime

Now that the result for the 5-loop β -function in $\overline{\text{MS}}$ is available [7, 8] it is enlightening to see what happens to the perturbative predictions as the flavor number is decreased towards the lower end of the conformal window. We have seen that the $N_f = 14$ fundamental and $N_f = 3$ sextet models are conformal with a perturbatively accessible fixed point. Here we will show the perturbative behavior of the $N_f = 12$ fundamental and $N_f = 2$ sextet models, both being actively studied on the lattice [11–21].

The perturbative β -function is shown in figure 5 for the two models. They both share the property that the 2-loop, 3-loop and 4-loop β -functions all have zeros, but at relatively large coupling, $g_*^2 > 5$. The 5-loop β -function is however without a fixed point suggesting QCD-like behavior, at least to this order of the loop expansion.

It is also straightforward to obtain the lower end of the conformal window at 5-loops order using the results [7, 8]. One obtains $N_f \approx 12.89$ and $N_f \approx 2.35$, for the fundamental and sextet rep-

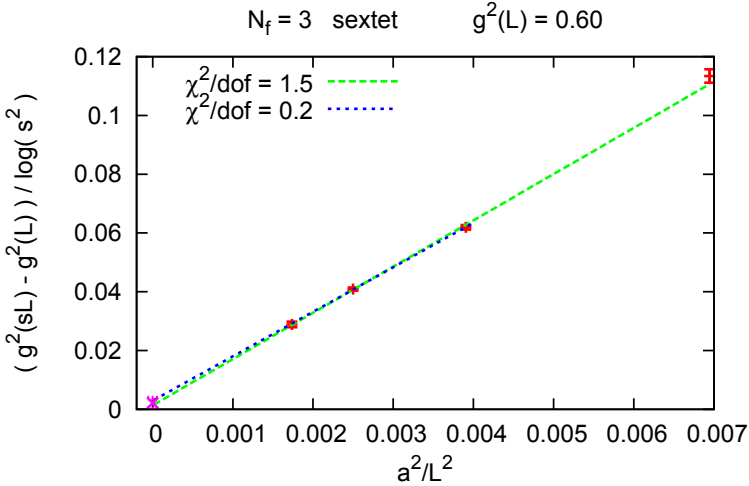


Figure 6. Continuum extrapolation of the discrete β -function in the $N_f = 3$ sextet model after interpolating to $g^2(L) = 0.60$. Two types of extrapolations were performed, either all 4 lattice spacings are used or the roughest one is dropped. The difference is taken into account as a systematic error. The χ^2/dof of the extrapolations are shown in the legend.

resentation, respectively. How trustworthy these perturbative predictions are can of course only be determined once all systematic effects are fully controlled in non-perturbative lattice simulations.

5 Conclusion

Studies of gauge theories close to the lower end of conformal window are plagued by large systematic effects. The difficulty of studying models far away from the weakly coupled CFT regime manifests itself both in the continuum via the loop expansion and non-perturbative simulations. In lattice studies the zeros of the β -function at finite lattice volumes may disappear towards the continuum signaling a qualitative change. Similarly, the zeros of the continuum β -function may disappear with increasing loop order again signaling a qualitative change in the supposed infrared dynamics. Hence it is very important to not completely trust the first few loop orders in the continuum calculation and also to go beyond small or medium size lattice volumes in lattice calculations.

In the weakly coupled CFT regime however the perturbative results for the β -function and its zero show little sensitivity to the order of the loop expansion. On the lattice the continuum extrapolation from finite lattice volumes is still challenging because the expected β -function is small and hence all measured errors must be small in absolute terms (not just relative) in order to resolve the result from zero. Nevertheless in our study of two CFT cases, $N_f = 14$ fundamental and $N_f = 3$ sextet we have identified trends which if they do not change further towards the continuum then guarantee the existence of a fixed point in the continuum.

Even though the continuum limit of the β -function is challenging it may very well be that the mass anomalous dimension γ_* can be obtained more reliably. We hope to return to this question in a future publication.

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