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Truncation of lattice $\mathcal{N} = 4$ super Yang-Mills

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Abstract. In twisted and orbifold formulations of lattice $\mathcal{N} = 4$ super Yang-Mills, the gauge group is necessarily $U(1) \times SU(N)$, in order to be consistent with the exact scalar supersymmetry Q. In the classical continuum limit of the theory, where one expands the link fields around a point in the moduli space and sends the lattice spacing to zero, the diagonal $U(1)$ modes decouple from the $SU(N)$ sector, and give an uninteresting free theory. However, lattice artifacts (described by irrelevant operators according to naive power-counting) couple the two sectors, so removing the $U(1)$ modes is a delicate issue. We describe how this truncation to an $SU(N)$ gauge theory can be obtained in a systematic way, with violations of Q that fall off as powers of $1/N^2$. We are able to achieve this while retaining exact $SU(N)$ lattice gauge symmetry at all N, and provide both theoretical arguments and numerical evidence for the $1/N^2$ suppression of Q violation.

1 Introduction

The twisted formulation of $\mathcal{N} = 4$ super Yang-Mills (SYM) appearing in [2] is based on the Marcus or Geometric-Langlands twist [3, 4] (which has been shown to be equivalent the the orbifolding approach [5, 6]; for alternative approaches see [7–12]) works with link fields that are in the algebra of the group $GL(N, \mathbb{C})$. This is readily apparent from the (twisted) scalar supersymmetry (SUSY) transformation

$$Q\mathcal{U}_m(x) = \psi_m(x), \quad m = 1, \ldots, 5 \tag{1.1}$$

where $\psi_m(x)$ is a twist fermion that transforms as a link variable

$$\psi_m(x) \to G(x)\psi_m(x)G^\dagger(x + e_m) \tag{1.2}$$

Since it is a fermion, it has an expansion in terms of generators,

$$\psi_m(x) = \sum_{A=0}^{N^2-1} \psi_m^A(x)t^A \tag{1.3}$$

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Here, t^0 is proportional to the unit matrix, and must be included if (1.1) is to hold, because the link field $\mathcal{U}_m$ on the left-hand side certainly has an expansion involving the unit matrix, if it is to yield the usual continuum limit, where the lattice spacing a is sent to zero, $a \to 0$, and the link fields are expanded about the unit matrix:

$$\mathcal{U}_m(x) = \mathbb{I} + a\mathcal{A}_m(x) + \cdots \tag{1.4}$$

(Here, $\mathcal{A}_m(x)$ is a complexification that contains both the gauge fields and scalars. For further details see the references cited above. Henceforth we will work in lattice units, $a = 1$.) On the other hand, SUSY should not convert a group valued field into a Lie algebra valued field, so in fact $\mathcal{U}_m$ should also have the expansion

$$\mathcal{U}_m = \sum_{A=0}^{N^2-1} \mathcal{U}_m^A t^A \tag{1.5}$$

with the U(1) mode $\mathcal{U}_m^0$ *fully dynamical.* The conclusion of this argument is that the scalar SUSY Q requires that the group to be $U(N) = U(1) \times SU(N)$, and not SU(N), with the bosonic link fields Lie algebra valued.

Unfortunately, U(1) modes fluctuations contained in $\mathcal{U}_m^0$ have presented many problems for us in our studies, for some time. In the continuum limit the entire U(1) sector decouples, and becomes an uninteresting free theory (all fields are in the "adjoint," hence neutral for U(1)). However on the lattice this sector is coupled to the SU(N) part through irrelevant operators, so we cannot completely ignore it. In fact, it seems that these irrelevant couplings are causing various problems.

One problem was the well-known phase transition of four-dimensional $U(1)$ gauge theory, where a proliferation of $U(1)$ monopoles leads to confinement. This was addressed through a $U(1)$ plaquette term that suppressed the monopoles [13]. The other problems have to do with $U(1)$ scalars.

In the original Q invariant formulation, the U(1) scalars were flat directions which had to be stabilized by breaking the SUSY with a mass term.

$$\mathcal{V} = \mu^2 \sum_{m,x} \left(\frac{1}{N} \operatorname{Tr} \overline{\mathcal{U}}_m \mathcal{U}_m - 1 \right)^2 \tag{1.6}$$

Later, it was realized that a Q invariant mass term could be formulated [14]. However, this has not been the end of the problem with scalars, as was much in evidence in the dimensional reduction of this lattice theory [15].

These quandries have led us to take a more aggressive approach with the U(1) sector. In one case we eliminate it entirely from the link fields $\mathcal{U}_m(x)$ through a brutal truncation. We will show that it is possible to do this while preserving SU(N) lattice gauge invariance. In the second case, we perform this truncation dynamically, by introducing a potential that strongly favors the link fields lying in $SL(N, \mathbb{C})$. In fact, through this latter approach we are able to show that the truncation leads to violations of Q SUSY invariance that are suppressed by $O(1/N^2)$. We present convincing numerical evidence that this is indeed the case.

In fact, these truncated approaches are yielding stable results at significantly larger values of the 't Hooft coupling λ, with far less SUSY breaking. Furthermore, measurements of the conformal dimensions of key primary operators are more stable and in better agreement with perturbation theory, at the values of λ that we are able to access. This illustrates that getting a better handle on the U(1) sector, essentially by eliminating it, clears up what were muddy waters.

2 Pristine formulation

The action in the original, undeformed formulation is given by

$$
\begin{aligned}
S &= \frac{1}{2g^2}(Q\Lambda + S_{\rm closed}) \\
\Lambda &= \sum_x \mathrm{Tr}\,(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a - \frac{1}{2}\eta d) \\
S_{\rm closed} &= -\frac{1}{4}\sum_x \epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c^{(-)}\chi_{ab}(x)
\end{aligned}
\tag{2.1}
$$

where the explicit expressions for the terms involving covariant derivatives are given by

$$
\begin{aligned}
\mathcal{F}_{ab}(x) &= \mathcal{D}_a^{(+)}\mathcal{U}_b(x) = \mathcal{U}_a(x)\mathcal{U}_b(x+e_a) - \mathcal{U}_b(x)\mathcal{U}_a(x+e_b) \\
\overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(x) &= \mathcal{U}_a(x)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x-e_a)\mathcal{U}_a(x-e_a) \\
\epsilon_{abcde}\chi_{de}\overline{\mathcal{D}}_c^{(-)}\chi_{ab}(x) &= \epsilon_{abcde}\chi_{de}(x+e_a+e_b)[\chi_{ab}(x)\overline{\mathcal{U}}_c(x-e_c) \\
&\quad -\overline{\mathcal{U}}_c(x-e_c+e_a+e_b)\chi_{ab}(x-e_c)]
\end{aligned}
\tag{2.2}
$$

Notice that these expressions involve fields which are associated to the links of an A_4^* lattice which possesses five (linearly dependent) basis vectors and an associated S^5 point group symmetry. To complete the specification of the action we also need the action of Q on the lattice fields, which is given by

$$
\begin{aligned}
&Q\mathcal{U}_a = \psi_a, \quad Q\psi_a = 0, \quad Q\overline{\mathcal{U}}_a = 0 \\
&Q\chi_{ab}(x) = -\overline{\mathcal{F}}_{ab}(x) \equiv \overline{\mathcal{U}}_b(x+e_a)\overline{\mathcal{U}}_a(x) - \overline{\mathcal{U}}_a(x+e_b)\overline{\mathcal{U}}_b(x) \\
&Q\eta = d, \quad Qd = 0
\end{aligned}
\tag{2.3}
$$

It can be checked that the classical continuum limit of this lattice action yields the usual Marcus twist of $\mathcal{N} = 4$ SYM if the lattice fields are decomposed into their irreducible components under the S^5 symmetry (see [6]) and the link fields expanded according to[1]

$$
\mathcal{U}_a(x) = 1 + \mathcal{A}_a(x), \quad \overline{\mathcal{U}}_a(x) = 1 - \overline{\mathcal{A}}_a(x) \tag{2.4}
$$

This formulation, along with the mass term (1.6), and further modifications described in [13, 14] have been realized in efficient parallel code [16, 17] that is publically available on the USQCD software website. We have added features to implement the truncation described in the next section.

3 Dynamic truncation: the “detlink” action

Here we give the action an additional contribution

$$
\Delta S = \kappa\kappa_{\rm link}\sum_{x,a} |\det\mathcal{U}_a(x) - 1|^2 \tag{3.1}
$$

which dynamically drives the links into $SL(N,\mathbb{C})$. This action is invariant under the $SU(N)$ subgroup of the original $U(N)$ gauge symmetry, because

$$
\det\mathcal{U}_a(x) \to \det(G(x)\mathcal{U}_a(x)G^\dagger(x+e_a)) = \det G(x)\det\mathcal{U}_a(x)\det G^\dagger(x+e_a) = \det\mathcal{U}_a(x) \tag{3.2}
$$

[1] We work with antihermitian generators of the $U(N)$ gauge group.

It does violate the diagonal $U(1)$ subgroup of $U(N)$, but as stated in the introduction, we are willing to sacrifice this symmetry in order to address stability problems. As in our previous works, $\kappa = N/2\lambda$; it is important to note that this lattice λ is related to the one in the continuum by $\lambda_{\text{cont}} = \lambda/2\sqrt{5}$, due to the A_4^* lattice and conventions for the lattice action. The new coupling κ_{link} can be adjusted to achieve the dynamic truncation to $SL(N, \mathbb{C})$ to varying degrees. Generally we must scale up κ_{link} as we increase λ, because of the competition between these two factors. Having this adjustable parameter also helps in making theoretical arguments about the effect of ΔS. One can begin the analysis at small κ_{link} and work outward from there.

In the limit $\kappa_{\text{link}} \to \infty$ we can completely remove the $U(1)$ modes—both gauge and scalar by restricting the links to $SL(N, \mathbb{C})$. Writing $\det U_a(x) = (\mathbb{I} + h_a(x))e^{i\theta_a(x)}$ and expanding to quadratic order in the fluctuations it is easy to see that

$$\Delta S = \kappa\kappa_{\text{link}} \sum_{x,a} \left(\theta_a^2 + h_a^2\right) + \cdots \tag{3.3}$$

The term thus serves to generate masses for the $U(1)$ modes.

This modification violates (scalar) $\mathcal{Q}$ supersymmetry. Since we are introducing an effect that splits off the $U(1)$ modes, which are 1 in N^2, we expect $\mathcal{Q}$ violations that are suppressed by $1/N^2$. We numerically test its restoration in the large N limit in two ways. One is measurements of the bosonic action S_B. Redefining this quantity by dividing it by $(9/2)VN^2$, where V is the four-dimensional volume, it should be equal to 1 is the $\mathcal{Q}$ supersymmetry is restored. In Fig. 1 we show that this is achieved in the large N limit up to a per cent level, fitting to the form

$$f(N) = A + B/N^2 \tag{3.4}$$

where in this case $f(N) = S_B$. Residual violations are expected because we continue to have the mass parameter $\mu \neq 0$ and anti-periodic boundary conditions (to keep the Pfaffian real and positive) which violate $\mathcal{Q}$ by finite volume effects. Another is through the Ward identity that has been used in our previous work [13], corresponding to

$$\langle \mathcal{Q}\,\text{Tr}\,\eta \sum_m \mathcal{U}_m \overline{\mathcal{U}}_m \rangle = 0 \tag{3.5}$$

Using the supersymmetry transformations of the fields and the equation of motion for the auxiliary field, we find that the following quantity should vanish:

$$W = \sum_x \langle \text{Tr}\,(\sum_n (\mathcal{U}_n \overline{\mathcal{U}}_n - \overline{\mathcal{U}}_n \mathcal{U}_n) \sum_m \mathcal{U}_m \overline{\mathcal{U}}_m) \rangle - B \tag{3.6}$$

where

$$B = \sum_x \langle \text{Tr}\,\eta \sum_m \psi_m \overline{\mathcal{U}}_m \rangle \tag{3.7}$$

In practice we normalize by the bilinear B, and measure the ratio

$$R = W/B \tag{3.8}$$

The results are shown in the Figs. 2 and 3. The plots differ in the range of N that is fit to the form (3.4). We find that this only has a marginal effect.

As mentioned in the Introduction, with the truncation (3.1) added to the action, we find that we are able to simulate to larger values of the 't Hooft coupling λ than in previous studies. For $SU(2)$ gauge group we can extend to values $\lambda > 10$. For $SU(N)$ with $N \geq 3$ the range is smaller, $\lambda \lesssim 8$.

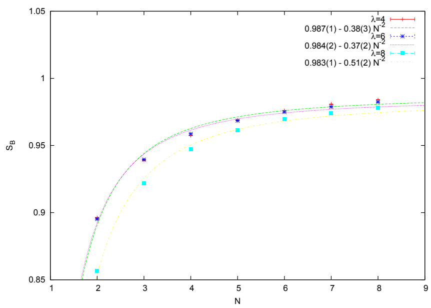

Figure 1. The bosonic action, scaled such that it should be exactly equal to 1 if the Q symmetry is fully restored. It can be seen that the N dependence falls of as $1/N^2$, as expected. The difference from 1 in the large N limit is anticipated from finite volume effects with anti-periodic boundary conditions.

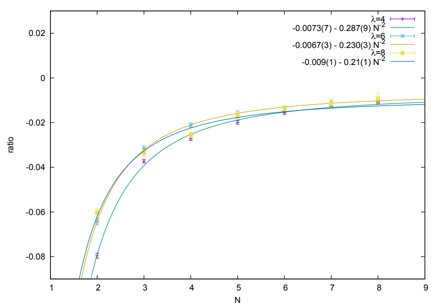

Figure 2. Combining results for the 8^4 lattice with detlink action, $\lambda = 4, 6, 8$, $\mu = 0.1$. Here $\kappa_{\text{link}} = 5, 5, 10$ respectively. The fits give χ^2/dof of 3.72, 0.22, 2.86 respectively.

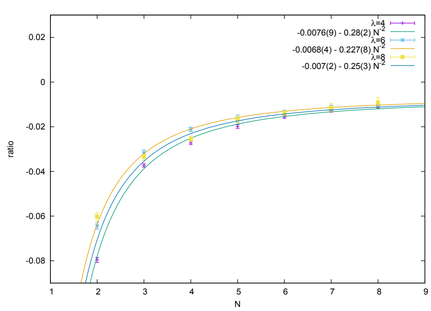

Figure 3. Again, combining results for the 8^4 lattice with detlink action. However, here we only fit $N = 3, \dots, 8$. The fits now give χ^2/dof of 4.49, 0.27, 1.93 respectively.

4 Conclusions

We have presented an approach to removing the $U(1)$ modes of lattice $\mathcal{N} = 4$ super Yang-Mills in a controlled way, which preserves the $SU(N)$ subgroup of lattice gauge invariance at every step. Arguments were presented that this leads to a violation of the otherwise exact Q supersymmetry by $1/N^2$ effects. Numerical evidence for this conclusion has been obtained through Monte Carlo simulations, summarized in the figures given above. Since the $U(1)$ modes have been especially problematic to our studies, this truncation of the lattice theory represents an important step toward stable, first principles studies of maximally supersymmetric Yang-Mills using the lattice technique. A more detailed presentation of our findings will appear in a forthcoming article [18].

Because we are now able to extend to larger values of λ, we are able to conduct studies where we are sensitive the the anomalous dimension of the Konishi operator. Forthcoming work will compare this to non-lattice predictions.

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