Erratum to: On the relation between pole and running heavy quark masses beyond the four-loop approximation

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Abstract. The effective charges motivated method is applied to the relation between pole and $\overline{\text{MS}}$-scheme heavy quark masses to study high order perturbative QCD corrections in the observable quantities proportional to the running quark masses. The non-calculated five- and six-loop perturbative QCD coefficients are estimated. This approach predicts for these terms the sign-alternating expansion in powers of number of lighter flavors n_l, while the analyzed recently infrared renormalon asymptotic expressions do not reproduce the same behavior. We emphasize that coefficients of the quark mass relation contain proportional to π^2 effects, which result from analytical continuation from the Euclidean region, where the scales of the running masses and QCD coupling constant are initially fixed, to the Minkowskian region, where the pole masses and the running QCD parameters are determined. For the t-quark the asymptotic nature of the non-resummed PT mass relation does not manifest itself at six-loops, while for the b-quark the minimal PT term appears at the probed by direct calculations four-loop level. The recent infrared renormalon based studies support these conclusions.

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1 Introduction

It is well-known that the masses of charm, bottom and top-quarks are one of the most important QCD parameters, which are relevant for processing different data, obtained at LHC and

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Tevatron. The pole and $\overline{\text{MS}}$-scheme running heavy quark masses are the generally accepted definitions for these parameters. The first ones are determined by the position of pole of the renormalized fermion propagator at Minkowskian region $k^2 = M_q^2$. The scale-dependence of the second ones are defined from the solution of the following renormalization-group equation:

$$\frac{\overline{m}_q(s)}{\overline{m}_q(\mu^2)} = \exp\left(\int\limits_{a_s(\mu^2)}^{a_s(s)} \frac{\gamma_m(x)}{\beta(x)} \, dx \right).$$

(1)

Here $\overline{m}_q(\mu^2)$ are the running masses of heavy quarks, normalized at the Minkowskian scale μ^2, $a_s(\mu^2) = \alpha_s(\mu^2)/\pi$ is the renormalized QCD coupling constant in the $\overline{\text{MS}}$-scheme, s is the energy time-like variable. The RG β-function and anomalous mass dimension γ_m are defined as:

$$\mu^2 \frac{\partial a_s}{\partial \mu^2} = \beta(a_s) = -\sum_{n=0}^{\infty} \beta_n a_s^{n+2}, \qquad \mu^2 \frac{\partial \log(\overline{m}_q)}{\partial \mu^2} = \gamma_m(a_s) = -\sum_{n=0}^{\infty} \gamma_n a_s^{n+1}.$$

(2)

Their five-loop approximations in the $\overline{\text{MS}}$-scheme can be found in works of [1, 2] and [3] correspondingly. It is worth to emphasize that the renormalization scale μ^2 in Eqs.(1, 2) may be initially defined in the Euclidean region. As was shown in [4–9] the pole quark mass relation is sensitive to the long-distance infrared renormalon (IRR) effects. This IRR sensitivity leads to the $O(\Lambda_{\text{QCD}})$ renormalon ambiguity in the determination of heavy quark masses, which is related to the first $\delta=1/2$ IRR pole in the δ-plane of the Borel transform for the PT QCD relation between pole and running heavy quark masses [5]. Therefore, the significant attention is paid to the determinations of masses in the $\overline{\text{MS}}$-scheme, which do not imply the need for knowledge of the long-distance contributions. Indeed, their definition within dimensional regularization presumes taking into account of the UV divergent poles only. Therefore sometimes the running $\overline{\text{MS}}$ heavy quark masses are called the short-distance masses, which unlike pole masses have nothing in common with the IRR effects.

In view of all mentioned above it is of interest to consider the relation between pole and running masses of heavy quarks:

$$M_q = \overline{m}_q(\overline{m}_q^2) \sum_{n=0}^{\infty} t_n^M a_s^n(\overline{m}_q^2).$$

(3)

The choice $\mu^2 = \overline{m}_q^2$ corresponds to the commonly accepted way of fixation of the renormalization scale in the Minkowskian region, wherein coefficients t_n^M are polynomials in powers of the number of massless flavors n_l

$$t_n^M = \sum_{k=0}^{n-1} t_{nk}^M n_l^k$$

(4)

with the initial condition $t_0^M = 1$. Note that we consider the approximation when heavy flavor number n_f of active quarks are defined as $n_f = n_l + 1$. The results of calculations of one, two and three-loop corrections to (3) were obtained in [10], [11–13], [14, 15] respectively. The four-loop contributions t_{43}^M and t_{42}^M are known from the analytical calculations, performed in [16]. The rest $O(a_s^4)$-corrections, namely t_{41}^M and t_{40}^M-terms, are not yet computed in analytical form, but are evaluated numerically with the corresponding theoretical uncertainties. The total expressions for the t_4^M-term at fixed numbers of massless flavors $n_l = 3, 4, 5$ were evaluated in [17]. Using these numerical results, supplemented with the analytical information on

t_{43}^M and t_{42}^M-terms, the authors of Ref.[18] presented the first estimates of the unknown previously t_{41}^M and t_{40}^M-coefficients by means of the renormalon-inspired approach. Independently, the numerical values of these terms were determined in [19, 20] with the help of mathematically self-consistent least squares (LS) method, which is well-defined procedure of solving the overdetermined system of linear equations with the fixation of theoretical inaccuracies of the central values of the obtained results. Recently the updated results of the numerical evaluation of t_4^M-terms were obtained by means of the Monte-Carlo methods in [21] not for three values of n_l only but at extra 18 values as well, which corresponds to the additional points in the studied interval $0 \leq n_l \leq 20$. The results of these updated computations were obtained with considerably smaller uncertainties than the ones, presented in [17]. This prompted us to reconsider the results of [19] and to get new values of t_{41}^M, t_{40}^M-terms and their uncertainties using the same LS method [22]. The central values of new results have changed slightly (see the Note added to [19] as well). This is related to slight change of the central values of the improved more precise numbers from Ref.[21]. Moreover, thanks to the increase of the number of analyzed by LS method equations, fixed by the results from [21], which have smaller numerical errors than the outcomes of previous calculations [17], the obtained in [22] solutions of larger system of linear equations turned out to be drastically more precise than the ones, obtained previously in Ref.[19]. This feature is in agreement with general property of the mathematical LS approach. Leaving the discussions of the technical issues of this method aside we will return now to the consideration of the asymptotic structure of Eq.(3).

Due to the manifestation of the IRRs in the Borel image for the pole-running heavy quark mass relation, it is possible to conclude that the mass conversion formula of Eq.(3) is asymptotic one with the sign-constant factorially growing high order PT coefficients t_n^M. This means that at some orders of PT the series in (3) start to diverge. Indeed, from the results of direct calculations, performed at the two-loop level in the works [11–13] and at the three-loop level in Refs.[14, 15] it is possible to conclude that the perturbative QCD relation between pole and running charm-quark mass diverges from the second (or the third) order of PT. The situation with the mass conversion formula for the *b*-quark is more delicate. Indeed, its PT high-order contributions decrease up to four-loop level, although quite slowly. The four-loop contribution, numerically evaluated in [17, 21], is very close to its three-loop term. This allows to affirm that the renormalon nature of the PT series for the $\overline{\text{MS}}$-on-shell mass relation for bottom-quark is manifesting itself from the $O(a_s^4)$ contribution. However, in the case of the pole-running top-quark mass relation the perturbative outburst of the corresponding series is not manifesting itself at the four-loop level. In order to understand when the truncated perturbative series can be still used for case of the *t*-quark pole mass, it is necessary to estimate high-order corrections to its relation. This problem is analyzed by us below.

2 The effective-charges motivated method: from the Euclidean to Minkowski region

Let us now apply the used in [23] approach for estimations of the high-order PT QCD corrections to the relation between different definitions of heavy quarks. This approach is following the lines of the developed in [24] method of probing the values of high-order perturbative corrections to the renormalization-group invariant quantities, which in its turn is based on the concepts of the effective-charges (ECH) of Ref.[25]. The considered in [23] approach was already used in [26] for estimating four-loop PT QCD corrections to the expression between pole and $\overline{\text{MS}}$ running heavy quark masses, numerically evaluated later on in [17]. As was shown in Refs.[23, 24], from general grounds it is more theoretically justified to use the ECH-motivated procedure to the physical quantities, defined in the Euclidean space-time region,

and then, if necessary, to translate the expressions of the corresponding PT corrections to the Minkowskian region. This leads to the appearance of the proportional to powers of π^2-terms in the coefficients of the PT series, which relate the quantities, defined in the Minkowskian region. In our case these quantities are the pole and running heavy quark masses.

To clarify how this procedure is working we start from the following formal dispersion representation for the pole masses of the heavy quarks, first considered in [23]:

$$M_q = \frac{1}{2\pi i} \int_{-\overline{m}_q(\overline{m}_q^2)-i\epsilon}^{-\overline{m}_q(\overline{m}_q^2)+i\epsilon} ds' \int_0^\infty \frac{T(s)}{(s+s')^2} ds .$$

(5)

The function $T(s)$ has the meaning of the spectral density and it is defined as $T(s) = \overline{m}_q(s) \sum_{n=0}^\infty t_n^M a_s^n(s)$, where coefficients t_n^M coincide with the ones, which enter Eqs.(3) and (4). The expression (5) is similar to relation between the determined in the Euclidean region the e^+e^- annihilation Adler function and the Minkowskian time-like ratio $R(s)$, which is one of the main characteristics of the e^+e^- annihilation to hadrons process. The used in Ref.[23] dispersion relation of (5) leads to the conclusion that the analogue of the Adler function, namely the Euclidean quantity $F(Q^2)$ may be defined as

$$F(Q^2) = Q^2 \int_0^\infty ds \frac{T(s)}{(s+Q^2)^2} .$$

(6)

Within PT QCD it is expressed through the following series

$$F(Q^2) = \overline{m}_q(Q^2) \sum_{n=0}^\infty f_n^E a_s^n(Q^2) .$$

(7)

Taking into account the scale dependence of the $\overline{\text{MS}}$-scheme coupling constant and of the running heavy quark masses and using expansions (5-7), we can fix the relations between the coefficients of the Minkowskian series t_n^M and the determined in the the Euclidean region terms f_n^E as:

$$f_n^E = t_n^M + \Delta_n .$$

(8)

In accordance with equation (6) Δ_n-contributions contain the proportional to π^2 effects of the analytical continuation. Their detailed derivation at the six-loop level is given in [22]. Here we present the numerical expressions of these contributions for the case of $SU(3)$ color gauge group only. They have the following form:

$$\Delta_0 = 0 , \qquad \Delta_1 = 0 ,$$

(9)

$$\Delta_2 = 5.89434 - 0.274156 n_l ,$$

$$\Delta_3 = 105.6221 - 10.04477 n_l + 0.198002 n_l^2 ,$$

$$\Delta_4 = 2272.002 - 403.9489 n_l + 20.67673 n_l^2 - 0.315898 n_l^3 ,$$

$$\Delta_5 = 56304.639 - 13767.2725 n_l + 1137.17794 n_l^2 - 37.745285 n_l^3 + 0.427523 n_l^4 ,$$

$$\Delta_6 = 1633115.62 \pm 347.65 + (-518511.694 \pm 56.723) n_l + (61128.1666 \pm 4.7791) n_l^2$$
$$+ (-3345.0818 \pm 0.1371) n_l^3 + 85.37937 n_l^4 - 0.818446 n_l^5 .$$

The analytical expressions for $\Delta_0 - \Delta_4$-terms can be found in [23, 26], whereas the analytical expressions for Δ_5 and Δ_6 were obtained in [22]. The uncertainties, which enter in the numerical expression for Δ_6-term are determined by the corresponding inaccuracies of the four-loop

numerical t_{41}^M and t_{40}^M-contributions to Eq.(4), fixed in [22] by the LS method. Using Eqs.(9) one can find that the numerical values of the Δ_n-terms increase considerably with the growth of the order n of PT.

At the next stage of application of the ECH-motivated approach we fix the effective charge $a_s^{eff}(Q^2)$ for the introduced in (7) Euclidean quantity $F(Q^2)/\overline{m}_q(Q^2)$

$$\frac{F(Q^2)}{\overline{m}_q(Q^2)} = f_0^E + f_1^E a_s^{eff}(Q^2) , \qquad a_s^{eff}(Q^2) = a_s(Q^2) + \sum_{n=2}^{\infty} \phi_n a_s^n(Q^2) , \qquad (10)$$

where $\phi_n = f_n^E/f_1^E$. The coefficients of the ECH β-function, which is defined as $\beta^{eff}(a_s^{eff}) = -\sum_{n\geq 0}\beta_n^{eff}(a_s^{eff})^{n+2}$, are related to the coefficients β_n determined in the $\overline{\text{MS}}$-scheme β-function of Eq.(2) by the following renormalization-scheme invariant equations:

$$\beta_0^{eff} = \beta_0 , \qquad \beta_1^{eff} = \beta_1 , \qquad \beta_2^{eff} = \beta_2 - \phi_2\beta_1 + (\phi_3 - \phi_2^2)\beta_0 , \qquad (11a)$$

$$\beta_3^{eff} = \beta_3 - 2\phi_2\beta_2 + \phi_2^2\beta_1 + (2\phi_4 - 6\phi_2\phi_3 + 4\phi_2^3)\beta_0 , \qquad (11b)$$

$$\beta_4^{eff} = \beta_4 - 3\phi_2\beta_3 + (4\phi_2^2 - \phi_3)\beta_2 + (\phi_4 - 2\phi_2\phi_3)\beta_1 \qquad (11c)$$
$$\quad + (3\phi_5 - 12\phi_2\phi_4 - 5\phi_3^2 + 28\phi_2^2\phi_3 - 14\phi_2^4)\beta_0 ,$$

$$\beta_5^{eff} = \beta_5 - 4\phi_2\beta_4 + (8\phi_2^2 - 2\phi_3)\beta_3 + (4\phi_2\phi_3 - 8\phi_2^3)\beta_2 \qquad (11d)$$
$$\quad + (2\phi_5 - 8\phi_2\phi_4 + 16\phi_2^2\phi_3 - 3\phi_3^2 - 6\phi_2^4)\beta_1$$
$$\quad + (4\phi_6 - 20\phi_2\phi_5 - 16\phi_3\phi_4 + 48\phi_2\phi_3^2 - 120\phi_2^3\phi_3 + 56\phi_2^2\phi_4 + 48\phi_2^5)\beta_0 .$$

The starting point of the ECH-motivated estimating procedure of Ref.[24] is the ansatz $\beta_n^{eff} = \beta_n$, which should be applied separately at each order of PT beginning from the three-loop one. In the case of the QCD relation between different definitions of heavy quark masses it was used at the three-loop level in Refs.[23, 26] and allowed to get the estimates for the coefficients $\phi_3 = f_3^E/f_1^E$ from Eq.(11a). Further application of the relation (8) with the given in Eq.(9) numerical expressions for the typical to the Minkowski region term Δ_3 leads to good agreement of the obtained in [23, 26] approximate expression for the t_3^M-coefficient (we denote it as $t_3^{M,\,ECH}$) with the explicit three-loop result, obtained in [14, 15]. It turned out later that the estimated in Ref.[26] by the similar way values of the coefficient $t_4^{M,\,ECH}$ at $n_l = 3, 4, 5$ are also in reasonable agreement with the results of the numerical calculations, performed in [17] (see the work of Ref.[21] as well).

These facts serve a-posteriori arguments in favor of the applicability of this ECH-inspired method, supplemented with the explicit expressions for the proportional to π^2 effects of analytical continuation, at higher orders of PT as well. In Ref.[22], which is summarized in brief here, we applied the conditions $\beta_4^{eff} = \beta_4$ and $\beta_5^{eff} = \beta_5$ for Eqs.(11c) and (11c) and got the numerical estimates for the coefficients $t_5^{M,\,ECH}$ and $t_6^{M,\,ECH}$ after taking into account the analytical continuation contributions Δ_5 and Δ_6. The concrete numerical results are presented and discussed in Sec.5. Here we only note, that while getting the estimate for the coefficient t_6^M from Eqs.(11d) and (8) in addition to the theoretical ansatz $\beta_5^{eff} = \beta_5$, which of course has definite unfixed theoretical uncertainties, the obtained by the ECH-motivated method at the five-loop level estimate for f_5^E-coefficient was used. Its application at the six-loop order leads to the additional theoretical ambiguity of the estimated value of the $O(a_s^6)$-correction to the pole-running heavy quark masses relation, which is not possible to fix. However, in order to study whether there may be extra theoretical ambiguities in the results of applications of the widely spread IRR approach, recently used in [27] to analyze the uncertainties of the asymptotic QCD predictions for the coefficients of heavy quark masses relation, we

will compare in Sec.5 its outcomes with the five and six-loop estimates of the same terms, obtained by means not related to the IRR-approach ECH-inspired methods.

3 The effective-charges motivated method: direct application in the Minkowski region

Since the pole masses of heavy quarks are defined in the Minkowski region, it is also worth to consider the predictions of the coefficients in the pole-running heavy quark mass relation applying the ECH-motivated approach of [24] in the time-like region directly. This was first done in [23] (see [26] as well) constructing the Minkowskian analogs of Eqs.(10-11b) for the spectral function of Eq.(5)

$$T(s) = \overline{m}_q(s) \sum_{n=0}^{\infty} t_n^M a_s^n(s) \, . \tag{12}$$

Defining the ECH $a_s^{eff}(s)$ for the quantity $T(s)/\overline{m}_q(s)$ and the corresponding $\tilde{\beta}^{eff}(a_s^{eff})$-function with coefficients $\tilde{\beta}_n^{eff}$, fixed by the replacements $\phi_n \to \phi_n^M = t_n^M/t_1^M$ in Eqs.(10-11d), one can get the estimates for coefficients $t_n^{M, ECH\,direct}$ of pole-running heavy quark mass relation directly in the Minkowskian region after using the ansatz $\tilde{\beta}_n^{eff} = \beta_n$ at $n \geq 2$. In Ref.[26] it was noticed that the numerical expressions of the third and fourth coefficients $t_3^{M, ECH\,direct}$ and $t_4^{M, ECH\,direct}$ of the relations between pole and running charm, bottom and top-quark masses are in satisfactory agreement with the values of $t_3^{M, ECH}$ and $t_4^{M, ECH}$-terms, obtained as described above in Sec.2 (for the detailed comparison see [22]). The estimated five- and six-loop terms, obtained in the Minkowski region directly, have the following form

$$
t_5^{M, ECH\,direct} \approx \frac{1}{3\beta_0(t_1^M)^3}\Big[3t_2^M(t_1^M)^3\beta_3 + t_3^M(t_1^M)^3\beta_2 - 4(t_2^M t_1^M)^2\beta_2 \tag{13a}
$$
$$
+ 2t_3^M t_2^M(t_1^M)^2\beta_1 - t_4^M(t_1^M)^3\beta_1 + 12t_4^M t_2^M(t_1^M)^2\beta_0 + 5(t_3^M t_1^M)^2\beta_0
$$
$$
+ 14(t_2^M)^4\beta_0 - 28t_3^M(t_2^M)^2 t_1^M\beta_0\Big]
$$

$$
t_6^{M, ECH\,direct} \approx \frac{1}{12\beta_0^2(t_1^M)^4}\Big[48t_4^M t_3^M(t_1^M)^3\beta_0^2 + 72t_4^M(t_1^M t_2^M)^2\beta_0^2 + 12t_2^M(t_1^M)^4\beta_0\beta_4 \tag{13b}
$$
$$
+ 136(t_2^M)^5\beta_0^2 - 200t_3^M t_1^M(t_2^M)^3\beta_0^2 - 20t_4^M t_2^M(t_1^M)^3\beta_0\beta_1 - (t_1^M)^3(t_3^M)^2\beta_0\beta_1
$$
$$
+ 48t_3^M(t_1^M t_2^M)^2\beta_0\beta_1 - 10t_1^M(t_2^M)^4\beta_0\beta_1 - 44t_2^M(t_1^M t_3^M)^2\beta_0^2 + 6t_3^M(t_1^M)^4\beta_0\beta_3
$$
$$
+ 36(t_1^M)^3(t_2^M)^2\beta_0\beta_3 - 56(t_1^M)^2(t_2^M)^3\beta_0\beta_2 + 2t_4^M(t_1^M)^4\beta_1^2 - 4t_3^M t_2^M(t_1^M)^3\beta_1^2
$$
$$
+ 8t_3^M t_2^M(t_1^M)^3\beta_0\beta_2 - 6t_2^M(t_1^M)^4\beta_1\beta_3 - 2t_3^M(t_1^M)^4\beta_1\beta_2 + 8(t_1^M)^3(t_2^M)^2\beta_1\beta_2\Big]
$$

and like the two-, three- and four-loop analogs can be expressed as

$$t_n^{M, ECH\,direct} = f_n^E - \tilde{\Delta}_n \quad . \tag{14}$$

The made in Refs.[22, 26] observations that $t_3^{M, ECH\,direct} \approx t_3^{M, ECH}$ and $t_4^{M, ECH\,direct} \approx t_4^{M, ECH}$ mean that $\Delta_3 \approx \tilde{\Delta}_3$ and $\Delta_4 \approx \tilde{\Delta}_4$, where terms in the l.h.s. are the exactly calculable analytical continuation effects. The presented in Sec.5 comparison of five- and six-loop estimates, obtained within both approaches, demonstrates that the above mentioned approximate equality remains true at the 5 and 6-loops level as well. This demonstrates compatibility of the estimates for heavy quark mass relations coefficients, obtained by applying ECH-inspired procedure both in the Euclidean and Minkowskian regions.

4 The estimates by the IRR-based approach

The most wide-spreaded modern approach of the analysis of high-order PT QCD corrections to physical quantities is based on application of the renormalon technique (for the previous developments see e.g. Refs.[5–9], [28–30]). It is related in part to large β_0-expansion (for the application of the latter one see e.g. [7, 8, 29, 30]). As was already mentioned above the asymptotic structure of the PT QCD expression of the pole heavy quark masses through the $\overline{\text{MS}}$-scheme running ones is governed by the leading IRR contribution [4, 5], which makes the coefficients of this relation growing factorially with the increase of order of PT. Therefore it is important to analyze the region of applicability of the corresponding asymptotic PT series. For this aim we consider the IRR-based formula [6], which predicts the following factorial behavior of the coefficients t_n^M:

$$t_n^{M,\,r-n} \xrightarrow{n\to\infty} \pi N_m (2\beta_0)^{n-1} \frac{\Gamma(n+b)}{\Gamma(1+b)} \left(1 + \sum_{k=1}^{3} \frac{s_k}{(n+b-1)\dots(n+b-k)} + O(n^{-4})\right), \quad (15)$$

where $\Gamma(x)$ is the Euler Gamma-function, $b = \beta_1/(2\beta_0^2)$ and the values of the sub-leading coefficients s_k can be found in [27, 31]. Note that our notations and normalizations differ from those introduced in Refs.[5, 6, 27]. In the presented below discussions the coefficients of the RG β-function depend on $(n_l - 1)$ numbers of flavors.

The normalization factor N_m in Eq.(15) is the function of n_l and of the order n of PT. Unfortunately, its explicit form is not known. Moreover, the way of fixation of N_m-values is different in various works on the subject (see e.g. [32–34] and the detailed work [31]). This fact introduces the important uncertainty in the IRR-based analysis. In our analysis we use the given in Table 1 numerical results for n_l-dependence of N_m, obtained in the process of four-loop analysis of Ref.[27][1]:

Table 1. The n_l dependence of N_m at the the fourth order of PT.

n_l	3	4	5	6	7	8
N_m	0.54	0.51	0.46	0.39	0.28	0.06

In the next section we will study whether the application of this n_l-dependent value of N_m is allowing to get IRR-based estimates, which agree with the five- and six-loop coefficients t_5^M and t_6^M, evaluated within both Euclidean and Minkowskian ECH-motivated approaches, and respect the following from the large β_0-expansion sign-alternating behavior of their representation through powers of n_l. In order to find the answer to this problem we should estimate the expressions for t_5^M and t_6^M-coefficients not only for the physical numbers of light flavors $n_l = 3, 4, 5$, which corresponds to the cases of consideration of the charm, bottom at top-quark masses, but for unphysical values of "light" flavors $6 \le n_l \le 8$ as well.

5 Numerical results and their interpretation

We now summarize theoretical discussions of Sec.2-Sec.4 by comparing the estimated expressions for the five and six-loop coefficients in the the relation between pole and $\overline{\text{MS}}$-scheme running heavy quark masses, obtained by the defined in the Euclidean and Minkowskian regions ECH-motivated methods and by the IRR-based asymptotic formula

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[1]Note that in [22] while applying the asymptotic expression of Eq.(15) for estimates of the five and six-loop corrections to the $\overline{\text{MS}}$-on-shell mass relation for charm, bottom and top-quarks the same approximate average value $N_m \approx 0.5$ was fixed, which in fact is too far from the result, obtained in Refs. [31, 32].

of Eq.(15), which is supplemented with the n_l-dependent value of the normalization factor N_m. The concrete n_l-dependent results for the numerical estimates of t_5^M and t_6^M-contributions to (3), obtained with the help of the discussed above three methods, are given in Table 2.

Table 2. The estimates of t_5^M and t_6^M-contributions by three considered methods.

n_l	$t_5^{M,\,ECH}$	$t_5^{M,\,ECHdirect}$	$t_5^{M,\,r-n}$	$t_6^{M,\,ECH}$	$t_6^{M,\,ECHdirect}$	$t_6^{M,\,r-n}$
3	28435	26871	34048	476522	437146	829993
4	17255	17499	22781	238025	255692	511245
5	9122	10427	13882	90739	133960	283902
6	3490	5320	7466	8412	57920	137256
7	-127	1871	3119	-29701	15798	50520
8	-2153	-196	344	-39432	-2184	4747

One can first observe that for the physical values n_l=3, 4, 5 the estimates for the coefficients t_5^M and t_6^M, obtained by two different realizations of the ECH-based technique, are in reasonable agreement [2]. However, *it is surprising* that the IRR-based approach with taken from [27] *four-loop values of N_m* does not reproduce the *supported by large β_0-approximation sign-alternating n_l-dependence structure [8]* of the corresponding estimated expression for t_5^M and t_6^M-coefficients, obtained within both realizations of the ECH-approach. Indeed, combining the values for $t_5^M(n_l)$ as given in three first columns of Table 2 with the representation of Eq.(4), we obtain the following expressions:

$$t_5^{M,\,ECH} = 2.5n_l^4 - 136n_l^3 + 2912n_l^2 - 26976n_l + 86620 \,, \tag{16a}$$

$$t_5^{M,\,ECHdirect} = 1.2n_l^4 - 77n_l^3 + 1959n_l^2 - 20445n_l + 72557 \,, \tag{16b}$$

$$t_5^{M,\,r-n} = -22n_l^4 + 416n_l^3 - 1669n_l^2 - 11116n_l + 72972 \,. \tag{16c}$$

The similar *surprising feature* is observed at the six-loop level, namely:

$$t_6^{M,\,ECH} = -4.9n_l^5 + 352n_l^4 - 9708n_l^3 + 131176n_l^2 - 855342n_l + 2096737 \,, \tag{17a}$$

$$t_6^{M,\,ECHdirect} = -2.2n_l^5 + 148n_l^4 - 4561n_l^3 + 71653n_l^2 - 538498n_l + 1519440 \,, \tag{17b}$$

$$t_6^{M,\,r-n} = 99n_l^5 - 2903n_l^4 + 30109n_l^3 - 99563n_l^2 - 305378n_l + 2040263 \,. \tag{17c}$$

This paradox of application of the IRR asymptotic formula of Eq.(15) is not clear to us.

Let us now consider the asymptotic structure of the relation between pole and running masses of real heavy quarks. One can see from Table 2 that for $n_l = 3, 4, 5$ three methods of estimates of high-order corrections give comparable values for the five-loop coefficients, while the six-loop large coefficients, estimated by the ECH-motivated approaches, are lower than the existing asymptotic IRR-renormalon predictions by the factor 2 only. Since for rather approximate estimate we do not consider this difference seriously, we will use these estimated by three methods numbers as inputs of our numerical studies. We fix the values of the the running masses of c, b and t-quarks following the presented in [22] considerations as $\overline{m}_c(\overline{m}_c^2) = 1.275$ GeV, $\overline{m}_b(\overline{m}_b^2) = 4.180$ GeV, $\overline{m}_t(\overline{m}_t^2) = 164.3$ GeV and take the following values of the $\overline{\text{MS}}$-scheme strong coupling constant, normalized at these running masses, viz $\alpha_s(\overline{m}_c^2) = 0.3947$, $\alpha_s(\overline{m}_b^2) = 0.2256$, $\alpha_s(\overline{m}_t^2) = 0.1085$. Taking into account the known results of direct diagram calculations [10–15, 21] and using the data, presented in Table 2, we find

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[2]It is worth emphasizing that our ECH-inspired results for t_5^M-coefficient at $n_l = 4$ agree rather well with the approximate value of this term, independently obtained in Ref.[35] as one of the outcomes of the global fits of characteristics of the bottomonium spectrum studied in non-relativistic QCD up to N^{3}LO.

that within the both ECH-motivated methods and the IRR-based approach the asymptotic PT expressions for the pole masses of charm, bottom and top-quarks has the following form:

$$\frac{M_c}{1\,\text{GeV}} \approx 1.275 + 0.214 + 0.208 + 0.295 + 0.541 \tag{18a}$$

$$+ \left\{ \underbrace{1.135 + 2.389}_{\text{ECH}};\ \underbrace{1.072 + 2.192}_{\text{ECH direct}};\ \underbrace{1.359 + 4.162}_{\text{IRR},\ N_m = 0.54} \right\},$$

$$\frac{M_b}{1\,\text{GeV}} \approx 4.180 + 0.400 + 0.200 + 0.146 + 0.137 \tag{18b}$$

$$+ \left\{ \underbrace{0.137 + 0.137}_{\text{ECH}};\ \underbrace{0.140 + 0.147}_{\text{ECH direct}};\ \underbrace{0.182 + 0.293}_{\text{IRR},\ N_m = 0.51} \right\},$$

$$\frac{M_t}{1\,\text{GeV}} \approx 164.300 + 7.566 + 1.614 + 0.498 + 0.196 \tag{18c}$$

$$+ \left\{ \underbrace{0.074 + 0.025}_{\text{ECH}};\ \underbrace{0.084 + 0.037}_{\text{ECH direct}};\ \underbrace{0.112 + 0.079}_{\text{IRR},\ N_m = 0.46} \right\}.$$

Based on these results we conclude that five-loop corrections to pole mass of charm-quark are rather close in all three considered estimate methods. However, the six-loop corrections, predicted with help of the IRR technique, differ significantly from the ones, obtained by the both ECH procedures. With reference to the b-quark pole mass the situation is more interesting. Indeed, the ECH-motivated method that takes into account the transition from the Euclidean to Minkowskian regions demonstrates output to some kind of plateau (four, five and six-loop corrections coincide), whereas the direct ECH and the IRR approaches indicate the growth of these corrections. These facts testify to the unconditional manifestation of the asymptoticity of the corresponding PT series for bottom-quark starting with five-loop order. For case of t-quark all three considered estimate procedures outline the decrease of the five and six-loop corrections. This means that the asymptotic structure of this PT series is not yet manifesting itself at these levels. Therefore the conception of pole mass of top-quark can be safely used even at the $O(a_s^6)$ level.

6 Conclusion

We apply three approximate methods for estimation of the five and six-loop corrections to the $\overline{\text{MS}}$-on-shell heavy quark mass relation, namely two ECH-motivated methods, defined in the Euclidean and Minkowskian regions correspondingly, and the infrared renormalon based approach. By means of these methods we determine flavor dependence of the considered contributions in the $O(a_s^5)$ and $O(a_s^6)$ orders. Wherein the IRR-based technique with normalization N_m-factor, taken in the four-loop approximation, does not give questionable n_l-dependent results while the both ECH approaches predict not only close values of the corresponding coefficients but reproduce the sign-alternating structure of these corrections in expansion in powers of massless flavors. The numerical studies of all estimate procedures indicate the growth of the five and six-loop corrections to the pole mass of charm-quark. Whereas the ECH Euclidean method for b-quark pole mass leads to effect of plateau and the rest two methods outline also the increase of these corrections. In the case of t-quark the asymptotic nature of the corresponding PT series is not observed even at six-loop level. Therefore the concept of the pole mass of top-quark is applicable up to 6 order of PT for sure.

7 Acknowledgments

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