New deformations of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric mechanics

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Abstract. This is a review of two different types of the deformed $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric mechanics. The first type is associated with the world-line realizations of the supergroups $SU(2|1)$ (four supercharges), as well as of $SU(2|2)$ and $SU(4|1)$ (eight supercharges). The second type is the quaternion-Kähler (QK) deformation of the hyper-Kähler (HK) $\mathcal{N} = 4$ mechanics models. The basic distinguishing feature of the QK models is a local $\mathcal{N} = 4$ supersymmetry realized in $d = 1$ harmonic superspace.

1 Introduction

Supersymmetric Quantum Mechanics (SQM) [1] is the $d = 1$ supersymmetric theory. It:

- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;

- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc.

An extended $\mathcal{N} > 2, d = 1$ supersymmetry is specific: it reveals dualities between various supermultiplets,, nonlinear "cousins" of off-shell linear multipliets, etc (see, e.g., [2] and refs. therein). $\mathcal{N} = 4$ SQM, with $\{Q_\alpha, \bar{Q}^\beta\} = 2\delta^\beta_\alpha H\,, \alpha = 1, 2,$, is of special interest. In particular, a subclass of $\mathcal{N} = 4$ SQM models have as their bosonic target, Hyper-Kähler (HK) manifolds.

In this Talk, two different types of deformations of $\mathcal{N} = 4$ SQM models will be outlined.

2 From deformed $\mathcal{N} = 4$ SQM to its $\mathcal{N} = 8$ extensions

The first type of deformed SQM arises while choosing some (semi)simple supergroups instead of higher-rank $d = 1$ super-Poincaré:

A. Standard extension: $(\mathcal{N} = 2\,,\ d = 1) \quad \Rightarrow \quad (\mathcal{N} > 2\,,\ d = 1 \text{ Poincaré})$,
B. Non-standard extension: $(\mathcal{N} = 2\,,\ d = 1) \equiv u(1|1) \quad \Rightarrow \quad su(2|1) \subset su(2|2) \subset \ldots$.

In the case **B**, the closure of supercharges contains, besides H, also internal symmetry generators. The deformed $\mathcal{N} = 4$ SQM is associated with the superalgebra $su(2|1)$:

$$\begin{aligned}
&\{Q^i, \bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,\\
&\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&\left[F, \bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.
\end{aligned} \tag{1}$$

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The parameter m is a deformation parameter: when $m \to 0$, the standard $\mathcal{N} = 4, d = 1$ super-Poincaré is restored.

The simplest models with a worldline realization of $su(2|1)$ were considered in [3], [4], [5]. The systematic superfield approach to $su(2|1)$ supersymmetry was worked out in [6], [7], [8], [9]. The models built on the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ were studied at the classical and quantum levels. Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized [10], [11], [12]. The common features of all these models are:

- The oscillator-type Lagrangians for the bosonic fields, with m^2 as the oscillator strength.
- The appearance of the Wess-Zumino terms for the bosonic fields, of the type $\sim im(\dot{z}\bar{z} - z\dot{\bar{z}})$.
- At the lowest energy levels, wave functions form *atypical* $su(2|1)$ multiplets, with unequal numbers of the bosonic and fermionic states.

As the next step, analogous deformations of $\mathcal{N} = 8$ SQM were studied. The flat $\mathcal{N} = 8$ superalgebra,

$$\{Q_A, Q_B\} = 2\delta_{AB} H\,, \quad A, B = 1, \ldots, 8\,,$$

admits two deformations with the minimal number of extra bosonic generators.

A. Superalgebra $su(2|2)$ [13]:

$$\begin{aligned} &\left\{Q^{ia}, S^{jb}\right\} = 2im\left(\varepsilon^{ab} L^{ij} - \varepsilon^{ij} R^{ab}\right) + 2\,\varepsilon^{ab}\varepsilon^{ij} C, \\ &\left\{Q^{ia}, Q^{jb}\right\} = 2\,\varepsilon^{ij}\varepsilon^{ab}(H + C_1), \qquad \left\{S^{ia}, S^{jb}\right\} = 2\,\varepsilon^{ij}\varepsilon^{ab}(H - C_1). \end{aligned} \tag{2}$$

In the limit $m \to 0$ a centrally extended flat $\mathcal{N} = 8$ superalgebra is reproduced, with two extra central charges C and C_1 and $SO(8)$ automorphisms broken to $SU(2) \times SU(2)$.

B. Superalgebra $su(4|1)$ [14]:

$$\begin{aligned} &\left\{Q^I, \bar{Q}_J\right\} = 2m\, L^I_J + 2\delta^I_J \mathcal{H}, \quad I, J = 1, \ldots, 4\,, \\ &\left[\mathcal{H}, Q^K\right] = -\frac{3m}{4}\, Q^K, \qquad \left[\mathcal{H}, \bar{Q}_L\right] = \frac{3m}{4}\, \bar{Q}_L\,. \end{aligned} \tag{3}$$

It respects $SU(4)$ automorphisms (instead of $SO(8)$ or $SU(2) \times SU(2)$).

In the case **A** we constructed, by analogy with $SU(2|1)$, the world-line superfield techniques and presented a few $SU(2|2)$ SQM models as deformations of flat $\mathcal{N} = 8$ models. These are based on the off-shell $SU(2|2)$ multiplets (**3, 8, 5**), (**4, 8, 4**) and (**5, 8, 3**). The actions for the $SU(2|2)$ multiplet (**5, 8, 3**) are massive deformations of those for the same multiplet in the flat case [15]. Another class of actions enjoys superconformal $OSp(4^*|4)$ invariance. This is the case for the general $SU(2|2)$ action of the multiplet (**3, 8, 5**).

Not all of the admissible multiplets of the flat $\mathcal{N} = 8$ SQM have $SU(2|2)$ analogs. It is most important that the so called "root" $\mathcal{N} = 8$ multiplet (**8, 8, 0**) does not. Meanwhile, all other flat $\mathcal{N} = 8$ multiplets and their invariant actions can be obtained from the root one and its general action through the appropriate covariant truncations [16]. How to construct a deformed version of the (**8, 8, 0**) multiplet?

It becomes possible in the models based on world-line realizations of the supergroup $SU(4|1)$. This multiplet is described by a chiral $SU(4|1)$ superfield

$$\Phi\,(t_{\rm L}, \theta_I) = \phi + \sqrt{2}\,\theta_K \chi^K + \theta_I\theta_J A^{IJ} + \frac{\sqrt{2}}{3}\,\theta_I\theta_J\theta_K \xi^{IJK} + \frac{1}{4}\,\varepsilon^{IJKL}\,\theta_I\theta_J\theta_K\theta_L B\,,$$

with the additional $SU(4|1)$ covariant constraints on the component fields

$$A^{IJ} = \sqrt{2}(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}),\ \xi^{IJK} = -\,\varepsilon^{IJKL}(i\dot{\bar{\chi}}_L - \frac{5m}{4}\,\bar{\chi}_L),\ B = \frac{2}{3}(\ddot{\bar{\phi}} + 2im\dot{\bar{\phi}}).$$

The $d = 1$ field content is just $8 = 2+6$ real bosonic fields (ϕ, y^{IJ}) in the $SU(4)$ representation $(\underline{1} \oplus \underline{6})$ and 4 complex fermionic fields χ^L in the fundamental of $SU(4)$.

The invariant action has the very simple form

$$S_{(\mathbf{8,8,0})} = \int dt\, \mathcal{L}_{\rm SK} = \int d\zeta_{\rm L}\, K(\Phi) + \int d\zeta_{\rm R}\, \bar{K}\big(\bar{\Phi}\big), \tag{4}$$

where, in the bosonic limit,

$$\mathcal{L}^{\rm bos}_{(\mathbf{8,8,0})} = g(\dot\phi\dot{\bar\phi} + \frac{1}{2}\dot y^{IJ}\dot y_{IJ} - \frac{m^2}{8}y^{IJ}y_{IJ}) - \frac{im}{4}(\dot\phi\,\partial_\phi g - \dot{\bar\phi}\,\partial_{\bar\phi}g)y^{IJ}y_{IJ} + 2im(\dot\phi\,\partial_{\bar\phi}\bar K - \dot{\bar\phi}\,\partial_\phi K)$$

$$g \sim \partial_\phi\partial_\phi K(\phi) + \partial_{\bar\phi}\partial_{\bar\phi}\bar K(\bar\phi).$$

Besides this action, there were constructed two more invariant actions which are not equivalent to each other. One of them exhibits the relevant superconformal symmetry $OSp(8|2)$.

It was also found that there exists another (twisted) multiplet (**8, 8, 0**), with the bosonic fields in $\underline{4}$ of $SU(4)$.

These results could possibly find applications in supersymmetric matrix models (see, e.g., [17]). In particular, the basic $SU(4|2)$ multiplet (**10, 16**) of matrix models can presumably be constructed from few $SU(2|2)$ or $SU(4|1)$ multiplets.

3 QK $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Another type of deformations of $\mathcal{N} = 4$ SQM models proceeds from the general Hyper-Kähler (HK) subclass of the latter. The deforrmed models are $\mathcal{N} = 4$ supesymmetrization of the Quaternion-Kähler (QK) $d = 1$ sigma models [18]. Both HK and QK $\mathcal{N} = 4$ SQM models can be derived from $\mathcal{N} = 4, d = 1$ harmonic superspace approach [19].

HK manifolds are bosonic targets of sigma models with **rigid** $\mathcal{N} = 2, d = 4$ supersymmetry [20]. After coupling these models to **local** $\mathcal{N} = 2, d = 4$ supersymmetry in the supergravity framework the target spaces are deformed into the so called Quaternion-Kähler (QK) manifolds [21]. QK manifolds are also $4n$ dimensional, but their holonomy group is a subgroup of $Sp(1)\times Sp(n)$. The deformation parameter is just Einstein constant κ, and in the "flat" limit $\kappa \to 0$, the appropriate HK manifolds are recovered.

What about $\mathcal{N} = 4$ Quaternion-Kähler SQM? The main question was as to how to ensure, in one or another way, a local supersymmetry and local $SU(2)$ automorphism symmetry.

In our recent paper with Luca Mezincescu [18], it was shown how to construct $\mathcal{N} = 4$ SQM with an arbitrary QK bosonic target. Like in constructing $\mathcal{N} = 4$ HK SQM, the basic tool is $d = 1$ harmonic superspace. Let us give a brief account of this approach.

The starting point is the ordinary $\mathcal{N} = 4, d = 1$ superspace:

$$(t, \theta^i, \bar\theta_k), \quad i, k, = 1, 2.$$

Its harmonic extension is introduced as :

$$(t, \theta^i, \bar\theta_k) \quad\Rightarrow\quad (t, \theta^i, \bar\theta_k, u^\pm_j),\ u^{+i}u^-_i = 1,\ u^\pm_i \in SU(2)_{Aut}.$$

Its main feature is the existence of the analytic basis,

$$(t_A, \theta^+, \bar\theta^+, u^\pm_k, \theta^-, \bar\theta^-) \equiv (\zeta, u^\pm, \theta^-, \bar\theta^-), \quad \theta^\pm = \theta^i u^\pm_i,\ \bar\theta^\pm = \bar\theta^k u^\pm_k,\ \ t_A = t + i(\theta^+\bar\theta^- + \theta^-\bar\theta^+),$$

in which one can single out an analytic subspace and define the analytic superfields:

$$D^+ = \frac{\partial}{\partial\theta^-},\ \bar D^+ = -\frac{\partial}{\partial\bar\theta^-}, \quad D^+\Phi = \bar D^+\Phi = 0 \ \Rightarrow\ \Phi = \Phi(\zeta, u^\pm)\,.$$

An important tool is the harmonic derivatives:

$$D^{\pm\pm} = u^{\pm}_{\alpha}\frac{\partial}{\partial u^{\mp}_{\alpha}} + \theta^{\pm}\frac{\partial}{\partial\theta^{\mp}} + \bar\theta^{\pm}\frac{\partial}{\partial\bar\theta^{\mp}} + 2i\theta^{\pm}\bar\theta^{\pm}\frac{\partial}{\partial t_{\rm A}}, \quad [D^{++}, D^{--}] = D^0\,,$$

$$[D^+, D^{++}] = [\bar D^+, D^{++}] = 0 \quad \Rightarrow \quad D^{++}\Phi(\zeta, u^{\pm}) \text{ is analytic.}$$

The basic building block of $\mathcal{N} = 4$ SQM models is $\mathcal{N} = 4, d = 1$ multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$. It is described off-shell by an analytic superfield $q^{+a}(\zeta, u)$:

$$(\mathbf{4},\mathbf{4},\mathbf{0}) \qquad \Longleftrightarrow \qquad q^{+a}(\zeta, u) \propto (f^{ia}, \chi^a, \bar\chi^a),\ a = 1, 2,$$

$$(a)\ D^+q^{+a} = \bar D^+q^{+a} = 0 \quad (Grassmann\ \ analyticity),$$
$$(b)\ D^{++}q^{+a} = 0 \quad (Harmonic\ \ analyticity),$$

$$(a) + (b) \quad \Longrightarrow \quad q^{+a} = f^{ka}u^+_k + \theta^+\chi^a - \bar\theta^+\bar\chi^a - 2i\theta^+\bar\theta^+\dot f^{ka}u^-_k.$$

The free off-shell action reads:

$$S_{free} \sim \int dt d^4\theta du\, q^{+a}q^-_a \sim \int dt\Big(\dot f^{ia}\dot f_{ia} - \frac{i}{2}\bar\chi^a\dot\chi_a\Big), \quad q^{-a} := D^{--}q^{+a}\,.$$

The general nonlinear $d = 1$ sigma model action is constructed as:

$$S_{free} \sim \int dt d^4\theta du\, \mathcal{L}(q^{+a}, q^{-b}, u^{\pm}).$$

In the bosonic sector it yields HKT ("Hyper-Kähler with torsion") sigma model. In components, the torsion appears in a term quartic in fermions.

How to construct general HK $\mathcal{N} = 4, d = 1$ sigma models? No torsion appears in this case, the geometry involves only Riemann curvature tensor. The answer was found in [22].

The basic superfields are still real analytic, $q^{+A}(\zeta, u) = f^{iA}u^+_i + ..., i = 1, 2, \ A = 1, \ldots 2n$, they encompass just $4n$ fields $f^{iA}(t)$ parametrizing the target bosonic manifold, $\widetilde{(q^+_A)} = \Omega^{AB}q^+_B$, with $\Omega^{AB} = -\Omega^{BA}$ a constant symplectic metric. The linear constraint $D^{++}q^{+A} = 0$ is promoted to a nonlinear one

$$D^{++}q^{+A} = \Omega^{AB}\,\frac{\partial L^{+4}(q^{+C}, u^{\pm})}{\partial q^{+B}}.$$

The superfield action is bilinear as in the free case,

$$S_{HK} \sim \int dt d^4\theta du\, \Omega^{AB}q^+_Bq^-_A = \int dt[g_{iA\,kB}(f)\dot f^{iA}\dot f^{kB} + \ldots]\,,$$

the whole interaction appearing only due to nonlinear deformation of the q^{+A}-constraint. The object L^{+4} is an analytic hyper-Kähler potential [23]: every L^{+4} produces the component HK metric $g_{iA\,kB}(f)$ and, *vice versa*, each HK metric originates from some HK potential L^{+4}.

The harmonic superspace approach supplies the natural arena for defining $\mathcal{N} = 4$ QK SQM. The new features of these models as compared to their HK prototypes are as follows.

1. QK SQM model corresponding to $4n$ dimensional QK manifold requires $(n + 1)$ multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ described by analytic superfields $q^{+a}(\zeta, w^{\pm}), (a = 1, 2), Q^{+r}(\zeta, w^{\pm}), (r = 1, \ldots 2n)$. An extra superfield $q^{+a}(\zeta, w^{\pm})$ is $d = 1$ analog of $\mathcal{N} = 2, d = 4$ "conformal compensator".

2. QK SQM actions are invariant under local $\mathcal{N} = 4, d = 1$ supersymmetry realized by the appropriate transformations of super coordinates, including harmonic variables $w^{\pm}_i$.

3. For ensuring local invariance it is necessary to introduce a supervielbein $E(\zeta, \theta^-, \bar\theta^-, w^{\pm})$ which is a general $\mathcal{N} = 4, d = 1$ superfield.

4. Besides the (q^+, Q^+) superfield part, the correct action should contain a "cosmological term" involving the vielbein superfield only.

By analogy with the $\mathcal{N} = 2, d = 4$ case we postulate that local $\mathcal{N} = 4, d = 1$ supersymmetry preserves the Grassmann analyticity,

$$\begin{aligned}
&\delta t = \Lambda(\zeta, w)\,, \;\; \delta\theta^+ = \Lambda^+(\zeta, w)\,, \;\; \delta\bar\theta^+ = \bar\Lambda^+(\zeta, w)\,,\\
&\delta\theta^- = \Lambda^-(\zeta, w, \theta^-, \bar\theta^-)\,, \quad \delta\bar\theta^- = \bar\Lambda^-(\zeta, w, \theta^-, \bar\theta^-)\\
&\delta w^+_i = \Lambda^{++}(\zeta, w)w^-_i\,, \quad \delta w^-_i = 0\,.
\end{aligned} \tag{5}$$

The explicit structure of the minimal set of analytic parameters is as follows

$$\begin{aligned}
\Lambda &= 2b + 2i(\lambda^i w^-_i \bar\theta^+ - \bar\lambda^i w^-_i \theta^+) + 2i\theta^+\bar\theta^+\tau^{(ik)}w^-_i w^-_k\,,\\
\Lambda^+ &= \lambda^i w^+_i + \theta^+[\dot b + \tau^{(ik)}w^+_i w^-_k]\,,\\
\Lambda^{++} &= \tau^{(ik)}w^+_i w^+_k - 2i(\dot\lambda^i w^+_i \bar\theta^+ - \dot{\bar\lambda}^i w^+_i \theta^+) - 2i\theta^+\bar\theta^+[\ddot b + \dot\tau^{(ik)}w^+_i w^-_k]\,,\\
\Lambda^- &= \lambda^i w^-_i + \theta^+\tau^{(ik)}w^-_i w^-_k + \theta^-[\dot b - \tau^{(ik)}w^-_i w^+_k]\\
&\quad - 2i\theta^-(\bar\theta^+\dot\lambda^i w^-_i - \theta^+\dot{\bar\lambda}^i w^-_i) + 2i\theta^+\bar\theta^+\theta^-\dot\tau^{(ik)}w^-_i w^-_k],.
\end{aligned} \tag{6}$$

Here, $b(t)$, $\tau^{(ik)}(t)$ and $\lambda^i(t)$, $\bar\lambda^i(t)$ are arbitrary local parameters, bosonic and fermionic.

How to generalize the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ superfields $q^{+A}(\zeta, w)$ to local supersymmetry? The simplest possibility is to keep the linear constraint

$$D^{++}q^{+a} = 0\,.$$

It is covariant under the transformations $\delta D^{++} = -\Lambda^{++}D^0$ and $\delta q^{+a} = \Lambda_0\, q^{+a}$, with

$$\Lambda^{++} = D^{++}\Lambda_0\,, \quad \Lambda_0 = \tau^{(ik)}w^+_i w^-_k - \dot b + 2i(\bar\theta^+\dot\lambda^i - \theta^+\dot{\bar\lambda}^i)w^-_i - 2i\theta^+\bar\theta^+\dot\tau^{(ik)}w^-_i w^-_k\,.$$

To construct invariant actions, one also needs the transformations of the integration measures $\mu_H := dtdwd^2\theta^+d^2\theta^-$, $\mu^{(-2)} := dtdwd^2\theta^+$ and the covariant derivative D^{--} ,

$$\delta\mu^{(-2)} = 0\,, \quad \delta\mu_H = \mu_H\, 2\Lambda_0\,, \quad \delta D^{--} = -(D^{--}\Lambda^{++})D^{--}\,.$$

The extra superfields $Q^{+r}(\zeta, w)\,, r = 1, 2, \ldots 2n$, encompassing n off-shell multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$, obey the same linear harmonic constraint $D^{++}Q^{+r} = 0$ and transform under local $\mathcal{N} = 4$ supersymmetry in the same way as q^{+a} . The basic part of the total invariant action reads

$$\begin{aligned}
&S_{(2)} = \int \mu_H\, E\, \mathcal{L}_{(2)}(q, Q)\,, \quad \mathcal{L}_{(2)}(q, Q) = \gamma q^{+a}q^-_a - Q^{+r}Q^-_r\,,\\
&q^-_a := D^{--}q^+_a\,, \quad Q^-_r := D^{--}Q^+_r\,,
\end{aligned} \tag{7}$$

where $\gamma = \pm 1$. The new object is a supervielbein E. It is harmonic-independent, $D^{++}E = D^{--}E = 0$, and possesses the following local $\mathcal{N} = 4$ supersymmetry transformation law

$$\delta E = (-4\Lambda_0 + 2D^{--}\Lambda^{++})E\,, \quad D^{++}(-4\Lambda_0 + 2D^{--}\Lambda^{++}) = 0\,.$$

This is not the end! One more important term should be added to $S_{(2)}$:

$$S_\beta = \beta \int \mu_H \sqrt{E}\,, \quad \delta S_\beta = \beta \int \mu_H D^{--}\Lambda^{++} \sqrt{E} = 0\,.$$

Thus the simplest locally $\mathcal{N} = 4$ supersymmetric action reads

$$S_{HP} \sim S_{(2)} + S_\beta = \int \mu_H [E\,\mathcal{L}_{(2)} + \beta\sqrt{E}]. \tag{8}$$

Why should the "cosmological constant" term S_β be added?

To answer this question, we pass to the bosonic limit:

$$q^{+a} \;\Rightarrow\; f^{ia} w_i^+ - 2i\theta^+\bar\theta^+ \dot f^{ia} w_i^-\,, \quad Q^{+r} \;\Rightarrow\; F^{ir} w_i^+ - 2i\theta^+\bar\theta^+ \dot F^{ir} w_i^-\,,$$

$$\begin{aligned} E \;\Rightarrow\;& e + \theta^+\theta^- M - \bar\theta^+\bar\theta^- \bar M + \theta^+\bar\theta^-(\mu - i\dot e) + \bar\theta^+\theta^-(\mu + i\dot e) \\ &+ 4i(\theta^+\bar\theta^+ w_i^- w_k^- - \theta^+\bar\theta^- w_i^- w_k^+ - \theta^-\bar\theta^+ w_i^- w_k^+ + \theta^-\bar\theta^- w_i^+ w_k^+) L^{(ik)} \\ &+ 4\theta^+\bar\theta^+\theta^-\bar\theta^-[D + 2\dot L^{(ik)} w_i^+ w_k^-]. \end{aligned}$$

In the bosonic limit,

$$\begin{aligned} L_{HP} \;\Rightarrow\;& \frac{1}{2} e\left(\dot F^{ir}\dot F_{ir} - \gamma\, \dot f^{ia}\dot f_{ia}\right) + L_{ik}\Big[F^{(ir}\dot F_r^{k)} - \gamma f^{(ia}\dot f_a^{k)}\Big] \\ &+ \frac{1}{4} D\Big(\gamma f^{ia} f_{ia} - F^{ir}F_{ir} + \beta\frac{1}{\sqrt{e}}\Big) + \frac{\beta}{4}\,\frac{1}{e^{3/2}}\Big[L^{ik}L_{ik} - \frac{1}{8}(M\bar M + \mu^2 + \dot e^2)\Big]. \end{aligned} \tag{9}$$

The auxiliary fields $M, \bar M$ and μ fully decouple. Also, $e(t)$ is an analog of $d = 1$ vierbein, so it is natural to choose the gauge

$$e = 1\,.$$

Then the bosonic Lagrangian becomes

$$L_{HP} \;\Rightarrow\; \frac{1}{2}\Big(\dot F^{ir}\dot F_{ir} - \gamma\, \dot f^{ia}\dot f_{ia}\Big) + L_{ik}\Big[F^{(ir}\dot F_r^{k)} - \gamma f^{(ia}\dot f_a^{k)}\Big] + \frac{1}{4} D\Big(\gamma f^{ia} f_{ia} - F^{ir}F_{ir} + \beta\Big) + \frac{\beta}{4}\, L^{ik}L_{ik}.$$

At $\beta \neq 0$ L^{ik} can be eliminated by its algebraic equation of motion, while D serves as the Lagrange multiplier giving rise to the constraint relating f^{ia} and F^{ir}:

$$L^{ik} = -2\frac{1}{\beta}\Big[F^{(ir}\dot F_r^{k)} - \gamma f^{(ia}\dot f_a^{k)}\Big], \quad \gamma f^{ia} f_{ia} - F^{ir}F_{ir} + \beta = 0\,. \tag{10}$$

Assuming that f^{ia} starts with a constant (compensator!), one uses local $SU(2)$ freedom, $\delta f^{ia} = \tau^i{}_l f^{la}$, to gauge away the triplet from f^{ia},

$$f^{(ia)} = 0 \;\rightarrow\; f_a^i = \sqrt{2}\,\delta_a^i\,\omega\,.$$

Then the constraint can be solved as

$$(a)\,\gamma = 1 \;\Rightarrow\; \beta < 0\,, \quad \omega = \frac{|\beta|^{1/2}}{2}\sqrt{1 + \frac{1}{|\beta|}F^2}\,,$$

(and analogously for $\gamma = -1$). The final form of the bosonic action for $\gamma = 1$ is

$$L_{HP} = \frac{1}{2}\Big[(\dot F\dot F) + \frac{2}{|\beta|}\,(F_{r(i}\dot F_{j)}{}^r)(F_s^{(i}\,\dot F^{sj)}) - \frac{1}{|\beta|}\,\frac{1}{1 + \frac{1}{|\beta|}F^2}(F\dot F)(F\dot F)\Big]. \tag{11}$$

The case of $\gamma = -1$ is recovered by the replacement $|\beta| \to -|\beta|$.

These actions describe $d = 1$ nonlinear sigma models on non-compact and compact maximally "flat" $4n$ dimensional QK manifolds, respectively,

$$\widetilde{\mathbb{HP}}^{\rm n} = \frac{Sp(1,n)}{Sp(1)\times Sp(n)} \qquad \text{and} \qquad \mathbb{HP}^{\rm n} = \frac{Sp(1+n)}{Sp(1)\times Sp(n)}\,. \tag{12}$$

Thus $\mathcal{N} = 4$ mechanics constructed is just a superextension of these QK sigma models.

4 Generalizations

The basic step in generalizing to $\mathcal{N} = 4$ mechanics with an arbitrary QK manifold is to pass to nonlinear harmonic constraints

$$D^{++}q^{+a} - \gamma\frac{1}{2}\frac{\partial}{\partial q^{+}_{a}}\Big[\hat{\kappa}^2(w^-\cdot q^+)^2\mathcal{L}^{+4}\Big] = 0\,,\; D^{++}Q^{+r} + \frac{1}{2}\frac{\partial}{\partial Q^{+}_{r}}\Big[\hat{\kappa}^2(w^-\cdot q^+)^2\mathcal{L}^{+4}\Big] = 0\,,$$

$$\mathcal{L}^{+4} \equiv \mathcal{L}^{+4}\Big(\frac{Q^{+r}}{\hat{\kappa}(w^-q^+)}, \frac{q^{+a}}{(w^-q^+)}, w^-_i\Big), \quad \hat{\kappa} := \frac{\sqrt{2}}{|\beta|^{1/2}}.$$

The invariant superfield action looks the same as in the $\mathbb{HP}^{\rm n}$ case

$$\mathbb{S}_{QK} \sim \Big[\tilde{S}_{(2)} + S_\beta\Big] = \int \mu_H\Big[E\,\tilde{\mathcal{L}}_{(2)} + \beta\sqrt{E}\Big], \tag{13}$$
$$\tilde{\mathcal{L}}_{(2)} = \gamma q^{+a}q^-_a - Q^{+r}Q^-_r\,, \quad q^-_a = D^{--}q^{+a}\,, \quad Q^-_r = D^{--}Q^{+r}\,.$$

The bosonic action precisely coincides with $d = 1$ reduction of the general QK sigma model action derived from $\mathcal{N} = 2, d = 4$ supergravity-matter action in [24]. This coincidence proves that we have indeed constructed the most general QK $\mathcal{N} = 4$ mechanics.

One more possibility is to consider the following generalization of the $\mathbb{HP}^{\rm n}$ action

$$S^{loc}(q,Q) = \int \mu_H\,\sqrt{E}\mathcal{F}(X,Y,w^-),\; X := \sqrt{E}\,(q^{+a}q^-_a)\,,\; Y := \sqrt{E}\,(Q^{+r}Q^-_r), \tag{14}$$
$$D^{++}q^{+a} = D^{++}Q^{+r} = 0 \quad \Rightarrow \quad D^{\pm\pm}X = D^{\pm\pm}Y = 0\,.$$

When $E{=}const$, it is reduced to the particular form of the HKT action $\int \mu_H\mathcal{F}(q^{+A}, q^{-B}, w^{\pm})$, while for $\mathcal{F}(X,Y,w^-) = \gamma X - Y + \beta$ just to $\mathbb{HP}^{\rm n}$ action. So the target geometry associated with $S^{loc}(q,Q)$ is expected to be a kind of QKT, i.e. "Quaternion-Kähler with torsion" [25]. To date, not too much known about such geometries...

5 Summary and Outlook

Two different deformations of $\mathcal{N} = 8$ supersymmetric mechanics based on the supergroups $SU(2|2)$ and $SU(4|1)$ as a generalization of the $SU(2|1)$ mechanics were sketched.

$\mathcal{N} = 4, d = 1$ harmonic superspace methods were used to construct a new class of $\mathcal{N} = 4$ supersymmetric mechanics models, those with $d = 1$ Quaternion-Kähler sigma models as the bosonic core.

A few generalizations of QK mechanics were proposed, in particular "Quaternion-Kähler with torsion" (QKT) models.

Some further lines of study:

(a). To construct the Hamiltonian formalism for the new class of mechanical systems, including $\mathcal{N} = 4$ supercharges. To perform quantization, at least for the simplest case of $\mathbb{HP}^{\rm n}$ mechanics, to find the energy spectrum.

(b). To explicitly construct some other particular $\mathcal{N} = 4$ QK SQM models, e.g., those associated with symmetric QK manifolds ("Wolf spaces").

(c). To construct locally supersymmetric versions of other off-shell $\mathcal{N} = 4, d = 1$ multiplets (such as $(\mathbf{3}, \mathbf{4}, \mathbf{1})$, $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, etc) and the associated SQM systems.

(d). To establish links between the two types of SQM deformations described in this talk.

Acknowledgements

I thank the Organizers of QUARKS-2018 for giving me the opportunity to present this talk. It is based mainly on the joint works with my co-authors Olaf Lechtenfeld, Luca Mezincescu and Stepan Sidorov. I am very grateful to them for fruitful collaboration.

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