EPJ Web of Conferences **175**, 10006 (2018) https://doi.org/10.1051/epjconf/201817510006
Lattice 2017

α_s from the Hadronic Vacuum Polarisation

Renwick Hudspith[1]⋆, *Randy* Lewis[1], *Kim* Maltman[2,3], and *Eigo* Shintani[4]

[1] *Dept. of Physics and Astronomy, York University, Toronto, Ontario, M3J 1P3, Canada*
[2] *Dept. of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada*
[3] *CSSM, University of Adelaide, Adelaide SA 5005, Australia*
[4] *RIKEN Advanced Institute for Computational Science, Kobe, Hyogo 650-0047, Japan*

Abstract. We present our result for the strong coupling constant computed from the u-d vector Hadronic Vacuum Polarisation function. We use $n_f = 2 + 1$ flavours of Domain Wall fermions at 3 lattice spacings, generated by the RBC-UKQCD collaboration. We identify several possible pitfalls in this method for determining the coupling and illustrate how to resolve them.

1 Introduction

The strong coupling constant of QCD α_s is a fundamental input parameter of the Standard Model of particle physics and it is known to the lowest precision of almost all fundamental constants despite decades of effort measuring it.

α_s is a purely perturbative quantity, and its precise measurement is of great importance for accurate perturbative calculations. It is commonly quoted in the $\overline{\mathrm{MS}}$ scheme at some particular scale, namely for five active quark flavours at the Z-boson mass M_Z. Being a perturbative quantity it must be determined by comparing some measured quantity to the perturbative prediction. A review of the methods used to do this and the world average can be found in the PDG [1].

Lattice QCD measurements of the coupling have dominated the world average in terms of statistical precision over the past twenty years with several complementary evaluations being performed, reviews of which can be found in [2]. In principle, all of these techniques should agree within their respective systematics. In this work we have chosen to investigate a determination of the coupling from the lattice Hadronic Vacuum Polarisation (HVP) function as performed in [3, 4]. We find this measurement technique compelling as it has good theoretical motivations and seems like a natural competitor and perhaps even successor for τ-decay based analyses. However, the result of [4] was found to be around three sigma below the world average and we wish to investigate whether this is due to systematics in the procedure.

We will improve upon previous determinations of the coupling using the HVP in the following ways: first by introducing several lattice spacings to investigate the cut-off dependence of the result, secondly by working at a scale that does not require fitting the arguably poorly-behaved $D^{(2)}$ series or higher-order condensates and finally by performing a multiple renormalisation scale analysis.

⋆Speaker e-mail: renwick.james.hudspith@gmail.com

2 Background

2.1 The running coupling

The running of the coupling α_s with the renormalisation scale μ in $\overline{\text{MS}}$ continuum perturbation theory is governed by [1],

$$\frac{d\alpha(\mu)}{d\ln(\mu)} = \sum_{i=0}^{5}\beta_i\left(\frac{\alpha(\mu)}{\pi}\right)^{2+i}. \tag{1}$$

We will be working at scales relevant to our lattice simulation $\mu \approx 2$ GeV, but it is customary to present the value of the coupling at a fixed scale, namely the Z-boson mass M_Z in the five-flavour theory. To achieve this value we will measure the coupling in our $n_f = 3$ simulations at some scale and run the coupling by numerically solving equation 1, matching through appropriate flavour thresholds where necessary [5].

2.2 Continuum HVP and the OPE

In the continuum, the HVP for some quark current j^a_μ is defined as,

$$\Pi_{\mu\nu}(Q) = \int d^4x\, e^{iQ\cdot x}\delta^{ab}\langle j^a_\mu(x) j^b_\nu(0)^\dagger\rangle, \tag{2}$$

which has the following Lorentz decomposition into transverse $\Pi^{(1)}(Q^2)$ and longitudinal $\Pi^{(0)}(Q^2)$ components,

$$\Pi_{\mu\nu}(Q) = (g_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi^{(1)}(Q^2) - Q_\mu Q_\nu \Pi^{(0)}(Q^2). \tag{3}$$

In this work we will use vector currents, which are purely transverse.

The HVP is not a physical quantity, but the Adler function $\left(D(Q^2) = -Q^2\frac{\partial \Pi^{(1)}(Q^2)}{\partial Q^2}\right)$ is. This has the following general operator product expansion (OPE),

$$D(Q^2,\mu^2) = D^{(0)}(Q^2,\mu^2) + \cancel{\frac{m^2}{Q^2}D^{(2)}(Q^2,\mu^2)} + \sum_{i=2}^{\infty}\frac{C_{2i}}{Q^{(2i)}}. \tag{4}$$

The series for $D^{(0)}$ is mass-independent and known to 5 loops [8–11], the series $D^{(2)}$ is known to 3 loops [12] and heavily suppressed at the scales we will choose to work at and so can be ignored, which is good because it appears to be poorly convergent. The dimensionless $D = 2i$ condensates C_{2i} strictly have infinite series expansions in the coupling, but can be approximated by constants up to α_s-suppressed logarithmic corrections.

We will work with the fixed-scale representation of the series $D^{(0)}$ [13],

$$D^{(0)}(Q^2,\mu^2) = \frac{1}{4\pi^2}\sum_{i=0}\left(\frac{\alpha(\mu)}{\pi}\right)^i \sum_{j=0}^{i-1} d_{ij}t^j, \quad t = \ln\left(\frac{Q^2}{\mu^2}\right). \tag{5}$$

Upon integration of the Adler function we obtain the series for the HVP, this quantity is directly measurable from a lattice simulation [14]. The HVP is both a scheme and regularisation dependent quantity due to the integration constant C,

$$\Pi(Q^2,\mu^2) = C + \frac{1}{4\pi^2}\sum_{i=0}\left(\frac{\alpha(\mu)}{\pi}\right)^i \sum_{j=0}^{i-1} d_{ij}\frac{t^{j+1}}{j+1}. \tag{6}$$

[1] with β function coefficients that can be found in [5–7]

We define a quantity Δ that cancels this integration constant and is equal to the $\frac{\alpha(\mu)}{\pi}$ at leading order,

$$\begin{aligned}\Delta(Q_1^2, Q^2, \mu^2) &= 4\pi^2 \left(\Pi(Q_1^2, \mu^2) - \Pi(Q^2, \mu^2)\right) - 1, \\ &= \sum_{i=1} \left(\frac{\alpha(\mu)}{\pi}\right)^i \sum_{j=0}^{i-1} d_{ij} \frac{t_1^{j+1} - t^{j+1}}{(j+1)(t_1 - t)}.\end{aligned} \tag{7}$$

This is what we will compute on the lattice and directly compare to perturbation theory.

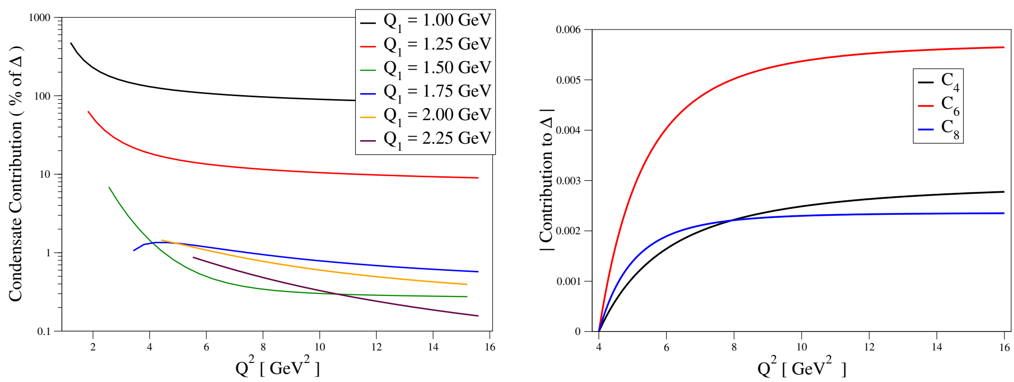
2.3 Lessons from FESRs

Continuum finite energy sum rules (FESRs) can provide information about the magnitude and sign of higher-order condensates. The usual FESR analysis relates the continuum spectral function $\rho(s)$ measured in experiment at some momentum $s = -Q^2$ up to some cut-off s_0 to the OPE for the HVP $\Pi(s)$ with particularly chosen analytic weight functions $w(s)$,

$$\int_0^{s_0} \mathrm{ds} \left(\frac{s}{s_0}\right)\rho(s) = -\frac{1}{2\pi i}\oint_{|s|=s_0} \mathrm{ds}\; w\left(\frac{s}{s_0}\right)\Pi(s). \tag{8}$$

A polynomial weight of degree k in the variable $\frac{s}{s_0}$ accesses higher dimension condensate contributions up to $D = 2k + 2$ as follows from the relation,

$$-\frac{1}{2\pi i s_0}\oint_{|s|=s_0} \mathrm{ds} \left(\frac{s}{s_0}\right)^n \frac{C_{2i}}{(-s)^n} = (-1)^{n+1}\frac{C_{2n+2}}{s_0^{n+1}}\delta_{i,n+1}. \tag{9}$$



(a) Contribution to Δ (b) Fixed $Q_1 = 2$ GeV magnitudes

Figure 1: Contribution to Δ of the effective condensates from the FESR relations.

If we use a rough estimate for $C_4 = 0.0012\ \mathrm{GeV}^4$ and $C_6 = -0.0093\ \mathrm{GeV}^6$, $C_8 = 0.0153\ \mathrm{GeV}^8$ from [15] we find that contributions from these higher order condensates are sub-percent corrections to the perturbative expression of Δ at $Q = 2$ GeV and above. At low scales however, say $Q \approx 1$ GeV, the contribution from condensates is of order 100% or more, this is illustrated in the left-hand graph of figure 1. It is evident from equation 9 and the right hand side of figure 1, which plots the magnitude of the condensate contributions, that not only do the condensates appear to become relevant at similar scales but large cancellations occur at higher orders indicating that fitting these condensates at low Q^2 to lattice data with some low-order truncation to the series of condensates will introduce significant systematics. This, to us, defines a window above which we are safe to measure the coupling with only the $D^{(0)}$ series and neglect the condensate contributions. The previous study of [4] performed a fit in a regime where the condensate contributions to the OPE are dangerously large.

3 Lattice Implementation

We use Domain Wall Fermion (DWF) ensembles generated by the RBC-UKQCD collaboration [16, 17] at three different lattice spacings as listed in table 1. A great benefit to this method is that at the scales where we expect perturbation theory to be compatible with our data, the use of the $(u-d)$ currents ensures that the mass-dependence of the HVP will be negligible for all the ensembles we consider. The heavier pion mass ensembles can be averaged with lighter pion-mass ensembles as they are observed to produce consistent results.

	Coarse			Fine			Superfine
Extent	$24^3\times64\times32$			$32^3\times64\times32$			$32^3\times64\times32$
a^{-1} (GeV)	1.7848(50)			2.3833(86)			3.148(17)
am_l	0.005	0.01	0.02	0.004	0.006	0.008	0.0047
m_π (GeV)	0.33	0.42	0.54	0.28	0.33	0.38	0.37
measurements	685	110	109	510	352	80	920
am_{res}	0.003076(58)			0.0006643(82)			0.0006296(58)
Z_V	0.71408(58)			0.74404(181)			0.77700(8)

Table 1: Simulation parameters for the ensembles used in this study. a^{-1}, Z_V and m_{res} have been measured in [17].

In direct analogy to the continuum HVP of equation 2 we define the lattice HVP (The factor of Z_V is required to renormalise the local vector current)

$$\Pi_{\mu\nu}(Q) = Z_V \sum_x e^{iQ\cdot x}\langle \mathcal{V}_\mu(x) V_\nu(0)^\dagger\rangle. \tag{10}$$

We will use conserved($\mathcal{V}$) - local(V) [18, 19] vector currents (the factor of Z_V is required to renormalise the local current), leading to an exact lattice Ward Identity on the (by convention, conserved) index μ,

$$\sum_\mu \hat{Q}_\mu e^{iaQ_\mu/2}\Pi_{\mu\nu}(Q) = 0, \quad \hat{Q}_\mu = 2\sin(aQ_\mu/2), \quad Q_\mu = \frac{2\pi n_\mu}{L_\mu}, \tag{11}$$

with lattice momentum $\hat{Q}_\mu$ and Fourier modes n_μ.

In general we expect the following structure of higher-order discretisation terms,

$$\Pi_{\mu\nu}(\hat{Q}^2 : a^2) = \Pi_{\mu\nu}(Q^2) + \sum_{m,n;m+n\geq 2} C_{mn} a^{m+n} \hat{Q}_\mu^m \hat{Q}_\nu^n. \tag{12}$$

We implement a technique that we call "reflection projection" [20] to eliminate some of the possible rotation-breaking H_4-invariant terms,

$$\Pi(\hat{Q}^2 : a^2) = \frac{1}{12}\sum_\mu\sum_{\nu\neq\mu} \frac{\Pi_{\mu\nu}(\hat{Q} : a^2) - \Pi_{\mu\nu}(r_\mu\hat{Q} : a^2)}{2\hat{Q}_\mu\hat{Q}_\nu}, \tag{13}$$

where $r_\mu = -1$ is a reflection in the μ direction of the momentum $\hat{Q}_\mu$, to use this we must have non-zero momenta in all directions. We then perform a cylinder cut [21, 22] keeping only momenta within a narrow body-diagonal cylinder of radius $w \approx 0.24$ in the momentum-space lattice,

$$|aQ_\mu - (aQ \cdot n)n_\mu| < aw\frac{2\pi}{L}. \tag{14}$$

We do this to reduce further the contributions from rotation-breaking terms.

3.1 Modelling of the remaining cut off effects

Ideally we would directly fit the lattice data to the continuum perturbative expression but even at the lowest scales where we expect perturbation theory to be valid we also expect there to exist discretisation effects in our lattice data. After the projection and the cylinder cut we expect the dominant remaining leading-order and next-to-leading-order cut off effects to be the rotation-preserving ones,

$$\Pi(\hat{Q}^2 : a^2) = \Pi(Q^2) \; + \; C_1(a^2\hat{Q}^2) \; + \; C_2(a^2\hat{Q}^2)^2. \tag{15}$$

We also expect a correction to the continuum coupling

$$\alpha(\mu : a^2) = \alpha(\mu)\left(1 + C_\alpha a^2\right). \tag{16}$$

We will globally fit the data in table 1 to the continuum perturbative expression of equation 13 with the corrections of equation 15 and equation 16. This will be a 4-parameter fit.

4 Results

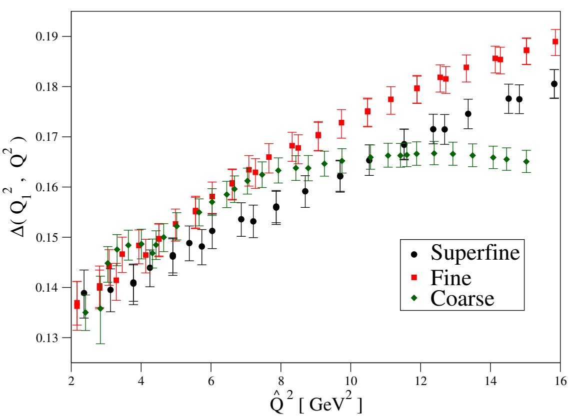
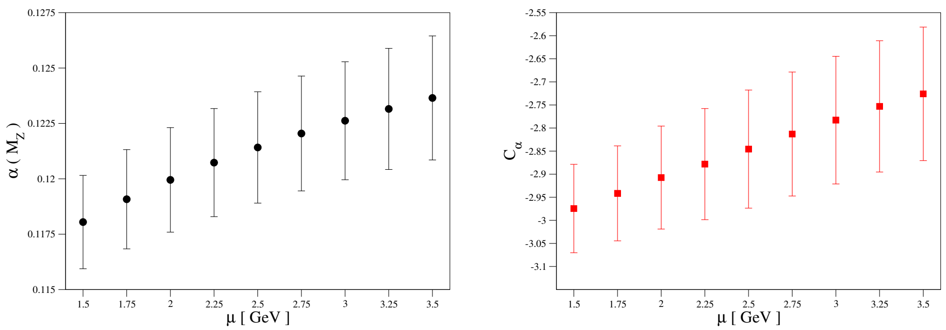


Figure 2: Plot of the lattice data for Δ.

In figure 2 we see the quality of our data (where the subtraction points $\hat{Q}_1^2$ were chosen to be as similar as possible within the constraint that each represent a Fourier mode within a given ensemble); after the cut and projection the momentum-dependence is quite smooth, which suggests that we have suppressed hypercubic artifacts. We do however see strong rotation-preserving artifacts both linear and quadratic in $a^2\hat{Q}^2$.

4.1 Single scale analysis

Following the work of [3, 4] we investigated a single renormalisation scale (μ) analysis. This means that we fixed the renormalisation scale to a particular reasonable value within our fit range and performed the fit to determine the coupling at that scale. We found that if we varied this scale the results when run to a common scale $\alpha_s^{(5)}(M_Z)$ showed very strong μ-dependence.

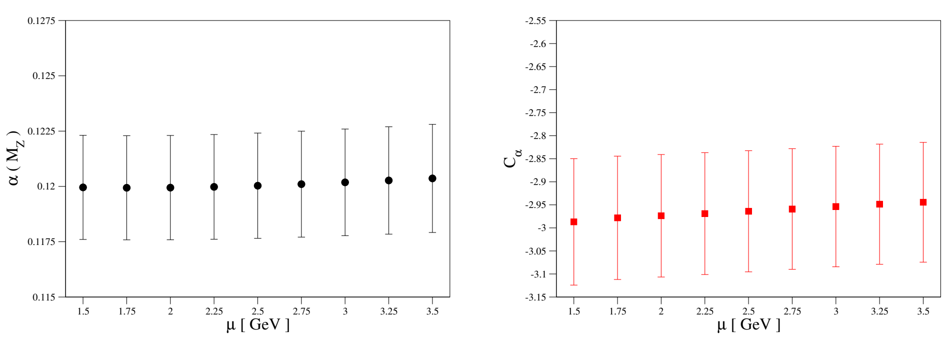


(a) $\alpha(M_Z)$ (b) C_α dependence

Figure 3: μ-dependence of the single scale analysis.

The perturbative expression for Δ does have some small, residual μ-dependence, but it is not of the same magnitude as what we see from the data and it is clear that the fit parameter dictating the lattice correction to the coupling C_α is completely correlated with the fit parameter giving α_s as can be seen by the direct relation between the two in figure 3. This correction term is somehow directly playing a rôle in absorbing the running of the coupling. This is unphysical and requires remediation.

4.2 Multiple scale analysis



(a) $\alpha(M_Z)$ (b) C_α dependence

Figure 4: μ-dependence of the multiple-scale analysis.

We can constrain the fit to remove the spurious μ-dependence we highlighted in the previous section. The idea is to perform the fit simultaneously at multiple, reasonable scales m run in perturbation theory to a common scale μ to obtain our final result for the coupling at that scale. As can be seen in figure 4, when we use auxiliary scales $m = 2,\ 2.25,\ 2.5$ and 2.75 GeV, we have stable results for the entire range $\mu \in [2, 3]$ GeV.

We find that with our data the coarse ensemble results can be discarded. This is because the applicable fit window for the coarse ensemble turns out to be very small and higher-order discretisation effects play a strong rôle even at low momentum scales with this data set. We therefore choose to not include the coarse ensemble in our final result and only use the fine and superfine data. We also found that, at the scale we perform our measurement at, perturbative truncation effects are well under control and even the 2-loop result is consistent with our highest truncation order result. The systematic from perturbation theory truncation is thus negligible.

5 Conclusions

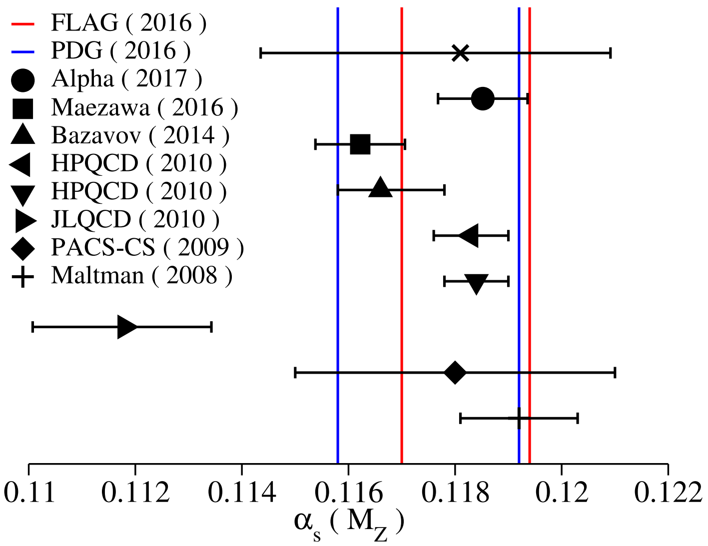


Figure 5: Comparison of our result with other determinations [4, 23–27], PDG result is without the lattice data [1]. Our *preliminary* result is the $\times$.

In figure 5 we show a comparison of our result ($\alpha_s(M_Z) = 0.1181^{+28}_{-37}$ with statistical and systematic errors added in quadrature) compared to a collection of other $n_f = 2 + 1$ lattice results [4, 23–27] as well as the most recent PDG (without lattice data) [1] and FLAG [2] averages. It appears that when taking the various inherent systematics of this method and our data into account our result is consistent with previous determinations and world averages. This should open the door for future measurements at finer lattice spacings to provide a competitive determination for the coupling and provide a complementary approach to the multitude of others that will help to accurately constrain this parameter.

6 Acknowledgements

Propagator inversions for this work were performed on the STFC funded DiRAC IBM BlueGene/Q system in the Advanced Computing Facility in Edinburgh. We would thank to the member of

RBC/UKQCD collaboration for discussion and comments. RJH, KM and RL are sponsored by the Natural Sciences and Engineering Research Council of Canada (NSERC). ES is grateful to RIKEN Advanced Center for Computing and Communication (ACCC) and The Mainz Institute for Theoretical Physics (MITP).

References

[1] C. Patrignani et al. (Particle Data Group), Chin. Phys. **C40**, 100001 (2016)

[2] S. Aoki et al., Eur. Phys. J. **C77**, 112 (2017), `1607.00299`

[3] E. Shintani, S. Aoki, T.W. Chiu, S. Hashimoto, T.H. Hsieh, T. Kaneko, H. Matsufuru, J. Noaki, T. Onogi, N. Yamada (TWQCD, JLQCD), Phys. Rev. **D79**, 074510 (2009), `0807.0556`

[4] E. Shintani, S. Aoki, H. Fukaya, S. Hashimoto, T. Kaneko, T. Onogi, N. Yamada, Phys. Rev. **D82**, 074505 (2010), [Erratum: Phys. Rev.D89,no.9,099903(2014)], `1002.0371`

[5] K. Chetyrkin, B.A. Kniehl, M. Steinhauser, Phys.Rev.Lett. **79**, 2184 (1997), `hep-ph/9706430`

[6] T. van Ritbergen, J. Vermaseren, S. Larin, Phys.Lett. **B400**, 379 (1997), `hep-ph/9701390`

[7] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn (2016), `1606.08659`

[8] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Phys. Lett. **B85**, 277 (1979)

[9] S.G. Gorishnii, A.L. Kataev, S.A. Larin, Phys. Lett. **B259**, 144 (1991)

[10] L.R. Surguladze, M.A. Samuel, Phys. Rev. Lett. **66**, 560 (1991), [Erratum: Phys. Rev. Lett.66,2416(1991)]

[11] P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn, Phys. Rev. Lett. **101**, 012002 (2008), `0801.1821`

[12] K.G. Chetyrkin, A. Kwiatkowski, Z. Phys. **C59**, 525 (1993), `hep-ph/9805232`

[13] K. Maltman, T. Yavin, Phys. Rev. **D78**, 094020 (2008), `0807.0650`

[14] T. Blum, Phys. Rev. Lett. **91**, 052001 (2003), `hep-lat/0212018`

[15] D. Boito, M. Golterman, K. Maltman, J. Osborne, S. Peris, Phys. Rev. **D91**, 034003 (2015), `1410.3528`

[16] R. Arthur et al. (RBC, UKQCD), Phys. Rev. **D87**, 094514 (2013), `1208.4412`

[17] T. Blum et al. (RBC, UKQCD), Phys. Rev. **D93**, 074505 (2016), `1411.7017`

[18] V. Furman, Y. Shamir, Nucl.Phys. **B439**, 54 (1995), `hep-lat/9405004`

[19] P. Boyle, L. Del Debbio, E. Kerrane, J. Zanotti, Phys. Rev. **D85**, 074504 (2012), `1107.1497`

[20] R.J. Hudspith, R. Lewis, K. Maltman, E. Shintani, PoS **LATTICE2015**, 268 (2016), `1510.04890`

[21] D.B. Leinweber, J.I. Skullerud, A.G. Williams, C. Parrinello (UKQCD collaboration), Phys.Rev. **D58**, 031501 (1998), `hep-lat/9803015`

[22] C. Alexandrou, P. de Forcrand, E. Follana, Phys. Rev. D **63**, 094504 (2001)

[23] M. Bruno, M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Schaefer, H. Simma, S. Sint, R. Sommer (2017), `1706.03821`

[24] Y. Maezawa, P. Petreczky, Phys. Rev. **D94**, 034507 (2016), `1606.08798`

[25] A. Bazavov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto, A. Vairo, Phys. Rev. **D90**, 074038 (2014), `1407.8437`

[26] C. McNeile, C.T.H. Davies, E. Follana, K. Hornbostel, G.P. Lepage, Phys. Rev. **D82**, 034512 (2010), `1004.4285`

[27] S. Aoki et al. (PACS-CS), JHEP **10**, 053 (2009), `0906.3906`