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Holographic RG flow at zero and finite temperatures

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Abstract. We consider a 5d holographic model with a dilaton potential representing a sum of exponential functions. We construct Poincaré invatiant and black brane solutions with AdS and non-AdS boundaries. Under the holographic duality these solutions can be interpreted as RG flows. We discuss the dependence of the running coupling on the energy through the constructed solutions.

1 Introduction

Holographic duality allows to perform various calculations for gauge theories at strong coupling where a perturbative approach is not applicable. The renormalization group flow in this approach has a geometric description in terms of Poincaré invariant gravity solutions[1, 2]. The scale factor of the solutions measures the energy scale and the dilaton that supports these backgrounds is non-trivial and corresponds to the running coupling. The confromal symmetry is restored only at boundaries, where dilaton is supposed to have a constant value, corresponding to fixed points of a dual gauge theory at zero temperature. The dynamics of the dilaton depends on its potential, which can be chosen in terms of sum exponential functions motivated by a form of p-brane actions in gauged supergravities.

Holographic RG geometries can be singular, but the only allowed curvature singularities are those which can be obtained as limits of regular black holes. A singularity that can be hidden by the horizon is a signal of non-trivial IR physics in the dual theory. So, if we get the "good" singularity in the limit of the regular black hole, the dual physics is that the IR cut off is removed [3].

In this letter we discuss a 5d holographic gravity model with a dilaton potential given as a sum of two exponential functions. The case of a single exponential term was discussed in [4]. The single exponential case has some limitations: the potential is monotonous, so it does not have a minimum; the RG flow in this case does not start from a UV fixed point, and in fact the dual theory is not well-behaved in the UV. The general analysis of RG flow with a single scalar, performed in [5].

The dilaton potential with two exponential functions has a minimum with $V_{min} < 0$. In the present work we discuss exact holographic RG flows for this model constructed in [6]. Geometries of these flows have AdS and non-AdS UV fixed points. Zero-temperature flows have curvature singularities and it will be shown that these singularities can be hidden by

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the black hole horizon. This paper is organized as follows. In Section 2 we present the holographic model, EOM, the general solutions to it and construct black holes. In Section 3 we explore the dynamics of the solutions as holographic RG flows and the behaviour of the running coupling. In Section 4 we give conclusions.

2 The holographic model and solutions

2.1 The setup

Our starting point is the action of the following form

$$\mathcal{S} = \frac{1}{16\pi G_5}\int d^5x\,\sqrt{|g|}\left(R - \frac{4}{3}\,(\partial\phi)^2 - V(\phi)\right) + G.H., \tag{1}$$

with the dilaton potential

$$V(\phi) = C_1 e^{2k_1\phi} + C_2 e^{2k_2\phi}, \tag{2}$$

where, k_i with $i = 1, 2$ are dilaton couplings. We choose the constants $C_1 < 0$ and $C_2 > 0$. In this case the potential has a minimum and regions of positive and negative sign. We suppose that the metric and the dilaton depend only on the u-coordinate.

The equations of motion which follow from the action (1) read

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{4}{3}\left(\partial_M\phi\partial_N\phi - \frac{1}{2}g_{MN}\partial_k\phi\partial^k\phi\right) - \frac{1}{2}g_{MN}V(\phi), \tag{3}$$

$$\Box\phi = \frac{3}{8}\frac{\partial V}{\partial\phi}. \tag{4}$$

Following the approach from [7] the generic form of the metric and the dilaton that solves EOM (3)-(4) were found in [6]

$$ds^2 = F_1^{\frac{8}{9k^2-16}}F_2^{\frac{9k^2}{2(16-9k^2)}}\left(-e^{2\alpha^1 u}dt^2 + e^{-\frac{2\alpha^1}{3}u}d\vec{y}^{\,2}\right) + F_1^{\frac{32}{9k^2-16}}F_2^{\frac{18k^2}{16-9k^2}}du^2, \tag{5}$$

$$\phi = -\frac{9k}{9k^2-16}\ln F_1 + \frac{9k}{9k^2-16}\ln F_2, \tag{6}$$

with F_1 and F_2 given by

$$F_1 = \sqrt{\left|\frac{C_1}{2E_1}\right|}\sinh(\mu_1\,(u-u_{01})),\ \mu_1 = \sqrt{\left|\frac{3E_1}{2}\left(k^2 - \frac{16}{9}\right)\right|}, \tag{7}$$

$$F_2 = \sqrt{\left|\frac{C_2}{2E_2}\right|}\sinh(\mu_2\,u),\ \mu_2 = \sqrt{\left|\frac{3E_2}{2}\left(\left(\frac{16}{9}\right)^2\frac{1}{k^2} - \frac{16}{9}\right)\right|}, \tag{8}$$

where $0 < k < 4/3$ and u is positive and $u > u_{01}$. Moreover, one has the constraint

$$E_1 + E_2 + \frac{2(\alpha^1)^2}{3} = 0, \quad E_1 < 0, \quad E_2 > 0. \tag{9}$$

From the form of the functions F_1 and F_2 given by (7)-(8) we can see that the solution has a pole at u_{01} and the degenerate case is $u_{01} = 0$.

2.2 Vacuum solutions and its boundaries

Solutions with $\alpha^1 = 0$ and have Poincaré invariance and correspond to zero-temperature case. Due to $|E_1| = |E_2|$ for these solutions the following relation always holds

$$\frac{\mu_2}{\mu_1} = \frac{4}{3k} > 1\,. \tag{10}$$

Let us see the asymptotics of the dilaton and the metric (5)-(6) at boundaries.

- at $u \to u_{01} + \epsilon$ the asymptotics are

$$ds^2 \sim z^{\frac{8}{9k^2-4}}\left(-dt^2 + dy_1^2 + dy_2^2 + dy_3^2 + dz^2\right), \tag{11}$$

$$\phi \sim \frac{9k}{4-9k^2}\log z \to -\infty, \tag{12}$$

where the conformal coordinate is $z \sim \frac{16-9k^2}{9k^2-4}(u-u_{01})^{\frac{4-9k^2}{16-9k^2}}$.
The scalar curvature has the following form

$$\begin{aligned} R &= \frac{(C_1/2E_1)^{\frac{16}{16-9k^2}}(\sqrt{C_2/2E_2}\sinh(\mu_2 u_{01}))^{\frac{-18k^2}{16-9k^2}}}{4(16-9k^2)^2}\Big(8(16-9k^2)(16\mu_1^2-9k^2\mu_2^2) \\ &+ 128(9k^2-10)(u-u_{01}))^{-2} - 864k^2\mu_2(u-u_{01})^{-1}\coth(\mu_2 u_{01}) \\ &+ 9k^2(128-45k^2)\mu_2^2\coth(\mu_2 u_{01})^2\Big)(\mu_1(u-u_{01}))^{\frac{32}{16-9k^2}}, \end{aligned} \tag{13}$$

or in the conformal coordinate

$$\begin{aligned} R &= \frac{(C_1/2E_1)^{\frac{16}{16-9k^2}}(\sqrt{C_2/2E_2}\sinh(\mu_2 u_{01}))^{\frac{-18k^2}{16-9k^2}}}{4(16-9k^2)^2}\Big(8(16-9k^2)(16\mu_1^2-9k^2\mu_2^2) \\ &+ 128(9k^2-10)(\frac{9k^2-4}{16-9k^2}z)^{\frac{2(16-9k^2)}{9k^2-4}} - 864k^2\mu_2\left(\frac{9k^2-4}{16-9k^2}z\right)^{\frac{16-9k^2}{9k^2-4}}\coth(\mu_2 u_{01}) \\ &+ 9k^2(128-45k^2)\mu_2^2\coth(\mu_2 u_{01})^2\Big)\left(\mu_1\frac{9k^2-4}{16-9k^2}z\right)^{\frac{32}{4-9k^2}}, \end{aligned} \tag{14}$$

that is regular.
- at $u \to +\infty$:

$$ds^2 \sim z^{2/3}\left(-dt^2 + dy_1^2 + dy_2^2 + dy_3^2 + dz^2\right), \tag{15}$$

$$\phi \sim \log z \to -\infty, \tag{16}$$

where z is defined by $z \sim -\frac{4+3k}{3\mu_1}e^{-\frac{3\mu_1 u}{4+3k}}$.

$$R = \left(\frac{C_1}{2E_1}\right)^{\frac{16}{16-9k^2}}\left(\frac{C_2}{2E_2}\right)^{-\frac{9k^2}{16-9k^2}}\frac{3(16\mu_1-9k^2\mu_2)^2}{4(16-9k^2)^2}e^{\frac{2(16\mu_1-9k^2\mu_2)}{16-9k^2}u}. \tag{17}$$

The scalar curvature (17) has a singularity due to $(16\mu_1 - 9k^2\mu_2) > 0$. In the conformal coordinates is

$$R = \left(\frac{C_1}{2E_1}\right)^{\frac{16}{16-9k^2}}\left(\frac{C_2}{2E_2}\right)^{-\frac{9k^2}{16-9k^2}}\frac{3(16\mu_1-9k^2\mu_2)^2}{4(16-9k^2)^2}\left(\frac{3\mu_1}{4+3k}z\right)^{-\frac{8}{3}}. \tag{18}$$

It is interesting to see asymptotics of the particular case with $u_{01} = 0$.

• In the limit with $u \to \pm\infty$ the asymptotics of the dilaton and the metric are the same as in (15)-(18).

• For $u \to 0$ one gets has the following form of the metric (5)

$$ds^2 \sim \frac{1}{z^2}(-dt^2 + dy_1^2 + dy_2^2 + dy_3^2 + dz^2), \tag{19}$$

where the conformal "radial" coordinate is defined as $z = 4u^{1/4}$ and $z \to 0$ as $u \to 0$. In (19) one can easily recognize the 5d AdS metric supported by the constant dilaton

$$\phi|_{u\to 0} = \frac{9k}{(16-9k^2)}\log\frac{3k}{4} + \frac{9k}{2(16-9k^2)}\log\left|\frac{C_1}{C_2}\right|, \tag{20}$$

which coincides with the minimum of the potential. As expected the scalar curvature of this solution with $u \to 0$ has a constant value

$$R = -\frac{5}{4}\left(\sqrt{\frac{C_1}{2E_1}}\mu_1\right)^{\frac{32}{16-9k^2}}\left(\sqrt{\frac{C_2}{2E_2}}\mu_2\right)^{\frac{18k^2}{9k^2-16}}. \tag{21}$$

2.3 The black brane solutions

From the previous section we see that the Poincaré invariant solution are singular in the IR limit and we aim to construct a black brane representation of the solution (5)-(6) here[1]. The metric can be rewritten in the following form

$$ds^2 = C\,\mathcal{X}(u)e^{(\kappa-\frac{2}{3}\alpha^1)(u-u_{01})}\left(-e^{\frac{8}{3}\alpha^1(u-u_{01})}dt^2 + d\vec{y}^2 + \mathcal{X}(u)^3C^3e^{(3\kappa+\frac{2}{3}\alpha^1)(u-u_{01})}du^2\right), \tag{22}$$

with

$$C \equiv \left(\frac{1}{2}\sqrt{\left|\frac{C_1}{2E_1}\right|}\right)^{\frac{8}{9k^2-16}}\left(\frac{1}{2}\sqrt{\left|\frac{C_2}{2E_2}\right|}e^{\mu_2 u_{01}}\right)^{\frac{9k^2}{2(16-9k^2)}}, \tag{23}$$

the function $\mathcal{X}(u)$ reads

$$\mathcal{X}(u) = (1-e^{-2\mu_1(u-u_{01})})^{-\frac{8}{16-9k^2}}(1-e^{-2\mu_2 u})^{\frac{9k^2}{2(16-9k^2)}} \tag{24}$$

and the exponent κ is

$$\kappa \equiv \frac{8}{\sqrt{6\,(16-9k^2)}}\left(-\sqrt{E_2 + \frac{2}{3}(\alpha^1)^2} + \frac{3}{4}k\sqrt{E_2}\right), \tag{25}$$

where we took into account the relations for μ_1 and μ_2, so for $0 < k < 4/3$ one has $\kappa < 0$.

Therefore, there is no conic singularity if the following constraints are satisfied

$$(i) \qquad \kappa - \frac{2}{3}\alpha^1 = 0, \tag{26}$$

$$(ii) \qquad \frac{4}{3C^{3/2}}\alpha^1\beta = 2\pi. \tag{27}$$

Plugging (25) in (26) we come to the condition to the parameters

$$E_2 = \frac{6k^2(\alpha^1)^2}{16-9k^2}, \tag{28}$$

[1] We note that the case with a single exponential function was solved in [8].

that corresponds to $\mu_1 = \mu_2$. Under the condition (26) the black brane metric has the form

$$ds^2 = C\mathcal{X}\Big(-e^{\frac{8}{3}\alpha^1(u-u_{01})}dt^2 + d\vec{y}^2\Big) + C^4\mathcal{X}(u)^4 e^{\frac{8}{3}\alpha^1(u-u_{01})}du^2, \tag{29}$$

where C and $\mathcal{X}(u)$ are given by

$$\mathcal{X} = (1 - e^{-2\mu(u-u_{01})})^{-\frac{8}{16-9k^2}}(1 - e^{-2\mu u})^{\frac{9k^2}{2(16-9k^2)}}, \tag{30}$$
$$C = \Big(\frac{1}{2}\sqrt{\Big|\frac{C_1}{2E_1}\Big|}\Big)^{\frac{8}{9k^2-16}}\Big(\frac{1}{2}\sqrt{\Big|\frac{C_2}{2E_2}\Big|}e^{\mu u_{01}}\Big)^{\frac{9k^2}{2(16-9k^2)}}.$$

Null geodesics imply

$$t - t_0 \sim \int_{u_0}^{u} d\bar{u}e^{(\frac{3}{2}\kappa-\alpha^1)\bar{u}}C^{3/2}\Big(1 + \frac{3\big(16e^{2\mu u_{01}} - 9k^2\big)}{4(16-9k^2)}e^{-2\mu\bar{u}}\Big) \underset{u\to\infty}{\to} \infty. \tag{31}$$

This calculation confirms that we have the horizon at $u = +\infty$ and the Hawking temperature

$$T = \frac{2}{3\pi}\frac{\alpha^1}{C^{3/2}}. \tag{32}$$

The scalar curvature and the Kretschmann scalar with $u \to +\infty$ are

$$R = \left(\frac{C_1}{2E_1}\right)^{\frac{16}{16-9k^2}}\left(\frac{C_2}{2E_2}\right)^{-\frac{9k^2}{16-9k^2}}\left(\frac{3(16\mu_1 - 9k^2\mu_2)^2}{4(16-9k^2)^2} - \frac{4}{3}\left(\alpha^1\right)^2\right)e^{\frac{2(16\mu_1-9k^2\mu_2)}{16-9k^2}u}, \tag{33}$$

$$K = \frac{\left(4\alpha^1(9k^2-16) + 27k^2\mu_2 - 48\mu_1\right)^2}{864(16-9k^2)^4}\left(\frac{C_1}{2E_1}\right)^{\frac{32}{16-9k^2}}\left(\frac{C_2}{2E_2}\right)^{\frac{18k^2}{9k^2-16}}e^{\frac{4(16\mu_1-9k^2\mu_2)}{16-9k^2}u}$$
$$\cdot\Big(304(\alpha^1)^2(16-9k^2)^2 + 168\alpha^1(9k^2-16)(16\mu_1 - 9k^2\mu_2) + 63(16\mu_1 - 9k^2\mu_2)^2\Big). \tag{34}$$

We note that with respect to the constraint to absence of the conic singularity (26) both the scalar curvature (33) and Kretschmann scalar (34) tend to zero with $u \to +\infty$, i.e. we cancel the curvature singularity in the IR region.

The dilaton supporting the geometry (29) reads

$$\phi = \frac{9k}{9k^2-16}\log\left[\sqrt{\left|\frac{E_1C_2}{E_2C_1}\right|}\frac{\sinh(\mu u)}{\sinh(\mu(u-u_{01}))}\right]. \tag{35}$$

and takes the constant value near horizon

$$\lim\phi_{u\to+\infty} = \frac{9k}{2(16-9k^2)}\log\left(\left|\frac{E_2C_1}{E_1C_2}\right|\right). \tag{36}$$

Now one can check if the Gubser's bound [3] for asymptotically non-AdS solutions holds

$$V(\phi(u_h)) < 0, \tag{37}$$

where in our case $u_h = u_\infty$. Plugging the solutions for the dilaton at the horizon (36) in (2) the inequality (37) takes the form

$$\frac{E_2}{|E_1|} - 1 < 0, \tag{38}$$

that is valid for our solution due to the constraint (9).

Now we turn to the special case of the non-vacuum solutions with $u_{01} = 0$ with $u > 0$. The metric (29) takes the form

$$ds^2 = C\left(1 - e^{-2\mu u}\right)^{-\frac{1}{2}}\left(-e^{-2\mu u}dt^2 + d\vec{y}^2\right) + C^4\left(1 - e^{-2\mu u}\right)^{-2}e^{-2\mu u}du^2, \tag{39}$$

where we took into account $\mu = -\frac{4}{3}\alpha^1$ and $C = (2\sqrt{2})^{1/2}\left(\frac{C_1}{E_1}\right)^{\frac{4}{9k^2-16}}\left(\frac{C_2}{E_2}\right)^{\frac{9k^2}{4(16-9k^2)}}$.

Due to the constraint $\mu_1 = \mu_2$ and $u_{01} = 0$ the dilaton becomes constant

$$\phi = \frac{9k}{2(16-9k^2)}\log\left|\frac{C_1E_2}{C_2E_1}\right|. \tag{40}$$

The curvature of the metric (39) is negative $R = -\frac{5\mu^2}{C^4}$. Doing the change of coordinates $z = z_h\left(1-e^{-2\mu u}\right)^{\frac{1}{4}}$, $C = z_h^{-2}$, one gets the usual from for the 5d AdS black brane

$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + d\vec{y}^2 + \frac{dz^2}{f(z)}\right), \quad f = 1-\left(\frac{z}{z_h}\right)^4. \tag{41}$$

For the dilaton potential we have the saturation of the Gubser's bound

$$V(\phi(u_h)) = V_{UV}, \tag{42}$$

that is in agreement with the suggestion from [3].

3 Holographic RG flow

3.1 RG flow at zero temperature and at finite temperature

To study the evolution of the solutions as holographic RG flows It is useful to come to so-called domain wall coordinates, which in the general form are

$$ds^2 = \frac{dw^2}{f(w)} + e^{2\mathcal{A}(w)}\left(-f(w)dt^2 + \eta_{ij}dx^idx^j\right), \tag{43}$$

which covers the vacuum case with $f(w) = 1$. Both temperature and vacuum solutions are characterized by the scale factor $e^{\mathcal{A}(w)}$, that measures the field theory energy scale, the blackening function $f(w)$ and by a scalar field profile $\phi(w)$, $\lambda = e^{\phi}$, which is interpreted as the running coupling. One can define the β-function holographically as

$$\beta(\lambda) = \frac{d\lambda}{d\log E} = \frac{d\lambda}{d\mathcal{A}}. \tag{44}$$

In general case the β-function satisfies the following RG equations [5]

$$\frac{dX}{d\phi} = -\frac{4}{3}\left(1-X^2+Y\right)\left(1+\frac{3}{8X}\frac{d\log V}{d\phi}\right), \tag{45}$$

$$\frac{dY}{d\phi} = -\frac{4}{3}\left(1-X^2+Y\right)\frac{Y}{X}, \tag{46}$$

where $X(\phi)$ is related with the β-function

$$X(\phi) = \frac{\beta(\lambda)}{3\lambda}. \tag{47}$$

In eqs. (45)-(46) the Y-variable is defined through the function f

$$Y(\phi) = \frac{1}{4}\frac{g'}{\mathcal{A}'}, \quad g = \log f, \tag{48}$$

representing the relation with the finite temperature.

It is possible to study eqs.(45)-(46) as a dynamical system merely using a relation for the dilaton potential. In particular, one can use the notion of the superpotential and its appropriate behaviour, see [5, 9]. At the same, knowing exact solutions for the dilaton and the scale factor, we can directly explore the behaviour of the functions (47)-(48).

Here we examine the evolution of the rescaled β-function X (47) as a function of *log* of the running coupling using exact solutions, while in [6] it is also done for eqs.(45)-(46). In Pic. 1 we present the behaviour of X using solutions for ϕ and $\mathcal{A}$ with $\alpha^1 = 0$. From Pic. 1 **A)** we see that the holographic β-functions at zero temperature constructed on the solutions $u > u_{01}$ can take both negative and positive values. In Fig. 1 **B)** we show the function $X(\phi)$

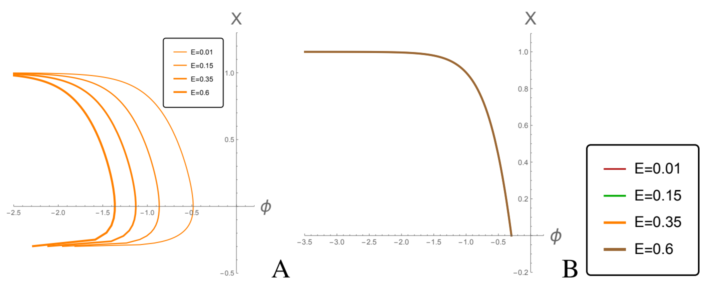

Figure 1. The X-function on the dilaton plotted on the solutions: **A)** $u > u_{01}$ $u_{01} = 1$, **B)** $u_{01} = 0$. For all plots $k = 0.4$, $C_1 = -2$, $C_2 = 2$, $|E_1| = |E_2|$, labeled as E on the legends.

on the dilation with $u_{01} = 0$. We see that the behaviour of $X(\phi)$ is the same for all E_2.

Now we turn to exploring the dependence of X on ϕ for the finite temperature case. In figures (2) we present the behaviour of X and Y for the black hole solutions with the metric (29) and the dilaton (35) with $u_{01} \neq 0$. The plots are drawn for different values of the parameter α^1 that defines the temperature (32). At the same time this parameter restricts the values of E_2 by (28) to omit the singularity. As a result, we observe in (2) **A)** that the β-function takes only negative value unlike to the zero-temperature case. In Fig. 2 **B)** we present the behavior of the function Y (48) on ϕ.

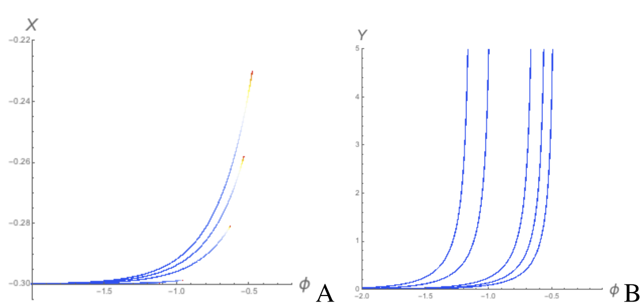

Figure 2. **A)** The X-function on ϕ plotted on the finite temperature solutions, **B)** The Y-function on ϕ plotted on the finite temperature solutions. For all plots $k = 0.4$, $C_1 = -2$, $C_2 = 2$, $\alpha = -0.5, -0.7, -1, -2, -2.5$.

3.2 The running coupling

It is useful and informative to see the behaviour of the running coupling $\lambda = e^{\phi}$ with the dependence on the energy scale A=$e^{\mathcal{A}}$. In Fig. 3 we show the dependence of λ on the energy scale A plotted on solutions (5)-(6) for zero and non-zero T varying α^1. For the solutions that

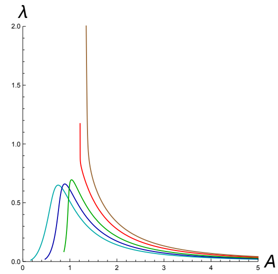

Figure 3. The dependence of λ on A= $e^{\mathcal{A}}$, $u_{01} = 1$ $\alpha^1 = 0$ (cyan), $\alpha^1 = -0.25$ (blue), $\alpha^1 = -0.5$ (green), $\alpha^1 = -0.8$ (red), $\alpha^1 = -1$ (brown).

do not obey the constraint (28) we observe a non-trivial behaviour of the running coupling in the IR region. But this dynamics can be changed by the appropriate values of parameters. So one can observe the correct behaviour of the running coupling $\lambda \to +\infty$ with A$\to 0$ if the constraint (28) is satisfy and we work with the regular black hole solutions. When the energy A is sent to big values the running coupling λ goes to 0 for all chosen parameters providing the UV freedom. Or in others words, we can mimic the QCD RG flow if the solutions have regular black hole generalization.

4 Conclusions

In this work we have considered the exact holographic RG flow for the dilaton potential with two exponential functions. The form of the potential allows for a rich variety of solutions. In particular, we have found a family of solutions depending on the dilaton coupling which interpolate between an AdS boundary in the UV limit given by and the hyperscaling violating boundary in the IR limit. The Poincaré invariant solutions can describe the RG flow at zero temperature, but have a singularity in the IR limit. However, we have shown that one can construct a black hole solution that satisfies the Gubser's bound and hides the singularity by the horizon. The dependence of the running coupling on the energy scale studied using the regular black hole solutions mimics the behaviour of the running coupling in QCD.

The next step is to understand better these solutions from the point of view of the dual field theory, and to investigate applications of the solutions to the physics of QCD-like theories. Particularly, it is of interest to see the picture from a gauged supergravity side.

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