Toward a QFT treatment of nonexponential decay

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Abstract. We study the properties of the survival probability of an unstable quantum state described by a Lee Hamiltonian. This theoretical approach resembles closely Quantum Field Theory (QFT): one can introduce in a rather simple framework the concept of propagator and Feynman rules, Within this context, we re-derive (in a detailed and didactical way) the well-known result according to which the amplitude of the survival probability is the Fourier transform of the energy distribution (or spectral function) of the unstable state (in turn, the energy distribution is proportional to the imaginary part of the propagator of the unstable state). Typically, the survival probability amplitude is the starting point of many studies of non-exponential decays. This work represents a further step toward the evaluation of the survival probability amplitude in genuine relativistic QFT. However, although many similarities exist, QFT presents some differences w.r.t. the Lee Hamiltonian which should be studied in the future.

1 Introduction

Quantum decays are a common phenomenon in particle, nuclear, and atomic physics [1–3]. A typical starting point for the discussion of the decay of the amplitude for the survival probability of a certain unstable state S,

$$a_S(t) = \int_{m_{th}}^{+\infty} \mathrm{dm} d_S(m) e^{-imt} \, , \tag{1}$$

where $d_S(m)$ is the so-called energy (or mass) distribution ($\mathrm{dm}d_S(m)$ is the probability that the unstable state has an energy (or mass) between m and $m + dm$ and m_{th} the lowest energy of the system). Under general assumptions, one can show that the survival probability $p_S(t) = |a_S(t)|^2$ is not exponential both at short times (where $p'_S(t \to 0) = 0$) and at long times (where a power low is realized). For these deviations to occur, it is enough that a low-energy threshold is present and that $d_S(m)$ is not of the Breit-Wigner type [4], see also Refs. [1, 2, 5–9] and refs. therein. The short-time behavior leads to the so-called Quantum Zeno Effect (QZE): multiple collapse measurements freeze the time evolution, thus preventing the decay to take place [10–13]. Note, very often $p_S(t)$ is expressed as $1 - t^2/\tau_Z^2 + ...(\tau_Z$ is the Zeno time), but the weaker requirement of a zero derivative of $p_S(t)$ at $t = 0$ is sufficient. Experimentally, deviations from the exponential law have been measured at short times

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in Ref. [14] (for the corresponding QZE see Ref. [15]) and at long times in Ref. [16] (for an indirect proof through data on beryllium decays, see Ref. [17]).

Lee Hamiltonians [18] (LH) represent a useful theoretical framework for the study of decays, e.g. Refs. [2, 6, 19–21] and refs. therein. This approach resembles very closely Quantum Field Theory (QFT). Similar Hamiltonians have been used in various areas of physics, which go from atomic physics and quantum optic [4, 22, 23] to QCD [24].

The issue of non-exponential decay in a pure QFT framework is still debated: while in Ref. [25] a negative result was found (also in the case of super-renormalizable Lagrangian), in Ref. [20, 26] a different result is obtained: it is argued that also in QFT Eq. (1) holds and short- and long-time deviations take place.

While the final goal is the derivation of Eq. (1), and hence of non-exponential decay, in a genuine QFT relativistic environment, in this work we take a more humble intent. We aim to recall in a detailed (and also didactical) way how Eq. (1) emerges when using Lee Hamiltonians. In particular, we shall also show that $d_S(m)$ is the mass distribution of the decaying particle. Moreover, we establish a link between a discrete and continuous base of final states and between the basis of the unperturbed and full Hamiltonians. This study is intended to be useful for further analyses on non-exponential decays.

The article is organized as follows: in Sec. 2 we present the Lee Hamiltonian, both in the discrete and in the continuous cases. Then, in Sec. 3 we study the time evolution of an unstable state: the amplitude of the survival probability is expressed first as the Fourier transformation of the propagator and then of the energy distribution. Finally, in Sec. 4 we present conclusions and outlook.

2 The Lee Hamiltonian's approach

We present here the Lee Hamiltonian (LH), first using an infinite discrete set of decay products and then performing the limit to the continuous case.

2.1 Discrete LH

Let us consider the quantum state $|S\rangle$ as the unstable state that we aim to investigate. In particular, we study its time evolution after its preparation at $t = 0$. The state $|S\rangle$ interacts with an 'infinity' of other states, denoted as:

$$|k_n\rangle \text{ with } k_n = \frac{2n\pi}{L} \text{ and } n = 0, \pm 1, \pm 2, \dots , \tag{2}$$

where the quantity L (with the dimension of energy^{-1}) can be thought as the length of the linear box in which we place our system. The physical results should not depend on L, if it is large enough. The quantities k_n 'look like' momenta, see below. Finally, the basis of the Hilbert space of our quantum problem reads:

$$\text{Basis of the Hilbert space } \mathcal{H}\text{: } \{|S\rangle, |k_0\rangle, |k_1\rangle, |k_{-1}\rangle, \dots\} \equiv \{|S\rangle, |k_n\rangle\} \tag{3}$$

with the usual orthonormal and completeness relations:

$$\langle S|S\rangle = 1\ ,\ \langle S|k_n\rangle = 0\ ,\ \langle k_n|k_m\rangle = \delta_{nm}\ ;\ |S\rangle\langle S| + \sum_n |k_n\rangle\langle k_n| = 1_{\mathcal{H}}\ . \tag{4}$$

The Lee Hamiltonian of the system consists of two pieces:

$$H = H_0 + H_1 \tag{5}$$

where H_0 describes the free (non-interacting) part while H_1 mixes $|S\rangle$ with all $|k_n\rangle$

$$H_0 = M_0 |S\rangle \langle S| + \sum_{n=0,\pm1,...} \omega(k_n) |k_n\rangle \langle k_n| \ ; H_1 = \sum_{n=0,\pm1,...} \frac{gf(k_n)}{\sqrt{L}} (|S\rangle \langle k_n| + |k_n\rangle \langle S|) \ . \tag{6}$$

Following comments are in order:

- The quantities M_0, $\omega(k_n)$, $gf(k_n)$ are real.
- The Hamiltonian H is Hermitian.
- Dimensions: M_0 and $\omega(k_n)$ have dimensions [energy], while g has dimensions [energy$^{+1/2}$]
- The energy M_0 is the bare energy of the level $|S\rangle$. In particle physics, it is the bare mass at rest.
- The energy $\omega(k_n)$ is the bare energy of the state $|k_n\rangle$, see below.
- The coupling constant g measures the strength of the interaction; the form factor $f(k_n)$ modulates the interaction. In practice, each mixing $|S\rangle \longleftrightarrow |k_n\rangle$ has its own coupling constant $gf(k_n)$.
- The factor $\sqrt{L}$ is introduced for future convenience: it is necessary for a smooth continuous limit $L \to \infty$.
- For notational simplicity, $\sum_{n=0,\pm1,...}$can be also expressed simply as $\sum_n$.

Further discussion concerns the interpretation and the energy $\omega(k_n)$.

Interpretation: The state $|S\rangle$ represents an unstable particle S in its rest frame and the state $|k_n\rangle$ represents a possible final state of the decay of S. In the simplest case of a two-body decay, the state $|k_n\rangle$ represents **two** particles emitted by S flying back-to-back:

$$S \to \varphi_1 + \varphi_2 \ . \tag{7}$$

In the case of one spacial dimension, k_n can be interpreted as the momentum of φ_1, while $-k_n$ is the momentum of φ_2. Schematically: $|k_n\rangle \equiv |\varphi_1(k_n), \varphi_2(-k_n)\rangle$. In this way, the total momentum of $|k_n\rangle$ is still zero, as it must. (The 3D extension is straightforward). As examples of such a process, we may think of: (i) The neutral pion π^0 decays into two photons: $\pi^0 \to \gamma\gamma$. Then, π^0 in its rest frame corresponds to $|S\rangle$, while $\gamma\gamma$ corresponds to $|k_n\rangle$ (one photon has momentum k_n, the other $-k_n$). (Note, a very large number of two-body decays is listed in the PDG [3]). (ii) An excited atom A^* decays into the-ground state atom A emitting a photon γ: $A^* \to A\gamma$. In this case, A^* is the sate $|S\rangle$, while $|k_n\rangle$ represents the joint system of the ground-state atom A and the photon.

Function $\omega(k_n)$: as mentioned above, the function $\omega(k_n)$ represents the energy of the state $|k_n\rangle$. In the case of a two-body decay its form is given by

$$\omega(k_n) = \sqrt{k_n^2 + m_1^2} + \sqrt{k_n^2 + m_2^2} \ , \tag{8}$$

where m_1 is the mass of φ_1 and m_2 of φ_2. Clearly, $\omega(k_n) \geq m_1 + m_2 = m_{th}$, where m_{th} represents the lowest energy of the $|k\rangle$ states. In the two-photon decay such as the process (i) described above, one has $m_1 = m_2 = 0$, hence $\omega(k_n) = 2|k_n| \geq 0 = m_{th}$. In an atomic decay of the type $A^* \to A + \gamma$, one has $m_1 = 0$, and $m_2 = M_A$, hence:

$$\omega(k_n) \simeq |k_n| + M_A. \tag{9}$$

In this case, one could also subtract a constant term, $H_0 \to H_0 - M_A 1_{\mathcal{H}}$, out of which $\omega(k_n) \simeq |k_n|$.

2.2 Continuous LH

The limit $L \to \infty$ implies that the variable k_n becomes continuos:

$$k_n = \frac{2\pi n}{L} \to k \subset (-\infty, +\infty). \tag{10}$$

As usual, when L is sent to infinity sums turn into integrals:

$$\sum_n = \frac{L}{2\pi} \sum_n \frac{2\pi}{L} \to \frac{L}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dk} = L \int_{-\infty}^{+\infty} \frac{\mathrm{dk}}{2\pi} \,, \tag{11}$$

where $\delta k = 2\pi/L$ has been introduced in order to generate the differential dk. Next, we turn to the kets $|k\rangle$ in the continuous limit, for which we expect that $\langle k_1|k_2\rangle = \delta(k_1 - k_2)$. To this end, let us write down the following L-dependent discrete representation of the Dirac-delta function:

$$\delta_L(k_n) = \int_{-L/2}^{L/2} \frac{\mathrm{dx}}{2\pi} e^{ik_n x} = \begin{cases} 0 \text{ for } n \neq 0 \\ \frac{L}{2\pi} \text{ for } n = 0 \end{cases} . \tag{12}$$

In the limit $L \to \infty$ one obtains (for an arbitrary function $u(k)$):

$$u(0) = \sum_n \delta k \delta_L(k_n) u(k_n) \to \int_{-\infty}^{+\infty} \mathrm{dk}\delta(k) u(k) = u(0) \tag{13}$$

showing that $\delta(k) = \lim_{L\to\infty} \delta_L(k_n)$ holds. Finally, the quite subtle link between $|k_n\rangle$ and $|k\rangle$ is given by:

$$|k_n\rangle \overset{L\to\infty}{=} \sqrt{\frac{2\pi}{L}}\, |k\rangle \ . \tag{14}$$

Namely:

$$\langle k_1|k_2\rangle = \lim_{L\to\infty} \frac{L}{2\pi} \langle k_{n_1}|k_{n_2}\rangle = \lim_{L\to\infty} \begin{cases} 0 \text{ for } n_1 \neq n_2 \\ \frac{L}{2\pi} = \delta_L(0) \text{ for } n_1 = n_2 \end{cases} = \delta(k_1 - k_2) \,, \tag{15}$$

as desired. (Note, in 3D we have $\sum_{\mathbf{k}} \to V \int \frac{d^3k}{(2\pi)^3}$,where $V = L^3$, and $|\mathbf{k} = 2\pi\mathbf{n}/L\rangle \to (2\pi)^{3/2}/\sqrt{V}\,|\mathbf{k}\rangle$). It is also quite peculiar that the dimension of the ket changes when considering the limit $L \to \infty$:

$$\dim[|k_n\rangle] = [\text{Energy}^0] \text{ (dimensionless)} \ , \ \dim[|k\rangle] = [\text{Energy}^{-1/2}] \ . \tag{16}$$

Then, the continuos Hilbert space is given by $\mathcal{H} = \{|S\rangle, |k\rangle\}$ with

$$\langle S|S\rangle = 1 \ , \ \langle S|k\rangle = 0 \ , \ \langle k_1|k_2\rangle = \delta(k_1 - k_2) \ . \tag{17}$$

We also check the completeness relation:

$$1_{\mathcal{H}} = |S\rangle\langle S| + \sum_n |k_n\rangle\langle k_n| = |S\rangle\langle S| + \sum_n \delta k \left(\sqrt{\frac{L}{2\pi}}\, |k_n\rangle\langle k_n| \sqrt{\frac{L}{2\pi}} \right) \overset{L\to\infty}{\to} |S\rangle\langle S| + \int_{-\infty}^{+\infty} dk\, |k\rangle\langle k| = 1_{\mathcal{H}} \tag{18}$$

Finally, we are ready to present the Lee Hamiltonian $H = H_0 + H_1$ in the continuous limit:

$$H_0 = M|S\rangle\langle S| + \int_{-\infty}^{+\infty} \mathrm{dk}\omega(k)\, |k\rangle\langle k| \ , H_1 = \int_{-\infty}^{+\infty} \mathrm{dk} \frac{gf(k)}{\sqrt{2\pi}} \left(|S\rangle\langle k| + |k\rangle\langle S| \right) \ .$$

One can verify that the dimensions is preserved. For instance:

$$\dim[dk\omega(k)\, |k\rangle\langle k|] = \dim[dk] \dim[\omega(k)] \dim^2[|k\rangle] = [\text{Energy}][\text{Energy}][\text{Energy}^{-1}] = [\text{Energy}] \ . \tag{19}$$

3 Determination of the survival probability

3.1 Time evolution operator

The Schrödinger equation (in natural units)

$$i\frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle \tag{20}$$

can be univocally solved for a certain given initial state

$$|\psi(0)\rangle = c_S |S\rangle + \sum_n c_n |k_n\rangle \overset{L\to\infty}{\equiv} c_S |S\rangle + \int_{-\infty}^{+\infty} \mathrm{d}k c(k) |k\rangle \tag{21}$$

with $c(k) \overset{L\to\infty}{\equiv} \sqrt{\frac{L}{2\pi}} c_n$. The normalization $\langle\psi(t)|\psi(t)\rangle = 1$ implies

$$1 = |c_S|^2 + \sum_n |c_n|^2 \overset{L\to\infty}{\equiv} |c_S|^2 + \int_{-\infty}^{+\infty} \mathrm{d}k\, |c(k)|^2 \ . \tag{22}$$

In particular, one is typically interested to the case $c_S = 1$ (but not only). A formal solution to the time evolution is obtained by introducing the time-evolution operator:

$$U(t) = e^{-iHt} \to |\psi(t)\rangle = U(t) |\psi(0)\rangle \ . \tag{23}$$

The time-evolution operator $U(t)$ can be expressed in terms of a Fourier transform (for $t > 0$):

$$U(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dE} \frac{1}{E - H + i\varepsilon} e^{-iEt} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dE} G(E) e^{-iEt} \text{ with } G(E) = \frac{1}{E - H + i\varepsilon} \ , \tag{24}$$

where ε is an infinitesimal quantity and $G(E)$ is the 'propagator operator', which can be expanded as:

$$G(E) = \frac{1}{E - H + i\varepsilon} = \sum_{n=0}^{\infty} \left(\frac{1}{E - H_0 + i\varepsilon} H_1 \right)^n \frac{1}{E - H_0 + i\varepsilon} \ , \tag{25}$$

where we have used that $(AB)^{-1} = B^{-1}A^{-1}$ (A, B arbitrary operators on the Hilbert space $\mathcal{H}$).

3.2 Propagator, Feynman rules, and survival probability

We are interested in the evaluation of the survival (or non-decay) probability amplitude $a_S(t) = \langle S| U(t) |S\rangle$, out of which the survival probability of the state S reads $p_S(t) = |a_S(t)|^2$. In the *trivial limit,* in which $H = H_0$ ($g \to 0$), one has

$$a_S(t) = \langle S| U(t) |S\rangle = \langle S| e^{-iH_0 t} |S\rangle = e^{-iM_0 t} \to p_S(t) = 1 \ . \tag{26}$$

Alternatively, one may use Eq. (24):

$$a_S(t) = \langle S| U(t) |S\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dE} \frac{1}{E - M_0 + i\varepsilon} e^{-iEt} = e^{-iM_0 t} \ , \tag{27}$$

where we have closed downwards and picked up the pole for $E = M_0 - i\varepsilon$ (one is obliged to close downwards to guarantee convergence). In passing by, we note that the object

$$G_S^{\text{free}}(E) = G_S^{(0)}(E) = \langle S| \frac{1}{E - H_0 + i\varepsilon} |S\rangle = \frac{1}{E - M_0 + i\varepsilon} \tag{28}$$

is the free propagator of the state S.

In the *interacting case* the evaluation of $a(t)$ proceeds as follow:

$$a_S(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} \mathrm{dE} G_S(E) e^{-iEt} \text{ , where } G_S(E) = \langle S|\, G(E)\, |S\rangle = \langle S|\, \frac{1}{E - H + i\varepsilon}\, |S\rangle \tag{29}$$

is the full propagator of S. It is now necessary to evaluate $G_S(E)$ explicitly though a lengthy but straightforward calculation [20]:

$$G_S(E) = \langle S|\, G(E)\, |S\rangle = \sum_{n=0}^{\infty} \langle S| \left(\frac{1}{E - H_0 + i\varepsilon} H_1\right)^n \frac{1}{E - H_0 + i\varepsilon}\, |S\rangle = \sum_{n=0}^{\infty} G_S^{(n)}(E) \tag{30}$$

with

$$G_S^{(n)}(E) = \langle S| \left(\frac{1}{E - H_0 + i\varepsilon} H_1\right)^n |S\rangle\, \frac{1}{E - M_0 + i\varepsilon}\ . \tag{31}$$

Let us evaluate the first three terms:

$$n = 0 \to G_S^{(0)}(E) = \langle S|\, 1\, |S\rangle\, \frac{1}{E - M_0 + i\varepsilon} = \frac{1}{E - M_0 + i\varepsilon}\ , \tag{32}$$

$$n = 1 \to \langle S|\, \frac{1}{E - H_0 + i\varepsilon} H_1\, |S\rangle\, \frac{1}{E - M_0 + i\varepsilon} = 0\ , \tag{33}$$

$$n = 2 \to G_S^{(1)}(E) = \langle S| \left(\frac{1}{E - H_0 + i\varepsilon} H_1\right)^2 |S\rangle\, \frac{1}{E - M_0 + i\varepsilon} \tag{34}$$

$$= \frac{1}{E - M_0 + i\varepsilon} \langle S|\, H_1 \frac{1}{E - H_0 + i\varepsilon} H_1\, |S\rangle\, \frac{1}{E - M_0 + i\varepsilon} = -\frac{\Pi(E)}{(E - M_0 + i\varepsilon)^3}\ . \tag{35}$$

The recursive quantity $\Pi(E)$ reads:

$$\Pi(E) = -\langle S|\, H_1 \frac{1}{E - H_0 + i\varepsilon} H_1\, |S\rangle\ . \tag{36}$$

We introduce $1_{\mathcal{H}} = |S\rangle\langle S| + \int_{-\infty}^{+\infty} dk\, |k\rangle\langle k|$ two times, obtaining:

$$\Pi(E) = -\langle S|\, H_1 1_{\mathcal{H}} \frac{1}{E - H_0 + i\varepsilon} 1_{\mathcal{H}} H_1\, |S\rangle = -\int_{-\infty}^{+\infty} \mathrm{dk} \int_{-\infty}^{+\infty} \mathrm{dq}\, \langle S|\, H_1\, |k\rangle \langle k|\, \frac{1}{E - H_0 + i\varepsilon}\, |q\rangle \langle q|\, H_1\, |S\rangle$$
$$= -\int_{-\infty}^{+\infty} \mathrm{dk} \int_{-\infty}^{+\infty} \mathrm{dq} \frac{g f(k)}{\sqrt{2\pi}} \frac{\delta(k - q)}{E - \omega(k) + i\varepsilon} \frac{g f(q)}{\sqrt{2\pi}} = -\int_{-\infty}^{+\infty} \frac{\mathrm{dk}}{2\pi} \frac{g^2 f(k)^2}{E - \omega(k) + i\varepsilon}\ ,$$

where $\langle S|\, H_1\, |k\rangle = g f(k)/\sqrt{2\pi}$ was used. Going further, for $n = 0, 1, 2, ...$ we get $G_S^{(2n+1)}(E) = 0$ and

$$G_S^{(2n)}(E) = \frac{[-\Pi(E)]^n}{(E - M_0 + i\varepsilon)^{n+1}}\ . \tag{37}$$

Finally:

$$G_S(E) = \sum_{n=0}^{\infty} G_S^{(2n)}(E) = \sum_{n=0}^{\infty} \frac{[-\Pi(E)]^n}{(E - M_0 + i\varepsilon)^{2n+1}} = \frac{1}{(E - M_0 + i\varepsilon)} \sum_{n=0}^{\infty} \frac{[-\Pi(E)]^n}{(E - M_0 + i\varepsilon)^n} \tag{38}$$

$$= \frac{1}{(E - M_0 + i\varepsilon)} \frac{1}{1 + \frac{\Pi(E)}{E - M_0 + i\varepsilon}} = \frac{1}{E - M_0 + \Pi(E) + i\varepsilon}\ . \tag{39}$$

At this point, we can identify 'Feynman rules' reminiscent of QFT:

$$\text{bare } S \text{ propagator } \rightarrow \frac{1}{E - M_0 + i\varepsilon} \tag{40}$$

$$\text{bare } k \text{ propagator } (k \text{ fixed}) \rightarrow \frac{1}{E - \omega(k) + i\varepsilon} \tag{41}$$

$$kS \text{ vertex} \rightarrow g f(k) \tag{42}$$

$$\text{internal } k \text{ line}(k \text{ not fixed}) \rightarrow -\Pi(E) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{g^2 f(k)^2}{E - \omega(k) + i\varepsilon} \tag{43}$$

Note, the latter can be understood by applying $gf(k)$ at each vertex and the k-propagator in the middle, and by an overall integration $\int_{-\infty}^{+\infty} \frac{dk}{2\pi}$ due to the fact that k is not fixed.

The full propagator of S determined above,

$$\text{full } S \text{ propagator } \rightarrow \frac{1}{E - M_0 + \Pi(E) + i\varepsilon}\,, \tag{44}$$

can be also obtained in a very elegant way by using the Bethe-Salpeter equation obtained by using the Feynman rules listed above:

$$G_S(E) = \frac{1}{E - M_0 + i\varepsilon} - \frac{1}{E - M_0 + i\varepsilon}\Pi(E) G_S(E)\,. \tag{45}$$

Finally, the survival amplitude (29) can be expressed as:

$$a_S(t) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dE} G_S(E) e^{-iEt} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dE} \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} e^{-iEt}\,. \tag{46}$$

3.3 Spectral function and survival probability

Let us denote the basis of eigenstates of the Hamiltonian H as $|m\rangle$ with

$$H\,|m\rangle = m\,|m\rangle \text{ for } m \geq m_{th} \ (m_{th} \text{ is the low-energy threshold})\,. \tag{47}$$

The existence of a minimal energy m_{th} is a general physical and mathematical property. The states $|m\rangle$ form an orthonormal basis of the Hilbert space $\mathcal{H}$ =$\{|m\rangle$ with $m \geq m_{th}\}$, whose elements fulfill standard relations:

$$1_{\mathcal{H}} = \int_{m_{th}}^{+\infty} \mathrm{dm}\,|m\rangle\,\langle m| \ ; \ \langle m_1 | m_2 \rangle = \delta(m_1 - m_2)\,. \tag{48}$$

The link between the 'old' basis $\{|S\rangle, |k\rangle\}$ (eigenstates of H_0) and the 'new' basis $\{|m\rangle\}$ (eigenstates of H) is not trivial. The state $|S\rangle$ can be expressed in terms of the basis $\{|m\rangle\}$ as

$$|S\rangle = \int_{m_{th}}^{\infty} \mathrm{dm} \alpha_S(m)\,|m\rangle \text{ with } \alpha_S(m) = \langle S|m\rangle\ . \tag{49}$$

The quantity

$$d_S(m) = |\alpha_S(m)|^2 = |\langle S|m\rangle|^2 \tag{50}$$

is called the **spectral function (or energy/mass distribution)** of the state S. The normalization of the state $|S\rangle$ implies the normalization of the mass distribution $d_S(m)$:

$$1 = \langle S|S\rangle = \int_{m_{th}}^{\infty} d_S(m) \mathrm{dm}\,. \tag{51}$$

The simple intuitive interpretation is that $d_S(m)$dm represents the probability that the state S has a energy (or mass) between m and $m+dm$. As a consequence, the time-evolution can be easily evaluated by inserting $1 = \int_{m_{th}}^{+\infty} \mathrm{dm}\, |m\rangle \langle m|$ two times:

$$a_S(t) = \langle S|\, U(t)\, |S\rangle = \langle S|\, e^{-iHt}\, |S\rangle = \int_{m_{th}}^{\infty} \mathrm{dm} d_S(m) e^{-imt} \, . \tag{52}$$

This is all formally correct, but it does not help us further as long as we do not have a way to calculate $d_S(m)$. This is possible by using the propagator of S studied in Sec. 3.2. In fact, the propagator can be re-expressed as (again, inserting $1 = \int_{m_{th}}^{+\infty} dm\, |m\rangle \langle m|$ two times):

$$G_S(E) = \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} = \langle S|\, 1 \frac{1}{E - H + i\varepsilon} 1\, |S\rangle = \int_{m_{th}}^{+\infty} \mathrm{dm} \frac{d_S(m)}{E - m + i\varepsilon} \, . \tag{53}$$

Its physical meaning can be understood by noticing that the dressed propagator $G_S(E)$ has been rewritten as the 'sum' of free propagators, whose weight function is $d_S(m)$. As a next step, we need to invert Eq. (53). Let us first consider the case $g = 0$. In this limit, it is evident from Eq. (53) that:

$$d_S(E) = \delta(E - M_0) \, . \tag{54}$$

This is expected because in this case the state $|S\rangle$ is an eigenstate of the Hamiltonian, hence the mass distribution is a delta-function peaked at M_0. When the interaction is switched on, we evaluate the imaginary part of Eq. (53):

$$\mathrm{Im}\, G_S(E) = \int_{m_{th}}^{+\infty} \mathrm{dm} \frac{-\varepsilon d_S(m)}{(E - m)^2 + \varepsilon^2} = -\int_{m_{th}}^{+\infty} \mathrm{dm} d_S(m) \pi \delta(E - m) = -\pi d_S(E) \, . \tag{55}$$

Hence $d_S(E)$ is calculated as:

$$d_S(E) = -\frac{\mathrm{Im}\, G_S(E)}{\pi} = \frac{1}{\pi} \frac{\mathrm{Im}\, \Pi(E)}{(E - M_0 + \mathrm{Re}\, \Pi(E))^2 + (\mathrm{Im}\, \Pi(E))^2} \, . \tag{56}$$

The normalization of $d_S(E)$, Eq. (51), can be also proven by using Eq. (56), see details in Ref. [27].

In the end, once the spectral function $d_S(m)$ is known, the survival amplitude can be re-expressed as its Fourier transform by using Eqs. (46) and (53):

$$\begin{aligned} a_S(t) &= \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{dE} G_S(E) e^{-iEt} = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}E \int_{m_{th}}^{+\infty} \mathrm{dm} \frac{d_S(m)}{E - m + i\varepsilon} e^{-imt} \\ &= \int_{m_{th}}^{+\infty} \mathrm{dm} d_S(m) e^{-imt} = \int_{m_{th}}^{+\infty} \mathrm{dm} d_S(m) e^{-imt} \, . \end{aligned} \tag{57}$$

The latter expression coincides with Eq. (1), whose detailed determination was our goal. From here on, all the usual strategy can be applied [1, 2, 12, 20]. In particular, the (unphysical) Breit-Wigner limit is obtained for $\omega(k) = k$ (unlimited from below) and $f(k) = 1$, out of which $d_S(m) = \frac{\Gamma}{2\pi} \left((m - M_0)^2 + \Gamma^2/4 \right)^{-1}$ with $\Gamma = g^2$. In this case, $a_S(t) = e^{-i(M_0 - i\Gamma/2)t}$ and $p_S(t) = e^{-\Gamma t}$ (see details in Refs. [21, 28]).

4 Conclusions

We have proven that Eq. (1) holds in the QFT-like approach of effective Lee Hamiltonians by showing all the main steps leading to it. However, a Lee Hamiltonian is not fully equivalent to QFT, since some features are still missing. In fact, the Lee approach does not contain transitions from the vacuum state to some particles (in genuine QFT, terms of the type $|0\rangle\langle S\varphi_1\varphi_2|$+hc are also part of the interacting Hamiltonian and affect the results for finite time intervals). Moreover, quadratic expressions are not present in the propagator(s) of the Lee Hamiltonian but naturally appear in QFT.

Hence, the main question for the future reads: is Eq. (1) as it stands valid also in QFT? If, as argued in Refs. [20, 26], this is true, non-exponential decay is realized in QFT both at short and long times. Moreover, in the interesting case of a super-renormalizable Lagrangian, the short-time behavior is independent on the cutoff (this is so because the energy distribution $d_S(m)$ scales as m^{-3} for large m). Note, this is different from the result of Ref. [25] obtained by using perturbation theory at second order (that is, without resumming the propagator).

A further interesting topic for the future is the study of the decay of a particle with a nonzero momentum [29–34]. Contrary to naive expectations, the usual relativistic time dilatation formula does not hold (even in the exponential limit, a different analytical result is obtained, see details in Ref. [34]). The full understanding of the study of decay in QFT can also help to shed light on decays of moving particles.

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