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Cylindrical symmetry: II. The Green's function in 3+ 1 dimensional curved space

Gopinath Kamath[1,*]

[1] Department of Mathematics, Indian Institute of Technology Tirupati, Renigunta Road, Tirupati 517 506, India

Abstract. An exact solution to the heat equation in curved space is a much sought after; this report presents a derivation wherein the cylindrical symmetry of the metric $g_{\mu\nu}$ in 3 + 1 dimensional curved space has a pivotal role. To elaborate, the spherically symmetric Schwarzschild solution is a staple of textbooks on general relativity; not so perhaps, the static but cylindrically symmetric ones, though they were obtained almost contemporaneously by H. Weyl, Ann. Phys. Lpz. 54, 117 (1917) and T. Levi-Civita, Atti Acc. Lincei Rend. 28, 101 (1919). A renewed interest in this subject in C.S. Trendafilova and S.A. Fulling, Eur.J.Phys. 32, 1663(2011) – to which the reader is referred to for more references – motivates this work, the first part of which (cf.Kamath, PoS (ICHEP2016) 791) reworked the Antonsen-Bormann idea – arXiv:hep-th/9608141v1 – that was originally intended to compute theheat kernel in curved space to determine – following D.McKeon and T.Sherry, Phys. Rev. D 35, 3584 (1987) – the zeta-function associated with the Lagrangian density for a massive real scalar field theory in 3 + 1 dimensional stationary curved space to one-loop order, the metric for which is cylindrically symmetric. Using the same Lagrangian density the second part reported here essentially revisits the second paper by Bormann and Antonsen – arXiv:hep 9608142v1 but relies on the formulation by the author in S. G. Kamath, AIP Conf.Proc.1246, 174 (2010) to obtain the Green's function directly by solving a sequence of first order partial differential equations that is preceded by a second order partial differential equation.

1 1 Introduction

This is the second of two papers associated with the Lagrangian density

$$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \tag{1}$$

* E-mail: gkamath01865@gmail.com

for a real massive scalar field ϕ in 3 + 1 dimensional curved space; it extends the scope of the first[1] to obtain the Green's function $G(x,x';\sigma)$ as a solution to

$$BG = -\frac{\partial G}{\partial \sigma} \tag{2}$$

using a method worked out previously [2]and presented briefly further below. The operator B In eq.(2) is defined from (1) as

$$B \equiv -\partial^{\mu}\left(g_{\mu\nu}\partial^{\nu}\right) - m^2 \tag{3}$$

and G is subject to the initial condition $G(x,x';\sigma \to 0) = \delta^{(4)}(x-x')$.
In this effort one begins with:

a. A write of $G(x,x';\sigma)$ as $G(x,x';\sigma) = G_0 e^{-T}$ where G_0 – the Green's function in flat space – is a solution to

   $$\left(-\eta^{ab}\partial_a\partial_b - m^2\right)G_0 = -\frac{\partial G_0}{\partial \sigma}$$ subject to $G_0(x,x';\sigma \to 0) = \delta^{(4)}(x-x')$, its

   Euclidean version being

   $$G_0 = (4\pi\sigma)^{-2} e^{-\frac{(x-x')^2}{4\sigma} - m^2\sigma} \tag{4}$$

   with $\eta^{ab} = diag(1-1-1-1)$.

b. Now representing T as $T = \frac{\tau_{-1}}{\sigma} + \sum_{k=1}^{\infty}\tau_k\sigma^k$, makes τ_{-1} a solution to

   $$\partial^a\partial_a\tau_{-1} - 2\partial^a\tau_0\partial_a\tau_{-1} - (x-x')^a\partial_a\tau_0 = 0 \tag{5}$$

   with τ_0 defined further below; equally, one gets:

   $$2\partial^a\tau_{-1}\partial_a\tau_{-1} + (x-x')^a\partial_a\tau_1 - f + \tau_1 + \partial^a\tau_0\partial_a\tau_0 - \partial^a\partial_a\tau_0 = 0 \tag{6}$$

and

$$2\partial^a\tau_{-1}\partial_a\tau_2 + (x-x')^a\partial_a\tau_1 + 2\tau_2 + 2\partial^a\tau_1\partial_a\tau_0 - \partial^a\partial_a\tau_1 = 0 \tag{7}$$

for τ_1 and τ_2 respectively, with similar coupled partial differential equations for τ_n, $n \geq 3$ .While eq.(5) is a second order partial differential equation for τ_{-1}, eqs.(6) and (7) – as well as those defining the subsequent τ_n – are of the first order thus allowing for an easy solution.

2 Calculating G

2.1 2 + 1 dimensions

As a prelude to determining G it pays to start with 2 + 1 dimensional curved space below , the metric for which is given by[3,4]

$$g_{00}=1, g_{01}=-\frac{\lambda y}{r^2}, g_{02}=\frac{\lambda x}{r^2}, g_{11}=-1+\left(\frac{\lambda y}{r^2}\right)^2, g_{12}=-\frac{\lambda^2 xy}{r^2}, g_{22}=-1+\left(\frac{\lambda x}{r^2}\right)^2 \quad (8)$$

with $2\pi\lambda=\kappa J, \kappa=8\pi G, G$ being the gravitational constant and $J=\left|\vec{J}\right|$ being the spin of the massless particle located at the origin[5]. Eq.(8) enables one to deduce

the vierbeins e_μ^a from $g_{\mu\nu}=\eta_{ab}e_\mu^a e_\nu^b$ and a convenient set is:

$$e_\mu^a : e_0^0=1, e_0^1=0=e_0^2; e_1^0=-\frac{\lambda y}{r^2}, e_1^1=\frac{1}{\sqrt{2}}=-e_1^2; e_2^0=\frac{\lambda x}{r^2}, e_2^1=-\frac{1}{\sqrt{2}}=e_2^2 \quad (9)$$

with their inverses e_a^μ determined from $e_\mu^a e_b^\mu=\delta_b^a$ as:

$$e_a^\mu : e_0^0=\frac{i\lambda y}{r^2}, e_0^1=i, e_0^2=0; e_1^0=i, e_1^1=0=e_1^2; e_2^0=\frac{\lambda x}{r^2}, e_2^1=0, e_2^2=-1 \quad (10)$$

From eqs.(10) one gets with $g^{\mu\nu}=\eta^{ab}e_a^\mu e_b^\nu, \eta^{ab}=diag(1-1-1-1)$ the following:

$$g^{00}=1-\frac{\lambda^2}{r^2}, g^{01}=-\frac{\lambda y}{r^2}, g^{02}=\frac{\lambda x}{r^2}, g^{11}=-1, g^{12}=0, g^{22}=-1 \quad (11)$$

With the vierbeins in (9) and (10) and the definition of $\tau_0=-\frac{1}{2}\int e_\alpha^n\partial_n\left(e_k^\alpha\right)dx^k$ as in Ref.2 it is easy to check that $\tau_0=0$. Eq.(5) now becomes $\partial_a\partial_a\tau_{-1}=0$ in 3 – dimensional Euclidean space, leading to $\tau_{-1}=\frac{C}{R}$ with $R=\sqrt{\left(x_0^2+x_1^2+x_2^2\right)}$ and C an integration constant. Again, with eqs.(9) and (10) and

$$f=\frac{1}{2}\eta^{ab}\partial_a\left(e_\alpha^m\partial_m\left(e_b^\alpha\right)\right)+\frac{1}{4}\eta^{ab}\left(e_\alpha^m\partial_m\left(e_a^\alpha\right)\right)\left(e_\alpha^m\partial_m\left(e_b^\alpha\right)\right) \quad (12)$$

as in Ref.2 one gets $f=0$ thereby reducing eq.(6) to

$$2\partial^a\tau_{-1}\partial_a\tau_{-1}+\left(x-x'\right)^a\partial_a\tau_1+\tau_1=0 \quad (13)$$

whose solution in Euclidean space with $x'=0$ is

$$\tau_1=D\left(2C-R^3\right)^{-1/3} \quad (14)$$

with another integration constant D .It is now easy to solve eq.(7) with $\tau_0=0$, the answer being

$$\tau_2=C_1\tau_1^2-\frac{24C}{5D^5}\tau_1^2\left(R\tau_1^5-\int dR\tau_1^5\right) \quad (15)$$

taking $x'=0$ in eq.(7), C_1 being an integration constant.

2.2 3 + 1 dimensions

It is tantalizing to assume that eqs.(5) – (7) will be just as easy to solve in 3 + 1 dimensions for which the metric[1] is taken as

$$g_{00}=\left(\frac{a}{r}\right)^4, g_{11}=-\frac{1}{r^2}\left(y^2+\frac{a^8x^2}{r^8}\right), g_{22}=-\frac{1}{r^2}\left(x^2+\frac{a^8y^2}{r^8}\right),$$
$$g_{12}=-\frac{xy}{r^2}\left(1-\frac{a^8}{r^8}\right), g_{33}=-\left(\frac{a}{r}\right)^4 \tag{16}$$

with a a non – zero constant, the remaining $g_{ij}, i\neq j$ in (16) being zero. For eqs.(16) it is convenient to adopt for the calculation here the vierbeins given by

$$e_\mu^a: \begin{array}{l} e_0^0=0, e_0^1=\frac{ia^2}{\sqrt{2}r^2}, e_0^2=-\frac{ia^2}{\sqrt{2}r^2}, e_0^3=0\,; e_1^0=\frac{ia^4x}{r^5}, e_1^1=0=e_1^2, e_1^3=\frac{y}{r}; \\ e_2^0=\frac{ia^4y}{r^5}, e_1^1=0=e_1^2, e_1^3=-\frac{x}{r}; e_3^0=0, e_3^1=\frac{a^2}{\sqrt{2}r^2}, e_3^2=\frac{a^2}{\sqrt{2}r^2}, e_3^3=0 \end{array} \tag{17}$$

e_a^μ :

$$e_0^0=0, e_0^1=-\frac{ixr^3}{a^4}, e_0^2=-\frac{iyr^3}{a^4}, e_0^3=0; e_1^0=-\frac{ir^2}{\sqrt{2}a^2}, e_1^1=0=e_1^2, e_1^3=\frac{r^2}{\sqrt{2}a^2}$$
$$e_2^0=-\frac{ir^2}{\sqrt{2}a^2}, e_2^1=0=e_2^2, e_2^3=\frac{r^2}{\sqrt{2}a^2}; e_3^0=0, e_3^1=\frac{y}{r}, e_0^2=-\frac{x}{r}, e_0^3=0 \tag{18}$$

Eqs.(17) and(18) yield

$$\tau_0=-\log r, f=\frac{1}{r^2} \tag{19}$$

thus making the solution of eqs.(5) – (7) less accessible relative to the 2 + 1 dimensional case . Yet, it pays to rework eq.(5) to the form given in Ref.2 but in Euclidean space, viz.

$$\partial_a\partial_a\chi_{-1}+\left(\partial_a\partial_a\tau_0-\partial_a\tau_0\partial_a\tau_0\right)\chi_{-1}=\left(x-x'\right)_a\partial_a\tau_0e^{-\tau_0} \tag{20}$$

using the definition $\tau_{-1}=\chi_{-1}e^{\tau_0}$.

With eq.(19),(20) becomes with $x'=0$,a second order inhomogeneous partial differential equation viz.

$$\partial_a\partial_a\chi_{-1}-\frac{1}{r^2}\chi_{-1}=-r \tag{21}$$

for which a solution via a Green's function suggests itself. To do so one starts with

$$\left(\partial_0^2+\partial_z^2+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\right)G=-\frac{\delta(r-r')}{r}\delta(\theta-\theta')\delta(z-z')\delta(x_0-x_0') \tag{22}$$

implying that $\chi_{-1}=\int dx_0'dz'd\theta'dr'G\left(x_0z\theta r\middle|x_0'z'\theta'r'\right)r'$

with the Fourier transform $G=\left(\frac{1}{2\pi}\right)^2\int\limits_{-\infty}^{\infty}dsd\omega e^{i\omega x_0'+isz'}H(r,\theta,s,\omega)$ (23)

Following Duffy[6], eq.(23) leads from (22) to

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\left(s^2+\omega^2+\frac{1}{r^2}\right)\right)H=-\frac{\delta(r-r')}{r}\delta(\theta-\theta')e^{-isz-i\omega x_0} \tag{24}$$

Going further[6] one gets to a Bessel – like equation for (24) viz.

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\left(s^2+\omega^2\right)-\frac{1}{r^2}\left(n^2+1\right)\right)H_n=-\frac{\varepsilon_n}{\pi r}\delta(r-r') \quad ,$$

$$\varepsilon_0=\frac{1}{2},\varepsilon_n=1,n\geq 1 \tag{25}$$

For $r\neq r'$ it is obvious that the general solution to (25) will be a modified Bessel function of order $m=\pm\sqrt{n^2+1}$; thus m is not an integer in general ,and this suggests that one should perhaps begin with the $\theta-$independent solutions to (23) first. Easily, they would be the $n=0$ solutions of eq.(24) a general version of which is

$$H_0(r,r')=c(r')I_1(kr)+d(r')K_1(kr),k^2=s^2+\omega^2 \tag{26}$$

with $I_1(kr),K_1(kr)$ being the modified Bessel functions of order 1.

Moving on, Ref.6 helps to get a sharper version of (25) namely,

$$H_0(r,r')=\theta(r'-r)I_1(kr)K_1(kr')+\theta(r-r')K_1(kr)I_1(kr') \tag{27}$$

with $\theta(x)$ the usual Heaviside function. Eq.(27) is the $n=0$ version of

$$H_n(r,r')=\theta(r'-r)I_m(kr)K_m(kr')+\theta(r-r')K_m(kr)I_m(kr'),m=\sqrt{n^2+1} \tag{28}$$

One can now fill in some of the steps now that have been passed over for eq.(24) et seq.; we have, following Ref.6 used the expansions

$$\delta(\theta-\theta')=\frac{1}{2\pi}+\frac{1}{\pi}\sum_1^\infty\cos[n(\theta-\theta')],$$

$$H=e^{-isz-i\omega x_0}\sum_0^\infty H_n(r,r')\varepsilon_n\cos[n(\theta-\theta')],\varepsilon_0=\frac{1}{2},\varepsilon_n=1,n\geq 1 \tag{29}$$

to obtain the results given in eqs.(27) and (28); using (27) in (23) thus gets a $\theta-$ independent form of χ_{-1} which can now yield the $\tau_k,k\geq 1$ from the solution of the first order partial differential equations given in eqs.(6) and(7).This will be presented in detail elsewhere.

References

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5. See eq.(20) in Clement,Ref.5
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