

## Duals of $U(N)$ LGT with staggered fermions

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**Abstract.** Various approaches to construction of dual formulations of non-abelian lattice gauge theories are reviewed. In the case of  $U(N)$  LGT we use a theory of the Weingarten functions to construct a dual formulation. In particular, the dual representations are constructed 1) for pure gauge models in all dimensions, 2) in the strong coupling limit for the models with arbitrary number of flavours and 3) for two-dimensional  $U(N)$  QCD with staggered fermions. Applications related to the finite temperature/density QCD are discussed.

### 1 Duals of lattice spin and gauge models

Dual representations proved to be a very useful concept in the context of abelian spin and gauge models. The application of dual representations ranges from the determination of the critical points in the self-dual abelian models to the proof of the confinement in the three-dimensional  $U(1)$  lattice gauge theory (LGT) [1, 2] and the numerical study of the  $U(1)$  LGT both at zero [3] and at finite temperature [4]. For abelian models the dual transformations are well-defined and described in many reviews and text books [5]. The status of the dual representations of non-abelian models is very different. During decades several different approaches have been attempted to construct dual representations.

- Dual representations based on the plaquette formulation [6–8]. Dual variables are introduced as variables conjugate to local Bianchi identities [6, 9, 10]. The dual model appears to be non-local due to the presence of connectors in the Bianchi identities for gauge models. An analogue of the plaquette formulation for the principal chiral model is so-called link representation [11, 12]. In this case one can construct a local dual theory for all  $U(N)$  and  $SU(N)$  principal chiral models [13].
- Dual representations based on 1) the character expansion of the Boltzmann weight and 2) the integration over link variables using Clebsch-Gordan expansion [14, 15]. This approach is not very useful in the context of principal chiral models as the summation over group indices cannot be performed locally. But in the case of LGT, due to the gauge invariance the summation over group (colour) indices can be done and this results in the local formulation in terms of invariant  $6j$  symbols. This dual form can be studied using Monte-Carlo simulations [16, 17].
- In the strong coupling limit the  $SU(N)$  LGT can be mapped onto monomer-dimer-closed baryon loop model [18].
- More recent interesting approaches are developed in [19] and in [20].

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Important application of dual formulations concerns gauge models at finite baryon chemical potential. In some cases the sign problem can be explicitly solved in frameworks of the dual approach. This is the case, *e.g.* for the massless two-dimensional  $U(1)$  LGT [21]. Also, the sign problem can be fully eliminated in the  $SU(3)$  spin model in the complex magnetic field in the flux representation for the partition function [22, 23]. This model is an effective Polyakov loop model which can be calculated from the QCD partition function at strong coupling and large quark masses.

In this contribution we present another approach to the duality transformations for  $U(N)$  spin and gauge models. Our approach is based on the Taylor expansion of the Boltzmann weight and an exact integration over original gauge or spin degrees of freedom. Integrals to be calculated had been studied in the large  $N$  limit in the end of the seventies [24]. Important functions which appear after such integration are called now the Weingarten functions, and their theory have been well developed during last decade [25–27]. In Sect. 2 we introduce our notations and present main results about group integrals which we need here. In Sects. 3–6 we list main applications related to the construction of dual formulations in several cases. The main results and perspectives are outlined in Sect. 7.

## 2 $U(N)$ group integrals

### 2.1 Notations and conventions

We work on a  $d$ -dimensional hypercubic lattice  $\Lambda = L^d$  with  $L$  - a linear extension and a unit lattice spacing.  $\vec{x} \equiv x = (x_1, \dots, x_d)$ ,  $x_i \in [0, L - 1]$  denote the sites of the lattice,  $l = (\vec{x}, \mu)$  is the lattice link in the  $\mu$ -direction and  $p = (\vec{x}, \mu < \nu)$  is the plaquette in the  $(\mu, \nu)$ -plane.  $e_\mu$  is a unit vector in the direction  $\mu$ . We impose the periodic boundary conditions (BC) in all directions. Let  $G = U(N)$ ;  $U(x), U_l = U_\mu(x) \in G$ , and  $dU$  denotes the Haar measure on  $G$ .  $\text{Tr}U$  will denote the fundamental character of  $G$ . We treat models with a local interaction whose partition functions can be written as

$$Z_\Lambda(\beta, h_r, h_i; N) \equiv Z_{\text{spin}} = \int \prod_x dU(x) \quad (1)$$

$$\times \exp \left[ \beta_s \sum_{x, \mu} \text{ReTr}U(x) \text{Tr}U^\dagger(x + e_\mu) + \sum_x (h_r \text{Tr}U(x) + h_i \text{Tr}U^\dagger(x)) \right]$$

in case of  $U(N)$  spin models and

$$Z_\Lambda(\beta; N) \equiv Z_{\text{gauge}} = \int \prod_l dU_l \exp \left[ \beta_g \sum_p \text{ReTr}U(p) \right] \quad (2)$$

in case of  $U(N)$  LGT, where the plaquette matrix reads

$$U_p = U_\mu(x) U_\nu(x + \mu) U_\mu^\dagger(x + \nu) U_\nu^\dagger(x). \quad (3)$$

$U(N)$  spin model can be considered as an effective model for the Polyakov loop in the finite-temperature  $U(N)$  LGT with  $N_f$  flavours of massive staggered fermions. One has

$$h_r = \frac{1}{2} \sum_{f=1}^{N_f} h_f e^{\mu_f}, \quad h_i = \frac{1}{2} \sum_{f=1}^{N_f} h_f e^{-\mu_f}, \quad h_f = (\cosh m_f)^{-1}. \quad (4)$$

The relation between couplings reads  $\beta_s \sim \beta_g^{N_t}$ , with  $N_t$  temporal extent of the lattice.

## 2.2 Group integrals and Weingarten function

The basic integral which we need below is of the form

$$\mathcal{I}_N(r, s) = \int dU \prod_{k=1}^r U^{i_k j_k} \prod_{n=1}^s U^{m_n l_n^*} = \delta_{r,s} \mathcal{I}_N(r). \quad (5)$$

Its large- $N$  asymptotic behaviour was investigated in [24] and calculated in [25] for  $r \leq N$  and extended to  $r > N$  in [26] (a simple proof can be found in Ref. [27])

$$\mathcal{I}_N(r) = \sum_{\tau, \sigma \in S_r} Wg^N(\tau^{-1} \sigma) \prod_{k=1}^r \delta_{i_k, m_{\sigma(k)}} \delta_{j_k, l_{\tau(k)}}. \quad (6)$$

$S_r$  is a group of permutations of  $r$  elements and  $Wg^N(\sigma)$  is the Weingarten function which depends only on the length of the cycles of a permutation  $\sigma$ . Its explicit form is given by

$$Wg^N(\sigma) = \frac{1}{(r!)^2} \sum_{\lambda} \frac{d^2(\lambda)}{s_{\lambda}(1)} \chi_{\lambda}(\sigma), \quad (7)$$

where  $d(\lambda), \chi_{\lambda}(\sigma)$  are the dimension and the character of the irreducible representation  $\lambda$  of  $S_r$ . The irreducible representations  $\lambda$  are enumerated by partitions  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)})$  of  $r$ , where  $l(\lambda)$  is the length of the partition and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{l(\lambda)} > 0$ . The sum in (7) is taken over all  $\lambda$  such that  $l(\lambda) \leq N$ .  $s_{\lambda}(X)$ ,  $X = (x_1, x_2, \dots, x_N)$ , is the Schur function ( $s_{\lambda}(1)$  is the dimension of  $U(N)$  representation).

In the context of  $U(N)$  spin model we need an integral of the form

$$Q_N(r) = \int dU [\text{Tr} U \text{Tr} U^*]^r. \quad (8)$$

Obviously,  $Q_1(r) = 1$ . For arbitrary  $N$  the compact and simple result can be obtained with the help of basic formula (6). Namely, expanding the traces as sums over diagonal elements one gets

$$\begin{aligned} Q_N(r) &= \sum_{i_1 i_2 \dots i_r=1}^N \sum_{j_1 j_2 \dots j_r=1}^N \int dU \prod_{k=1}^r U_{i_k i_k} U_{j_k j_k}^* = r! \sum_{i_1 i_2 \dots i_r=1}^N \sum_{\sigma \in S_r} Wg^N(\sigma) \prod_{k=1}^r \delta_{i_k, i_{\sigma(k)}} \\ &= r! \sum_{\sigma \in S_r} Wg^N(\sigma) P_{\sigma}(I) = \sum_{\lambda} d^2(\lambda), \end{aligned} \quad (9)$$

where  $P_{\sigma}(I) = N^{|\sigma|}$  is the power sum symmetric function of a unit argument and we used the relation

$$s_{\lambda}(X) = \sum_{\tau \in S_r} \chi_{\lambda}(\tau) P_{\tau}(X). \quad (10)$$

A few remarks are in order here. The constraint  $r = s$  appearing in (5) is essentially abelian one. One solves the constraint by introducing genuine dual variables, like in  $U(1)$  model. No other constraints are generated. For  $U(N)$ , in any dimension, summation over group (matrix) indices is factorized in every lattice site and can be done locally. Therefore, the dual theory is a theory with only local interaction. Last property holds also in the presence of fermions.

### 3 One link integrals

The simplest one-link integral which gives an exact solution of two-dimensional pure  $U(N)$  LGT

$$Z = \int dU \exp [\beta \text{ReTr} U ] = \det I_{i-j}(\beta) , \quad i, j = 1, \dots, N \quad (11)$$

can be easily computed. Expanding exponential in the Taylor series and using (9) one finds

$$Z = \sum_{r=0}^{\infty} \left(\frac{\beta}{2}\right)^{2r} \frac{1}{(r!)^2} Q_N(r) . \quad (12)$$

Another one-link integral appears in the strong coupling limit  $\beta_g = 0$  of  $U(N)$  LGT with  $N_f$  flavours of staggered fermions

$$Z_0 = \int dU \prod_{f=1}^{N_f} \exp \left[ \eta_{x,\mu} \left( \bar{\psi}_f^i(x) U^{ij} \psi_f^j(x + \mu) - \bar{\psi}_f^i(x + \mu) U^{\dagger ij} \psi_f^j(x) \right) \right] . \quad (13)$$

With the help of (6) it is straightforward to obtain

$$Z_0 = \sum_{r=0}^{NN_f} \sum_{\tau \in S_r} \frac{(-1)^\tau}{r!} W g^N(\tau) \sum_{f_k, v_k=1}^{N_f} \prod_{k=1}^r \sigma_{f_k, v_k}(x) \sigma_{v_k, f_{\tau(k)}}(x + \mu) , \quad (14)$$

where  $\sigma_{f,v}(x)$  is colourless meson field with  $f, v$  flavour indices  $\sigma_{f,v}(x) = \sum_{i=1}^N \bar{\psi}_f^i(x) \psi_v^i(x)$ . The notation  $(-1)^\tau$  means  $(-1)$  if the permutation  $\tau$  is odd and  $+1$  if it is even. For one flavour one recovers the well-known result [28].

### 4 Polyakov loop model

In this Section we present our results for the Polyakov loop model defined in (1). Expanding all exponentials in the Taylor series one again encounters the integral of the form (8). Computing all integrals with the help of Eq.(9) and making the change of summation variables suggested in [22] for  $SU(3)$  spin model we write down the partition function in the form

$$Z_{\text{spin}} = \sum_{r(l)=-\infty}^{\infty} \sum_{p(l)=0}^{\infty} \sum_{t(x)=0}^{\infty} \prod_l \left[ \left(\frac{\beta}{2}\right)^{|r(l)|+2p(l)} \frac{1}{(p(l) + |r(l)|)! p(l)!} \right] \prod_x \left[ \frac{(h_r h_i)^{t(x) + \frac{1}{2}|r(x)|}}{t(x)! (t(x) + |r(x)|)!} Q_N(s(x)) \right] , \quad (15)$$

$$s(x) = \sum_{i=1}^{2d} \left( p(l_i) + \frac{1}{2} |r(l_i)| \right) + t(x) + \frac{1}{2} |r(x)| , \quad r(x) = \sum_{n=1}^d \left( r_\mu(x) - r_\mu(x - \mu) \right) . \quad (16)$$

where  $l_i$  are  $2d$  links attached to a site  $x$ . The Boltzmann weight is strictly positive if  $h_r, h_i \geq 0$ . When external fields are vanishing  $h_r = h_i = 0$ , only configurations  $t(x) = r(x) = 0$  contribute to the partition function. The constraint  $r(x) = 0$  can be solved in any number of dimensions by introducing dual variables. For example, in two-dimensional model we find the following representation for the partition function on the dual lattice

$$Z_{\text{spin}} = \sum_{r(x)=-\infty}^{\infty} \sum_{p(l)=0}^{\infty} \prod_l \left[ \left(\frac{\beta}{2}\right)^{|r(x)-r(x+\mu)|+2p(l)} \frac{1}{(p(l) + |r(x) - r(x + \mu)|)! p(l)!} \right] \prod_p Q_N(s(p)) , \quad (17)$$

$$s(p) = \sum_{i=1}^4 \left( p(l_i) + \frac{1}{2} |r(l_i)| \right). \quad (18)$$

Here,  $l_i$  are 4 links forming dual plaquette  $p$  and  $r(l_1) = r(x) - r(x + \mu)$  and so on. For the  $U(1)$  model,  $Q_1(s) = 1$ , we recover the conventional dual form of the two-dimensional  $XY$  spin model

$$Z_{\text{spin}} = \sum_{r(x)=-\infty}^{\infty} \prod_l I_{r(x)-r(x+\mu)}(\beta), \quad I_r(x) - \text{the modified Bessel function}. \quad (19)$$

## 5 Pure gauge models

Here we turn our attention to pure gauge models. In order to use the integration method of Sect. 2 we expand the integrand of (2) in the Taylor series and express the traces of the plaquette matrices as sums over group indices. The integration over link variables leads to a complicated set of Kronecker deltas on each link of the lattice. The main observation is that on every link this set of deltas is divided into two subsets. Each subset can be identified with one of lattice sites a given link belongs to. Thus, all summations over group indices are factorized in every lattice site in any dimension. This is a direct consequence of the local symmetry. In each site the combination of the sets of deltas, which come from all links containing given site, defines a site permutation  $\gamma(x)$  on the set of all group indices corresponding to this site. This site permutation  $\gamma(x)$  is a function of the link permutations  $\sigma_l, \tau_l$ . The lengthy calculations of the corresponding sums will be presented in a separate publication. Here we present our results for three- and four-dimensional models. In 3d the constraint of Eq.(5)  $\delta_{r,s}$  takes a form  $r(l) = r(p_1) + r(p_2) - r(p_3) - r(p_4) = 0$ , where  $p_i$  are four plaquettes having link  $l$  in common, and can be solved by introducing dual variables and placing them in the centers of original cubes. Then, the summation over group indices and duality transformations lead to the following representation on the dual lattice

$$Z_{\text{gauge}} = \sum_{r(x)=-\infty}^{\infty} \sum_{k(l)=0}^{\infty} \sum_{\{\tau_p, \sigma_p\}} \prod_l \frac{\beta^{2k(l)+|r(x)-r(x+\mu)|}}{k(l)!(k(l) + |r(x) - r(x + \mu)|)!} \prod_p W g^N (\tau_p^{-1} \sigma_p) \prod_c P_{\gamma(c)}(1). \quad (20)$$

Here,  $\sigma_p, \tau_p$  are elements of a permutation group  $S_p$ ,  $P = \sum_{l \in p} (k(l) + |r(x) - r(x + \mu)|/2)$ .  $\prod_c$  runs over all elementary cubes of the dual lattice and the symmetric function  $P_{\gamma(c)}(1) = N^{|\gamma(c)|}$ .  $|\gamma(c)|$  is the number of cycles in combined permutations:  $\gamma(c) \in S_C$ ,  $C = \sum_{l \in c} (2k(l) + |r(x) - r(x + \mu)|)$ . Similar to the spin models one recovers the conventional dual form for  $U(1)$  LGT

$$Z = \sum_{r(x)=-\infty}^{\infty} \prod_l I_{r(x)-r(x+\mu)}(\beta). \quad (21)$$

Dual representation can be simplified by using orthogonality relation for the characters

$$\sum_{\omega \in S_r} \chi_{\mu}(\omega\tau) \chi_{\lambda}(\omega\sigma) = r! \delta_{\mu,\lambda} \frac{\chi_{\lambda}(\tau^{-1}\sigma)}{d(\lambda)}. \quad (22)$$

One then finds

$$Z_{\text{gauge}} = \sum_{r(x)=-\infty}^{\infty} \sum_{k(l)=0}^{\infty} \sum_{\{\omega_p\}} \prod_l \frac{\beta^{2k(l)+|r(x)-r(x+\mu)|}}{k(l)!(k(l) + |r(x) - r(x + \mu)|)!} \prod_c \left( \prod_{p \in c} \sum_{\sigma_p} B_p \right) N^{|\gamma(c)|}, \quad (23)$$

where we introduced notation

$$B_p = \sum_{\lambda} \left( \frac{d(\lambda)}{r!} \right)^{3/2} \frac{1}{(s_{\lambda}(1))^{1/2}} \chi_{\lambda}(\omega_p \sigma_p). \quad (24)$$

The above consideration can be directly generalized to the four-dimensional theory. The only important difference is that in four dimension the solution of the constraint for the original plaquette variables  $r(p)$  is given by the dual link variables. Therefore, the dual of the partition function can be written as

$$Z_{\text{gauge}} = \sum_{r(l)=-\infty}^{\infty} \sum_{k(p)=0}^{\infty} \sum_{\{\tau_c, \sigma_c\}} \prod_p \frac{\beta^{2k(p)+|r(p)|}}{k(p)!(k(p)+|r(p)|)!} \prod_c Wg^N(\tau_c^{-1} \sigma_c) \prod_h P_{\gamma(h)}(1), \quad (25)$$

where

$$r(p) = r(l_1) + r(l_2) - r(l_3) - r(l_4), \quad (26)$$

and  $l_i$  are four links forming a given oriented plaquette  $p$ .  $\prod_c$  runs over all elementary cubes of the dual lattice.  $\sigma_c, \tau_c$  are elements of a permutation group  $S_C$ ,  $C = \sum_{p \in c} (k(p) + |r(p)|/2)$ .  $\prod_h$  runs over all hypercubes of the dual lattice and the symmetric function  $P_{\gamma(h)}(1) = N^{|\gamma(h)|}$ .  $|\gamma(h)|$  is the number of cycles in combined permutations which belong to the permutation group  $S_H$ ,  $H = \sum_{p \in h} (2k(p) + |r(p)|)$ . The generalization of Eq.(23) to four dimension is also straightforward.

## 6 Two-dimensional U(N) QCD

Finally, we consider two-dimensional  $U(N)$  LGT with one flavour of the massive staggered fermions. The presence of the dynamical fermions does not destroy the main observation of the previous section, namely all summations over group indices are factorized around lattice sites and can be performed as before. The resulting permutation group becomes even more complicated as it includes now the link occupation numbers. All details of the integration and summation will be given in a separate work. We present here the final result for the partition function. Expressed in terms of the plaquette and link occupation numbers the partition function reads

$$\begin{aligned} Z = & \sum_{r(p)=-\infty}^{\infty} \sum_{t(p)=0}^{\infty} \sum_{k(l)=0}^N \sum_{n(l)=0}^N \sum_{s(x)=0}^N \sum_{\{\tau_l, \sigma_l\} \in S_X(l)} \prod_x m^{s(x)} N^{|\gamma(x)|} \\ & \prod_p \frac{(\beta/2)^{2t(p)+|r(p)|}}{t_p!(t_p+|r(p)|)!} \prod_l \left[ \frac{\eta_l(x)}{2} \right]^{k(l)+n(l)} e^{-\mu(k(l)-n(l))} Wg^N(\tau_l^{-1} \sigma_l) \\ & \times (\text{constraints}) \times (\text{sign factor}), \end{aligned} \quad (27)$$

where  $\eta_l(x)$  is staggered sign factor,  $\mu$  - chemical potential. Permutation group  $S_X(l)$  is fixed by

$$X(l) = t(p) + t(p') + \frac{1}{2} (|r(p)| + r(p) + |r(p')| - r(p')) + k(l). \quad (28)$$

Factor  $N^{|\gamma(x)|}$  arises after summation over all group indices,  $|\gamma(x)|$  is the number of cycles in combined permutations  $\sigma_l, \tau_l, l \in x$ . Allowed configurations of monomers, plaquette and link occupation numbers are constrained by the integration over gauge fields on every link

$$\prod_l \delta(r(p) - r(p') + k(l) - n(l)), \quad p, p' \text{ have common link } l, \quad (29)$$

and by the integration over fermion fields in every site

$$\prod_x \delta(s(x) + k(x) - N) \delta(s(x) + n(x) - N) ,$$

$$k(x) = \sum_{\nu=1}^2 [k_{\nu}(x) + n_{\nu}(x - \nu)] , \quad n(x) = \sum_{\nu=1}^2 [n_{\nu}(x) + k_{\nu}(x - \nu)] . \quad (30)$$

Sign factor appears due to the integration over fermions and is non-trivial only in the presence of closed fermion loops as allowed by the above constraints. It takes a form similar to the massless two-dimensional QED [21], namely

$$\prod_{\mathcal{L}} (-1)^{1+\frac{1}{2}|\mathcal{L}|} , \quad (31)$$

where  $\mathcal{L}$  is a closed fermion loop and  $|\mathcal{L}|$  is a length of a given loop. Detailed derivation of this formula as well as discussion of the general conditions for which the full Boltzmann weight is positive will be given elsewhere.

## 7 Discussion

In this paper we presented a new approach to construction of the dual formulations of non-abelian models. The existing integration methods, explained in Sect. 2, allow to make such dual transformations for  $U(N)$  lattice spin and gauge models. We have constructed dual forms of Polyakov loop spin models for arbitrary  $U(N)$  model with and without external field. Also, we have outlined our main findings for pure  $U(N)$  gauge theories and for two-dimensional  $U(N)$  QCD. Details of our calculations will be reported in future publications.

Some obvious extensions of this work can be done. For example, it should not be difficult to generalize the present approach for gauge models with Wilson fermions. More important and interesting problem concerns extension of this method to  $SU(N)$  LGT. The recent paper [29] can be helpful in this direction.

As for the applications of our dual formulations we think it can be interesting in, at least, two aspects. Since the asymptotic expansions and bounds on the Weingarten function are known it might give a new direction in the investigation of LGT at large values of  $N$ . It can be useful in the studying of the confinement problem, as well. It remains to be seen if the dual formulation can help in solving the sign problem in QCD, at least in two-dimensional theories.

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