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Gluing operation and form factors of local operators in $\mathcal{N} = 4$ Super Yang-Mills theory

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Abstract. The gluing operation is an effective way to get form factors of both local and non-local operators starting from different representations of on-shell scattering amplitudes. In this paper it is shown how it works on the example of form factors of operators from stress-tensor operator supermultiplet in Grassmannian and spinor helicity representations.

1 Introduction

$\mathcal{N} = 4$ Super Yang-Mills theory (SYM) turns out to be a very popular object for theoretical research. Mainly this is due to the development of new techniques, such as recursive methods for tree-level amplitudes, on-shell diagrams and so on (see [3] for review), allowing to obtain analytical results for the matrix elements of the S-matrix. The Grassmannian integral representation for the amplitude is of particular interest, since it makes symmetry properties of the amplitudes manifest and relates scattering amplitudes to on-shell diagrams [4]. After a while, many of the techniques, developed for scattering amplitudes, were generalized to more complicated class of quantities - form factors, which are matrix elements of the form

$$\langle 0|\mathcal{O}(x)|1, ..., n\rangle,$$

on which our attention is focused on. In particular, it was especially interesting to obtain the Grassmannian integral representation for form factors, which was done in [1] and [2] in slightly different ways. The common thing for both these two approaches is gluing of the minimal formfactor to the corresponding amplitude, whose integral representation is known [4]. The purpose of this article is to check formulae, obtained in [1] and [2] for different representations, i.e. for the Grassmannian integral and spinor helicity representations.

2 Form factors of the Stress-Tensor Supermultiplet operator

For the purposes, claimed in the previous section, only form factors of the chiral part of the stress-tensor supermultiplet operator would be considered. With the help of the so-called harmonic superspace [5] the operator itself can be written as

$$T(x, \theta^+) = \mathrm{tr}(\phi^{++}\phi^{++}) + ... + \frac{1}{3}(\theta^+)^4\mathcal{L}, \tag{1}$$

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where $\theta_\alpha^{+a} = \theta_\alpha^A u_A^{+a}$, $\theta_\alpha^{-a'} = \theta_\alpha^A u_A^{-a'}$ are projections of coordinates on the superspace to the harmonic superspace with projectors u_A^{+a} and $u_A^{-a'}$. The lowest component of $T(x, \theta^+)$ is scalar operator $\mathrm{tr}(\phi^{++}\phi^{++})$ with $\phi^{++} = \frac{1}{2}\epsilon_{ab} u_A^{+a} u_B^{+b} \phi^{AB}$. The indices a, a' and $\pm$ correspond to $SU(2) \times SU(2)' \times U(1) \subset SU(4)$. Let q and $\gamma_a^{-\alpha}$ be the momentum and supermomentum, carried by the operator $O(x)$ respectively. Then the super form factor is defined by

$$\mathcal{F}_{k,n} = \int d^4x d^4\theta^+ e^{-iqx - i\theta_\alpha^{+a}\gamma_a^{-\alpha}} \langle 1, ..., n| T(x, \theta^+)|0\rangle. \tag{2}$$

For $k = 2$, which corresponds to the Maximal Helicity Violating (MHV) case, the form factor is given by the following equation:

$$\mathcal{F}_{2,n}(1, ..., n, q, \gamma^-) = \frac{\delta^{(4)}(P)\hat{\delta}^{(4)}(Q^+)\hat{\delta}^{(4)}(Q^-)}{\langle 12\rangle\langle 23\rangle ... \langle n-1n\rangle\langle n1\rangle}, \tag{3}$$

where

$$P = \sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i - q, Q^+ = \lambda_i \eta_i^+, Q^- = \lambda_i \eta_i^- - \gamma^-. \tag{4}$$

3 Gluing operation & Grassmannian integral representation

As it was pointed out in the beginning, there were some difficulties with obtaining the grassmannian integral representation for the form factor, which was explained by the fact that form factors are partially off-shell quantities. A conjecture was made, that such a representation could be obtained using known integral representation for the scattering amplitude. Graphically it can be illustrated by the figure 3. The idea of the gluing operation is very simple, i.e. to take known representation of the

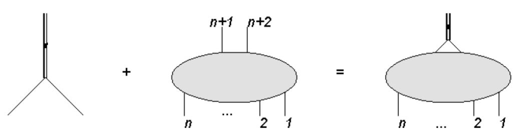

Figure 1. Graphical representation of the gluing operation

amplitude with two additional external legs and glue to it the minimal form factor, which contains delta functions, imposing proper conditions on kinematics, by integrating out these two extra degrees of freedom. Since the minimal form factor for the operator, considered here, has two external legs, one needs to glue it to $\mathcal{A}_{k,n+2}$. On mathematical language gluing reduces to the on-shell phase-space integration:

$$\mathcal{F}_{k,n} = \int \left(\prod_{i=n+1,n+2} \frac{d^2\lambda_i d^2\tilde{\lambda}_i}{Vol[GL(1)^2]} d^4\eta_i \right) \mathcal{F}_{2,2}\Big|_{\lambda_{n+1,n+2} \to -\lambda_{n+1,n+2}} \mathcal{A}_{k,n+2} + \text{other gluing positions}, \tag{5}$$

where $\mathcal{F}_{2,2} = \delta^2(\underline{\tilde{\lambda}_4})\delta^4(\underline{\eta_4})\delta^2(\underline{\tilde{\lambda}_5})\delta^4(\underline{\eta_5})$ and underscored variables impose twisted kinematics conditions (see eq. 2.21 of [1]):

$$\underline{\tilde{\lambda}_4} = \tilde{\lambda}_4 - \frac{\langle 5|q}{\langle 54\rangle} \qquad \underline{\eta_4^-} = \eta_4^- - \frac{\langle 5|\gamma^-}{\langle 54\rangle} \qquad \underline{\eta_4^+} = \eta_4^+$$

$$\underline{\tilde{\lambda}_5} = \tilde{\lambda}_5 - \frac{\langle 4|q}{\langle 45\rangle} \qquad \underline{\eta_5^-} = \eta_5^- - \frac{\langle 4|\gamma^-}{\langle 45\rangle} \qquad \underline{\eta_5^+} = \eta_5^+$$

After performing the integration, one arrives at the formula, valid for arbitrary k, n [1], [2]:

$$\mathcal{F}_{k,n} = \langle \xi_A \xi_B \rangle^2 \int \frac{d^{k\times(n+2)}C}{Vol[GL(k)]} \Omega_{k,n} \hat{\delta}^{2\times k}(C \cdot \underline{\underline{\tilde{\lambda}}}) \hat{\delta}^{4\times k}(C \cdot \underline{\underline{\eta}}) \hat{\delta}^{2\times(n+2-k)}(C^{\perp} \cdot \underline{\underline{\lambda}}) + \text{other gluing positions}, \quad (6)$$

where ξ_A and ξ_B are arbitrary reference spinors and

$$\Omega_{k,n} = \frac{Y(1-Y)^{-1}}{\prod_{i=1}^{n+2} M_i}, Y = \frac{(n-k+2, ...n, n+1)(n+2, 1, ..., k-1)}{(n-k+2, ...n, n+2)(n+1, 1, ..., k-1)}$$

and M_i are consecutive minors of the C matrix.

4 Examples of evaluation

In this section several examples of the application of the eq. (5), (6) will be provided.

The first example is 3-point MHV form factor in spinor helicity representation. Since $\mathcal{F}_{2,3}$ has 3 external legs, there are 3 gluing positions, but due to the fact, that all top-cell diagrams are equivalent in this case, all gluing positions should give the same results, so, to obtain answer for $\mathcal{F}_{2,3}$ one has to take into account only one top-cell diagram (i.e. one gluing position) since they are all equal. Substituting the expression for $\mathcal{A}_5^{\text{MHV}} = \frac{\hat{\delta}^{(8)}(\sum_{i=1}^{5} \lambda_i \eta_i)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$ to the integral (5) and considering gluing between legs 3 and 1, one has to integrate over $\lambda_{4,5}$, $\tilde{\lambda}_{4,5}$ and $\eta_{4,5}$. Integrations over $\tilde{\lambda}_{4,5}$ and $\eta_{4,5}$ could be done with the help of delta functions of the minimal form factor $\mathcal{F}_{2,2}$ which in turn mean that corresponding variables are just to be relabelled (see definitions of the underlined variables in the previous section). Thus, at this stage one can write:

$$\mathcal{F}_{2,3} = \int \left(\prod_{i=4,5} \frac{d^2\lambda_i}{Vol[GL(1)^2]} \right) \mathcal{A}_5^{\text{MHV}}(\lambda, \underline{\tilde{\lambda}}, \underline{\eta}).$$

To do the last interation one has to eliminate $GL(1)^2$ invariance, reparametrizing spinors as follows, introducing arbitrary reference spinors:

$$\lambda_4 = \xi_A - \beta_1 \xi_B$$
$$\lambda_5 = \xi_B - \beta_2 \xi_A.$$

This reparametrization is just change of variables, thus one can rewrite the result for $\mathcal{F}_{2,3}$ as integral over parameters $\beta_{1,2}$:

$$\mathcal{F}_{2,3} = -\langle \xi_A \xi_B \rangle^2 \frac{\delta^{(4)}(q - \sum_{i=1}^{3} p_i) \hat{\delta}^{(+4)}(Q^+) \hat{\delta}^{(-4)}(Q^-)}{\langle 12\rangle\langle 23\rangle} \cdot I$$

$$I = \frac{1}{\langle 3\xi_B\rangle\langle 1\xi_A\rangle\langle \xi_A\xi_B\rangle} \int \frac{d\beta_1 d\beta_2}{\left(\beta_1 - \frac{\langle 3\xi_A\rangle}{\langle 3\xi_B\rangle}\right)\left(\beta_2 - \frac{\langle 1\xi_B\rangle}{\langle 1\xi_A\rangle}\right)} = -\frac{1}{\langle 31\rangle\langle \xi_A\xi_B\rangle^2}.$$

The integral I is evaluated via the Cauchy residue theorem. For the case, when the minimal form factor is glued between legs i and j, one obtains for I:

$$I = -\frac{1}{\langle ij\rangle\langle \xi_A\xi_B\rangle^2}.$$

Thus the full answer reads [6]:

$$\mathcal{F}_{2,3} = \frac{\delta^{(4)}(q - \sum_{i=1}^{3} p_i)\hat{\delta}^{(+4)}(\sum_{i=1}^{3} \lambda_i \eta_i^+)\hat{\delta}^{(-4)}(\sum_{i=1}^{3} \lambda_i \eta_i^- - \gamma^-)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \quad (7)$$

The same answer could be derived via grassmannian integral (6). As can be easily shown, there are no integrations over grassmannian, since the number of integration variables is equal to zero in this case:

$$\#(\text{integrations}) = (k-2)(n-k)\Big|_{k=2,n=3} = 0,$$

which means that all coefficients of the C matrix are determined by the delta functions. Since

$$\hat{\delta}^6(C^\perp \cdot \underline{\underline{\lambda}}) = \prod_{i=3,4,5} \hat{\delta}^2(\underline{\underline{\lambda}}_i - c_{1i}\underline{\underline{\lambda}}_1 - c_{2i}\underline{\underline{\lambda}}_2) = \prod_{i=3,4,5} \frac{1}{\underline{\underline{\langle 12\rangle}}}\hat{\delta}\left(c_{1i} - \frac{\underline{\underline{\langle i2\rangle}}}{\underline{\underline{\langle 12\rangle}}}\right)\hat{\delta}\left(c_{2i} - \frac{\underline{\underline{\langle i1\rangle}}}{\underline{\underline{\langle 21\rangle}}}\right),$$

on the support of these delta functions one arrives at eq. (7) again, where double underscore mens twisted kinematics conditions (see eq. 3.9 of [1]).
The next example is 3-point next-to-MHV form factor. As in the previous case, all top-cell diagrams are equivalent here, so

$$\mathcal{F}_{3,3} = \frac{\langle \xi_A \xi_B\rangle^2 \delta^{(4)}(q - \sum p_i)\delta^4(\sum \lambda_i \eta_i^+)\delta^4(\sum \lambda_i \eta_i^- - \gamma^-)}{[12][23]\langle 12\rangle^4} \cdot I,$$

$$I = q^2 \delta^2(\eta_3^+) \int \frac{d\beta_1 d\beta_2}{1 - \beta_1\beta_2} \frac{\delta^2(q^2\eta_3^- - \langle 4|q|3]\eta_4^- - \langle 5|q|3]\eta_5^-)}{(\langle \xi_B|q|3] - \beta_1\langle \xi_A|q|3])(\langle \xi_A|q|1] - \beta_2\langle \xi_B|q|1])}.$$

Taking residues in simple poles of the integrand one gets

$$\mathcal{F}_{3,3} = \frac{q^2(\eta_3^+)^2\hat{\delta}^{(-2)}(q^2\eta_3^- + [3|q\gamma^-)}{\langle 12\rangle^4[12][23][31]}\delta^{(4)}\left(q - \sum_{i=1}^{3} p_i\right)\hat{\delta}^{(+4)}\left(\sum_{i=1}^{3} \lambda_i \eta_i^+\right)\hat{\delta}^{(-4)}\left(\sum_{i=1}^{3} \lambda_i \eta_i^- - \gamma^-\right) \quad (8)$$

Let's extract a specific component from this super form factor, namely the component at the maximal η degree ([6]), which is 12 in this case:

$$F^{max} = \frac{q^4}{[12][23][31]}. \quad (9)$$

This quantity would be necessary to check that results, derived using different approaches, agree.
In the full analogy with the first example it is possible to compute the same quantity via integral over the Grassmannian. Again, for $\mathcal{F}_{3,3}$ there are no integrations, so on the support of the

$$\hat{\delta}^6(C \cdot \underline{\underline{\tilde{\lambda}}}) = \prod_{i=1}^{3} \hat{\delta}^2(\underline{\underline{\tilde{\lambda}}}_i + c_{i4}\underline{\underline{\tilde{\lambda}}}_4 + c_{i5}\underline{\underline{\tilde{\lambda}}}_5)$$

one can solve the constraints to obtain coefficients of the C matrix:

$$c_{i4} = -\frac{\left[\underline{\underline{i5}}\right]}{\left[\underline{\underline{45}}\right]}, c_{i5} = -\frac{\left[\underline{\underline{i4}}\right]}{\left[\underline{\underline{54}}\right]}. \quad (10)$$

Substituting these values to the integrand yields:

$$\mathcal{F}_{3,3} = \frac{q^4}{[12][23][31]}\delta^{(4)}(q - \sum_{i=1}^{3} p_i)\hat{\delta}^{12}(C \cdot \underline{\underline{\eta}}). \tag{11}$$

By virtue of $\hat{\delta}^{12}(C \cdot \underline{\underline{\eta}}) = \left(\prod_{i=1}^{3}(\eta_i^+)^2\right) \times \prod_{i=1}^{3} \hat{\delta}^{(-2)}(\eta_i^- - ...)$ one can extract the component with the maximal η degree, which agrees with eq. (9).
The simplest example, containing non-trivial integrations over the Grassmannian manifold is Next-to-MHV form factor with 4 legs, or NMHV$_4$. In the case of 4 external legs there are 4 possible gluing positions, shown on the picture:

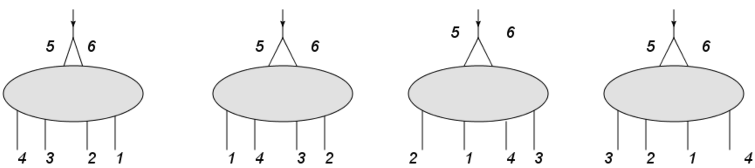

Figure 2. Gluing positions

It's worth mentioning here that a priori there is no prescription for choosing contour of integration, which gives the correct answer for arbitrary n, k. The integral over Grassmannian, corresponding to gluing between legs 1 and 4 has the form:

$$\mathcal{F}_{3,4} = \langle \xi_A \xi_B \rangle^2 \int \frac{d^{3\times 6}C}{Vol[GL(3)]} \Omega_{3,4} \hat{\delta}^{(6)}(C \cdot \underline{\underline{\tilde{\lambda}}}) \hat{\delta}^{(12)}(C \cdot \underline{\underline{\eta}}) \hat{\delta}^{(6)}(C^{\perp} \cdot \underline{\underline{\lambda}}). \tag{12}$$

Using the formula #(integrations) $= (k-2)(n-k)$, one notices that in this case the integral involves one non-trivial integration. The strategy of integration is described in [3], [7]: since the integral is 1-dimensional, there must exist one-parameter family of solutions $\hat{c}_{ij} = c_{ij}(\tau)$ that solve constraints, imposed by the delta functions of the integrand on the coefficients of the C-matrix. Then bosonic delta functions could be replaced by the momentum conserving delta function by localizing the integral on the solution of the constraints. Thus, the whole thing could be rewritten as follows:

$$\mathcal{F}_{3,4} = \langle \xi_A \xi_B \rangle^2 \delta^{(4)}(q - \sum_{i=1}^{4} p_i) \int d\tau \hat{\Omega}_{3,4} \prod_{i=2,4,6} \hat{\delta}^{(4)}(\eta_i + \sum_{j=1,3,5} \hat{c}_{ij}\eta_j). \tag{13}$$

In the latter formula the C-matrix is gauge-fixed to be

$$C = \begin{pmatrix} c_{21} & 1 & c_{23} & 0 & c_{25} & 0 \\ c_{41} & 0 & c_{43} & 1 & c_{45} & 0 \\ c_{61} & 0 & c_{63} & 0 & c_{65} & 1 \end{pmatrix} \tag{14}$$

This for of the matrix suits for calculation residues of «even» minors ((234) and (456)), while for «odd» minors ((123) and (561)) one has to use another gauge:

$$C = \begin{pmatrix} 1 & c_{12} & 0 & c_{14} & 0 & c_{16} \\ 0 & c_{32} & 1 & c_{34} & 0 & c_{36} \\ 0 & c_{52} & 0 & c_{54} & 1 & c_{56} \end{pmatrix} \tag{15}$$

Hats here indicate τ-dependence. Let's denote (ijk) - minor of the C-matrix, composed of i-th, j-th and k-th columns of C. The integrand has poles where $(ii+1i+2) = 0$. With the help of this condition one can express c_{ij} in terms of helicity spinors, assigned to external particles (see [3] for details). As it has already been mentioned, there is no prescription in which poles one has to take residues, so it has to be figured out somehow. It turns out that in this particular case only 2 of 4 gluing positions contribute to the answer, and only in 2 of 4 poles one must take residues in each gluing position, namely

$$\mathcal{F}_{3,4} = \text{Res}_{41}(123) + \text{Res}_{41}(561) + \text{Res}_{23}(123) + \text{Res}_{23}(561). \tag{16}$$

It's worth metioning that eq. (16) agrees with eq. 3.76 of [1]. Contributing diagrams are shown on the figure 3. Here the subscript on $\text{Res}_{ij}(klm)$ denotes legs, between which the minimal form factor is

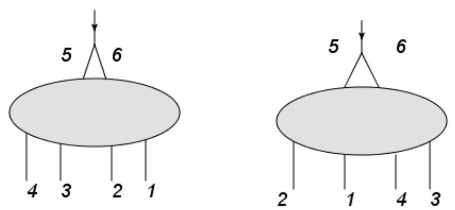

Figure 3. Diagrams contributing to $\mathcal{F}_{3,4}$

pasted in. Expressions for residues are given below:

$$\text{Res}_{41}(123) = \frac{\langle\xi_A\xi_B\rangle^2\langle 13\rangle^4[4\underline{\underline{6}}]^4\hat{\delta}^{(12)}(C_{(123)}\cdot\underline{\underline{\eta}})}{\langle 12\rangle\langle 23\rangle[\underline{\underline{56}}]^2\langle 3|1+2|4]\langle 1|2+3|4]P^2_{123}}$$

$$\text{Res}_{41}(561) = \frac{\langle\xi_A\xi_B\rangle^2\langle 1\underline{\underline{5}}\rangle^4[24]^4\hat{\delta}^{(12)}(C_{(561)}\cdot\underline{\underline{\eta}})}{[23][34]\langle\underline{\underline{56}}\rangle^2\langle 1|\underline{\underline{5}}+\underline{\underline{6}}|2]\langle 1|\underline{\underline{5}}+\underline{\underline{6}}|4]P^2_{234}}.$$

To obtain $\text{Res}_{23}(123)$ and $\text{Res}_{23}(561)$ one has to apply $s = 2$ shift to the latter formulas, i.e. relabel $1 \to 3$, $2 \to 4$, etc. $C_{(ijk)}$ denotes C-matrix evaluated at residue (ijk). Extracting $F(\phi_{12}, \phi_{12}, g^-, g^+)$ component (i.e. coefficient at $(\eta_1)^2(\eta_2)^2(\eta_3)^4$), one arrives at the following formula:

$$F(\phi_{12}, \phi_{12}, g^-, g^+) = c^2_{16}c^2_{52}\text{Res}_{41}(561) + c^2_{36}c^2_{54}\text{Res}_{23}(123). \tag{17}$$

Values of coefficients could be evaluated exactly as in sec. 9.3.2 of [3]. Thus, the final result for this particular component of 4-point NMHV super form factor, evaluated from Grassmannian integral, is represented by the the following formula ($q = p_1 + p_2 + p_3 + p_4$):

$$F(\phi_{12}, \phi_{12}, g^-, g^+) = \frac{1}{\langle 1|q|2]}\left[\frac{[24]^2\langle 1|q|4]}{[23][34]P^2_{234}} + \frac{\langle 13\rangle^2\langle 3|q|2]}{\langle 34\rangle\langle 41\rangle P^2_{134}}\right]. \tag{18}$$

The answer, obtained with the help of Grassmannian integral representation coincides with the formula 2.10 derived in [6] by means of BCFW recursion relations.

5 Conclusion

In recent papers [1], [2] it was shown that gluing operation could be used to derive different representations for form factors. In order to confirm their conjecture, several examples have been considered in the present article. The obtained results agree with those derived earlier. Also it's worth mentioning that presented results can be further generalized to the case of gluing non-local operators, such as Wilson line operators, to on-shell amplitudes to obtain Grassmannian integral representation for amplitudes with one leg off-shell, which will be the next step.

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