New deformations of $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric mechanics

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Abstract. This is a review of two different types of the deformed $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric mechanics. The first type is associated with the world-line realizations of the supergroups $SU(2|1)$ (four supercharges), as well as of $SU(2|2)$ and $SU(4|1)$ (eight supercharges). The second type is the quaternion-Kähler (QK) deformation of the hyper-Kähler (HK) $\mathcal{N} = 4$ mechanics models. The basic distinguishing feature of the QK models is a local $\mathcal{N} = 4$ supersymmetry realized in $d = 1$ harmonic superspace.

1 Introduction

Supersymmetric Quantum Mechanics (SQM) [1] is the $d = 1$ supersymmetric theory. It:
- Catches the basic features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc.

An extended $\mathcal{N} > 2, d = 1$ supersymmetry is specific: it reveals dualities between various supermultiplets, nonlinear “cousins” of off-shell linear multiplets, etc (see, e.g., [2] and refs. therein). $\mathcal{N} = 4$ SQM, with $\{Q, \bar{Q}\} = 2\delta^\theta_\alpha H, \alpha = 1, 2, \ldots$, is of special interest. In particular, a subclass of $\mathcal{N} = 4$ SQM models have as their bosonic target, Hyper-Kähler (HK) manifolds.

In this Talk, two different types of deformations of $\mathcal{N} = 4$ SQM models will be outlined.

2 From deformed $\mathcal{N} = 4$ SQM to its $\mathcal{N} = 8$ extensions

The first type of deformed SQM arises while choosing some (semi)simple supergroups instead of higher-rank $d = 1$ super-Poincaré:

A. Standard extension: $(\mathcal{N} = 2, d = 1) \Rightarrow (\mathcal{N} > 2, d = 1$ Poincaré$)$,

B. Non-standard extension: $(\mathcal{N} = 2, d = 1) \equiv u(1|1) \Rightarrow su(2|1) \subset su(2|2) \subset \ldots$.

In the case B, the closure of supercharges contains, besides $H$, also internal symmetry generators. The deformed $\mathcal{N} = 4$ SQM is associated with the superalgebra $su(2|1)$:

\[
\begin{align*}
\{Q^i, \bar{Q}^j\} &= 2m(T^i_j - \delta^i_j F) + 2\delta^i_j H, & \{I^i_j, I^k_l\} &= \delta^i_k I^j_l - \delta^j_k I^i_l, \\
\left[ I^i_j, \bar{Q}_l \right] &= \frac{1}{2} \delta^i_l \bar{Q}_j - \delta^j_l \bar{Q}_i, & \left[ I^i_j, Q^k \right] &= \delta^i_k Q^j - \frac{1}{2} \delta^j_k Q^i, \\
\left[ F, \bar{Q}_l \right] &= -\frac{1}{2} \bar{Q}_l, & \left[ F, Q^k \right] &= \frac{1}{2} Q^k.
\end{align*}
\]
The parameter \( m \) is a deformation parameter: when \( m \to 0 \), the standard \( \mathcal{N} = 4, d = 1 \) super-Poincaré is restored.

The simplest models with a worldline realization of \( su(2|1) \) were considered in [3], [4], [5]. The systematic superfield approach to \( su(2|1) \) supersymmetry was worked out in [6], [7], [8], [9]. The models built on the multiplets (1, 4, 3), (2, 4, 2) and (4, 4, 0) were studied at the classical and quantum levels. Recently, \( su(2|1) \) invariant versions of super Calogero-Moser systems were constructed and quantized [10], [11], [12]. The common features of all these models are:

- The oscillator-type Lagrangians for the bosonic fields, with \( m^2 \) as the oscillator strength.
- The appearance of the Wess-Zumino terms for the bosonic fields, of the type \( \sim im(\bar{z} \dot{z} - \dot{z} \bar{z}) \).
- At the lowest energy levels, wave functions form atypical \( su(2|1) \) multiplets, with unequal numbers of the bosonic and fermionic states.

As the next step, analogous deformations of \( \mathcal{N} = 8 \) SQM were studied. The flat \( \mathcal{N} = 8 \) superalgebra,

\[
\{Q_A, Q_B\} = 2\delta_{AB} H, \quad A, B = 1, \ldots, 8,
\]

admits two deformations with the minimal number of extra bosonic generators.

A. Superalgebra \( su(2|2) \) [13]:

\[
\begin{align*}
\{Q^a, S^{j b}\} &= 2i m \left( \epsilon^{a b} L^j - \epsilon^{j b} R^a \right) + 2 \epsilon^{a b} \epsilon^{j b} C, \\
\{Q^a, Q^{j b}\} &= 2 \epsilon^{a b} \epsilon^{j b} (H + C_1), \quad \{S^{a I}, S^{j b}\} = 2 \epsilon^{a I} \epsilon^{j b} (H - C_1).
\end{align*}
\]

In the limit \( m \to 0 \) a centrally extended flat \( \mathcal{N} = 8 \) superalgebra is reproduced, with two extra central charges \( C \) and \( C_1 \) and \( SO(8) \) automorphisms broken to \( SU(2) \times SU(2) \).

B. Superalgebra \( su(4|1) \) [14]:

\[
\begin{align*}
\{Q^I, \bar{Q}_J\} &= 2m L^j + 2\delta^I_J H, \quad I, J = 1, \ldots, 4, \\
[H, Q^K] &= -\frac{3m}{4} Q^K, \quad [H, \bar{Q}_L] = \frac{3m}{4} \bar{Q}_L.
\end{align*}
\]

It respects \( SU(4) \) automorphisms (instead of \( SO(8) \) or \( SU(2) \times SU(2) \)).

In the case A we constructed, by analogy with \( SU(2|1) \), the world-line superfield techniques and presented a few \( SU(2|2) \) SQM models as deformations of flat \( \mathcal{N} = 8 \) models. These are based on the off-shell \( SU(2|2) \) multiplets (3, 8, 5), (4, 8, 4) and (5, 8, 3). The actions for the \( SU(2|2) \) multiplet (5, 8, 3) are massive deformations of those for the same multiplet in the flat case [15]. Another class of actions enjoys superconformal \( OS p(4|4) \) invariance. This is the case for the general \( SU(2|2) \) action of the multiplet (3, 8, 5).

Not all of the admissible multiplets of the flat \( \mathcal{N} = 8 \) SQM have \( SU(2|2) \) analogs. It is most important that the so-called “root” \( \mathcal{N} = 8 \) multiplet (8, 8, 0) does not. Meanwhile, all other flat \( \mathcal{N} = 8 \) multiplets and their invariant actions can be obtained from the root one and its general action through the appropriate covariant truncations [16]. How to construct a deformed version of the (8, 8, 0) multiplet?

It becomes possible in the models based on world-line realizations of the supergroup \( SU(4|1) \). This multiplet is described by a chiral \( SU(4|1) \) superfield

\[
\Phi(\eta_L, \theta_I) = \phi + \sqrt{2} \theta_K \chi_K^\dagger + \theta_I \theta_J A^{IJ} + \frac{\sqrt{2}}{3} \theta_I \theta_J \theta_K \epsilon^{IJK} + \frac{1}{4} \epsilon^{IJKL} \theta_I \theta_J \theta_K \theta_L B,
\]

with the additional \( SU(4|1) \) covariant constraints on the component fields

\[
A^{IJ} = \sqrt{2}(iy^{IJ} - \frac{m}{2} y^{IJ}), \quad \epsilon^{IJK} = -\epsilon^{IJKL}(i\chi_L - \frac{5m}{4} \bar{\chi}_L), \quad B = \frac{2}{3}(\phi + 2im\hat{\phi}).
\]
The $d = 1$ field content is just $8 = 2 + 6$ real bosonic fields $(\phi, y^{IJ})$ in the $SU(4)$ representation $(1 \oplus 6)$ and $4$ complex fermionic fields $\chi^I$ in the fundamental of $SU(4)$.

The invariant action has the very simple form

$$S_{(8,8,0)}(\xi, \chi) = \int dt L_{SK} = \int d\xi K(\Phi) + \int d\chi R(\Phi),$$

where, in the bosonic limit,

$$L_{bos}^{(8,8,0)} = g(\dot{\phi} + \frac{1}{2} g^{IJ} y_{1J} - \frac{m^2}{8} y^{IJ} y_{1J}) - \frac{im}{4} (\phi \bar{\partial}\partial \phi - \bar{\phi} \partial\partial \bar{\phi} y_{1J} + 2 \text{im}(\phi \bar{\partial}\partial \bar{\phi} - \bar{\phi} \partial\partial \phi)).$$

Besides this action, there were constructed two more invariant actions which are not equivalent to each other. One of them exhibits the relevant superconformal symmetry $O(8|2)$.

It was also found that there exists another (twisted) multiplet $(8, 8, 0)$, with the bosonic fields in $4$ of $SU(4)$.

These results could possibly find applications in supersymmetric matrix models (see, e.g., [17]). In particular, the basic $SU(4|2)$ multiplet $(10, 16)$ of matrix models can presumably be constructed from few $SU(2|2)$ or $SU(4|1)$ multiplets.

### 3 Quaternion-Kähler $\mathcal{N} = 4$ SQM as a deformation of HK SQM models

Another type of deformations of $\mathcal{N} = 4$ SQM models proceeds from the general Hyper-Kähler (HK) subclass of the latter. The deformed models are $\mathcal{N} = 4$ supersymmetrization of the Quaternion-Kähler (QK) $d = 1$ sigma models [18]. Both HK and QK $\mathcal{N} = 4$ SQM models can be derived from $\mathcal{N} = 4, d = 1$ harmonic superspace approach [19].

HK manifolds are bosonic targets of sigma models with rigid $\mathcal{N} = 2, d = 4$ supersymmetry [20]. After coupling these models to local $\mathcal{N} = 2, d = 4$ supersymmetry in the supergravity framework the target spaces are deformed into the so called Quaternion-Kähler (QK) manifolds [21]. QK manifolds are also $4n$ dimensional, but their holonomy group is a subgroup of $Sp(1) \times Sp(n)$. The deformation parameter is just Einstein constant $\kappa$, and in the “flat” limit $\kappa \to 0$, the appropriate HK manifolds are recovered.

What about $\mathcal{N} = 4$ Quaternion-Kähler SQM? The main question was as to how to ensure, in one or another way, a local supersymmetry and local $SU(2)$ automorphism symmetry.

In our recent paper with Luca Mezincescu [18], it was shown how to construct $\mathcal{N} = 4$ SQM with an arbitrary QK bosonic target. Like in constructing $\mathcal{N} = 4$ HK SQM, the basic tool is $d = 1$ harmonic superspace. Let us give a brief account of this approach.

The starting point is the ordinary $\mathcal{N} = 4, d = 1$ superspace:

$$(t, \theta^I, \theta^I_k), \quad i, k = 1, 2.$$ 

Its harmonic extension is introduced as:

$$(t, \theta^I, \theta^I_k) \Rightarrow (t, \theta^I, \theta^I_k, u^I), \quad u^I u_I^* = 1, \quad u^I_i \in SU(2)_{Aut}.$$ 

Its main feature is the existence of the analytic basis,

$$(t^A, \theta^+, \theta^+, u^I, \theta^-) \equiv (\zeta^i, u^I, \theta^-), \quad \theta^\pm = \theta^I u^I_k, \quad \theta^\pm \equiv \theta^I u^I_k, \quad t_A = t + i(\theta^+ \theta^- + \theta^- \theta^+),$$

in which one can single out an analytic subspace and define the analytic superfields:

$$D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial \theta^+}, \quad D^+ \Phi = \bar{D}^+ \Phi = 0 \Rightarrow \Phi = \Phi(\zeta^i, u^I).$$
An important tool is the harmonic derivatives:

\[ D^\pm = u^\pm_a \frac{\partial}{\partial u^a} + \theta^\pm \frac{\partial}{\partial \theta^a} + \bar{\theta}^\pm \frac{\partial}{\partial \bar{\theta}^a} + 2i\theta^\pm \bar{\theta}^\pm \frac{\partial}{\partial \bar{\theta}^a}, \quad [D^{++}, D^{--}] = D^0, \]

\[ [D^+, D^{++}] = [\bar{D}^+, D^{++}] = 0 \Rightarrow D^{++} \Phi(\zeta, u^\pm) \text{ is analytic.} \]

The basic building block of \( N = 4 \) SQM models is \( N = 4, d = 1 \) multiplet \((4,4,0)\). It is described off-shell by an analytic superfield \( q^{+a}(\zeta, u) \):

\[(4,4,0) \iff q^{+a}(\zeta, u) \sim (f^i, \chi^a, \bar{\chi}^a), \quad a = 1, 2,\]

\[(a) D^+ q^{+a} = \bar{D}^+ q^{+a} = 0 \quad \text{(Grassmann analyticity)},\]

\[(b) D^{++} q^{+a} = 0 \quad \text{(Harmonic analyticity)},\]

\[(a) + (b) \Rightarrow q^{+a} = f^ka_k + \theta^a \chi^a - \bar{\theta}^a \bar{\chi}^a - 2i\theta^a \bar{\theta}^+ f^ka_k u^a.\]

The free off-shell action reads:

\[ S_{\text{free}} \sim \int dt d\theta du q^{+a}q^a \sim \int dt \left( f^i f_i - \frac{i}{2} \bar{\chi}^a \bar{\chi}_a \right), \quad q^{-a} := D^- q^{+a}. \]

The general nonlinear \( d = 1 \) sigma model action is constructed as:

\[ S_{\text{free}} \sim \int dt d\theta du L(q^{+a}, q^{-b}, u^\pm). \]

In the bosonic sector it yields HKT (“Hyper-Kähler with torsion”) sigma model. In components, the torsion appears in a term quartic in fermions.

How to construct general HK \( N = 4, d = 1 \) sigma models? No torsion appears in this case, the geometry involves only Riemann curvature tensor. The answer was found in [22].

The basic superfields are still real analytic, \( q^{+A}(\zeta, u) = f^i u^+_i + \ldots, i = 1, 2, A = 1, \ldots, 2n \), they encompass just \( 4n \) fields \( f^{iA}(t) \) parametrizing the target bosonic manifold, \( \langle q^+_A \rangle = \Omega^{AB} q^+_B \), with \( \Omega^{AB} = -\Omega^{BA} \) a constant symplectic metric. The linear constraint \( D^{++} q^+ = 0 \) is promoted to a nonlinear one

\[ D^{++} q^+ = \Omega^{AB} \frac{\partial L^{++}(q^{+C}, u^\pm)}{\partial q^{+B}}. \]

The superfield action is bilinear as in the free case,

\[ S_{\text{HK}} \sim \int dt d\theta du \Omega^{AB} q^{+a}_B q^a_A \sim \int dt \left[ g_{iAKB}(f) f^{iA} f^{KB} + \ldots \right], \]

the whole interaction appearing only due to nonlinear deformation of the \( q^{+A} \)-constraint. The object \( L^{++} \) is an analytic hyper-Kähler potential [23]: every \( L^{++} \) produces the component HK metric \( g_{iAKB}(f) \) and, vice versa, each HK metric originates from some HK potential \( L^{++} \).

The harmonic superspace approach supplies the natural arena for defining \( N = 4 \) QK SQM. The new features of these models as compared to their HK prototypes are as follows.

1. QK SQM model corresponding to \( 4n \) dimensional QK manifold requires \((n + 1)\) multiplets \((4, 4, 0)\) described by analytic superfields \( q^{+a}(\zeta, w^\pm), (a = 1, 2), Q^{+r}(\zeta, w^\pm), (r = 1, \ldots, 2n) \). An extra superfield \( q^{+2}(\zeta, w^\pm) = d = 1 \) analog of \( N = 2, d = 4 \) “conformal compensator”.


2. QK SQM actions are invariant under local $N = 4, d = 1$ supersymmetry realized by the appropriate transformations of super coordinates, including harmonic variables $w^i_\pm$.

3. For ensuring local invariance it is necessary to introduce a supervielbein $E(\zeta, \theta^-, \bar{\theta}^-, w^\pm)$ which is a general $N = 4, d = 1$ superfield.

4. Besides the $(q^+, Q^+)$ superfield part, the correct action should contain a “cosmological term” involving the vielbein superfield only.

   By analogy with the $N = 2, d = 4$ case we postulate that local $N = 4, d = 1$ supersymmetry preserves the Grassmann analyticity,
   \[
   \delta t = \Lambda(\zeta, w), \quad \delta \theta^+ = \Lambda^+(\zeta, w), \quad \delta \bar{\theta}^+ = \bar{\Lambda}^+(\zeta, w), \\
   \delta \theta^- = \Lambda^-(\zeta, w, \theta^-, \bar{\theta}^-), \quad \delta \bar{\theta}^- = \bar{\Lambda}^-(\zeta, w, \theta^-, \bar{\theta}^-) \\
   \delta w^+_i = \Lambda^{++}(\zeta, w)w^-_i, \quad \delta w^-_i = 0.
   \]
   (5)

   The explicit structure of the minimal set of analytic parameters is as follows
   \[
   \Lambda = 2b + 2i(\lambda^j w^-_j \bar{\theta}^- - \bar{\lambda}^j w^-_j \theta^+) + 2i\theta^+ \bar{\theta}^+ \tau^{(ik)} w^-_i w^-_k, \\
   \Lambda^+ = \lambda^j w^+_j + \theta^+[\bar{b} + \tau^{(ik)} w^+_i w^+_k], \\
   \Lambda^{++} = \tau^{(ik)} w^+_i w^+_k - 2i(\lambda^j w^+_j \bar{\theta}^- - \bar{\lambda}^j w^+_j \theta^+) - 2i\theta^+ \bar{\theta}^+ [\bar{b} + \tau^{(ik)} w^-_i w^-_k], \\
   \Lambda^- = \bar{\lambda}^j w^-_j + \theta^- \tau^{(ik)} w^-_i w^-_k + \theta^- [\bar{b} - \tau^{(ik)} w^+_i w^+_k] \\
   - 2i\theta^+ \bar{\theta}^+ \theta^+ \bar{\tau}^{(ik)} w^-_i w^-_k].
   \]
   (6)

   Here, $b(t)$, $\tau^{(ik)}(t)$ and $\lambda^j(t), \bar{\lambda}^j(t)$ are arbitrary local parameters, bosonic and fermionic.

   How to generalize the $(4, 4, 0)$ superfields $q^{+\alpha}(\zeta, w)$ to local supersymmetry? The simplest possibility is to keep the linear constraint
   \[
   D^{++} q^{+\alpha} = 0.
   \]
   It is covariant under the transformations $\delta D^{++} = -\Lambda^{++} D^0$ and $\delta q^{+\alpha} = \Lambda_0 q^{+\alpha}$, with
   \[
   \Lambda^{++} = D^{++} \Lambda_0, \quad \Lambda_0 = \tau^{(ik)} w^+_i w^+_k - 2i(\bar{\lambda}^j \theta^- - \bar{\lambda}^j \bar{\theta}^-) w^-_i - 2i\theta^+ \bar{\theta}^+ \tau^{(ik)} w^-_i w^-_k.
   \]

   To construct invariant actions, one also needs the transformations of the integration measures $\mu_H := dt dw^2 \theta^2 d^2 \theta^-$, $\mu^{(-2)} := dt dw^2 \theta^+$ and the covariant derivative $D^{-}$,
   \[
   \delta \mu^{(-2)} = 0, \quad \delta \mu_H = \mu_H 2\Lambda_0, \quad \delta D^{-} = -(D^{-} \Lambda^{++}) D^{-}.
   \]

   The extra superfields $Q^{+r}(\zeta, w), r = 1, 2, \ldots, 2n$, encompassing $n$ off-shell multiplets $(4, 4, 0)$, obey the same linear harmonic constraint $D^{++} Q^{+r} = 0$ and transform under local $N = 4$ supersymmetry in the same way as $q^{+\alpha}$. The basic part of the total invariant action reads
   \[
   S_{(2)} = \int \mu_H E L_{(2)}(q, Q), \quad L_{(2)}(q, Q) = \gamma q^{+\alpha} q^{+\alpha} - Q^{+r} Q^{-}_r, \\
   q^{-}_a := D^{-} q^{+\alpha}, \quad Q^{-}_r := D^{-} Q^{+r},
   \]
   (7)

   where $\gamma = \pm 1$. The new object is a supervielbein $E$. It is harmonic-independent, $D^{++} E = D^{-} E = 0$, and possesses the following local $N = 4$ supersymmetry transformation law
   \[
   \delta E = (-4\Lambda_0 + 2D^{-} \Lambda^{++}) E, \quad D^{++}(-4\Lambda_0 + 2D^{-} \Lambda^{++}) = 0.
   \]
This is not the end! One more important term should be added to $S_{(2)}$:

$$\frac{S_{\beta}}{\mu} = \beta \int \sqrt{E} \, \sqrt{\mathbf{g}}, \quad \delta S_{\beta} = \beta \int \mu (D^- \Lambda^+) \sqrt{E} = 0. \tag{8}$$

Thus the simplest locally $\mathcal{N} = 4$ supersymmetric action reads

$$S_{\text{HP}} \sim S_{(2)} + S_{\beta} = \int \mu_H (E L_{(2)} + \beta \sqrt{E}).$$

Why should the “cosmological constant” term $S_{\beta}$ be added? To answer this question, we pass to the bosonic limit:

$q^{(a)} \Rightarrow f^{(a)} w^+_i - 2i \theta^a \bar{\theta}^i \tilde{J}^{a} w^-_i, \quad Q^+ \Rightarrow F^{(a)} w^+_i - 2i \theta^a \bar{\theta}^i \tilde{J}^{a} w^-_i, \quad E \Rightarrow e + \theta^a \bar{\theta}^i \tilde{M} - \theta^a \bar{\theta}^i \tilde{M} + \theta^a \bar{\theta}^i (\mu - i \bar{\rho}) + \bar{\theta}^a \theta^i (\mu + i \bar{\rho}) + 4i (\theta^a \bar{\theta}^i w^+_j - \theta^a \bar{\theta}^i w^-_j - \theta^a \bar{\theta}^i w^+_j + \theta^a \bar{\theta}^i w^-_j) L^{(ik)}, \quad 4 \theta^a \theta^i \bar{\theta}^j \tilde{M} + 2 L^{(ik)} w^+_i w^-_j, \tag{9}$$

In the bosonic limit,

$$L_{\text{HP}} \Rightarrow \frac{1}{2} e (\tilde{F}^{(ir)} \tilde{F}_{(ir)} - \gamma f^{(ia)} \tilde{f}_{(ia)}^a) + L_{ab}[F^{(ir)} \tilde{F}_{(ir)} - \gamma f^{(ia)} \tilde{f}_{(ia)}^a] + \frac{1}{4} D\gamma f^{(ia)} \tilde{f}_{(ia)}^a - \gamma f^{(ia)} \tilde{f}_{(ia)}^a + \frac{1}{4} \beta \frac{1}{\sqrt{e}} + \frac{\beta}{4} \left[L^i L_{ik} - \frac{1}{8} (M \mathcal{M} + \mu^2 + \bar{\rho}^2) \right].$$

The auxiliary fields $M, \tilde{M}$ and $\mu$ fully decouple. Also, $e(t)$ is an analog of $d = 1$ vierbein, so it is natural to choose the gauge

$$e = 1. \tag{10}$$

Then the bosonic Lagrangian becomes

$$L_{\text{HP}} \Rightarrow \frac{1}{2} F^{(ir)} \tilde{F}_{(ir)} - \gamma f^{(ia)} \tilde{f}_{(ia)}^a + \frac{1}{4} D\gamma f^{(ia)} \tilde{f}_{(ia)}^a - \gamma f^{(ia)} \tilde{f}_{(ia)}^a + \frac{1}{4} \beta \frac{1}{\sqrt{e}} + \frac{\beta}{4} L^i L_{ik}. \tag{11}$$

Assuming that $f^{(ia)}$ starts with a constant (compensator!), one uses local $SU(2)$ freedom, $\delta f^{(ia)} = \tau^i f^{(ia)}$, to gauge away the triplet from $f^{(a)}$,

$$f^{(ia)} = 0 \Rightarrow f^i_a = \sqrt{2} \delta^i_a \omega. \tag{12}$$

Then the constraint can be solved as

$$(a) \gamma = 1 \Rightarrow \beta < 0, \quad \omega = \frac{|\beta|^{1/2}}{2} \sqrt{1 + \frac{1}{|\beta|} F^2}, \tag{13}$$

(and analogously for $\gamma = -1$). The final form of the bosonic action for $\gamma = 1$ is

$$L_{\text{HP}} = \frac{1}{2} (\tilde{F} \tilde{F}) + \frac{1}{|\beta|} (F^{(ir)} \tilde{F}_{(ir)}) (F^{(ia)} \tilde{f}_{(ia)}^a) + \frac{1}{|\beta|} \frac{1}{1 + \frac{1}{|\beta|} F^2} (\tilde{F} \tilde{F})(\tilde{F} \tilde{F}).$$
The case of $\gamma = -1$ is recovered by the replacement $|\beta| \to -|\beta|$. These actions describe $d = 1$ nonlinear sigma models on non-compact and compact maximally “flat” $4n$ dimensional QK manifolds, respectively.

$$\tilde{S}_{\text{HP}^n} = \frac{S\, p(1,n)}{S\, p(1) \times S\, p(n)} \quad \text{and} \quad \tilde{S}_{\text{HP}^n} = \frac{S\, p(1 + n)}{S\, p(1) \times S\, p(n)}. \quad (12)$$

Thus $\mathcal{N} = 4$ mechanics constructed is just a superextension of these QK sigma models.

## 4 Generalizations

The basic step in generalizing to $\mathcal{N} = 4$ mechanics with an arbitrary QK manifold is to pass to nonlinear harmonic constraints

$$D^{++} q^+ - \frac{1}{2} \frac{\partial}{\partial q^+_a} \left[ \hat{k}^2 (w^- \cdot q^+)^2 \mathcal{L}^{++} \right] = 0, \quad D^{++} \mathcal{Q}^{++} + \frac{1}{2} \frac{\partial}{\partial \mathcal{Q}^+_r} \left[ \hat{k}^2 (w^- \cdot q^+)^2 \mathcal{L}^{++} \right] = 0,$$

$$\mathcal{L}^{++} \equiv \mathcal{L}^{++} \left( \frac{Q^+}{\hat{k}(w^- \cdot q^+)}, \frac{q^+}{(w^- \cdot q^+)}, w^- \right), \quad \hat{k} := \frac{\sqrt{2}}{|\beta|^{1/2}}.$$

The invariant superfield action looks the same as in the $\text{HP}^n$ case

$$S_{\text{QK}} \sim [\tilde{S}_{(2)} + S_\beta] = \int \mu_H \left[ E\, \tilde{L}_{(2)} + \beta \sqrt{E} \right], \quad (13)$$

$$\tilde{L}_{(2)} = \gamma q^a q^- - \mathcal{Q}^+ Q^- , \quad q^- = D^{-+} q^+ , \quad Q^- = D^{-+} Q^+.$$

The bosonic action precisely coincides with $d = 1$ reduction of the general QK sigma model action derived from $\mathcal{N} = 2, d = 4$ supergravity-matter action in [24]. This coincidence proves that we have indeed constructed the most general QK $\mathcal{N} = 4$ mechanics.

One more possibility is to consider the following generalization of the $\text{HP}^n$ action

$$S_{\text{loc}}^{(q, \mathcal{Q})} = \int \mu_H \sqrt{E} \mathcal{F} (X, Y, w^-), \quad X := \sqrt{E} (q^+ q^-), \quad Y := \sqrt{E} (\mathcal{Q}^+ \mathcal{Q}^-), \quad (14)$$

$$D^{++} q^+ = D^{++} \mathcal{Q}^+ = 0 \quad \Rightarrow \quad D^{++} X = D^{++} Y = 0.$$

When $E=\text{const}$, it is reduced to the particular form of the HKT action

$$\int \mu_H \mathcal{F} (q^{+A}, q^{-B}, w^+), \quad \text{while for} \quad \mathcal{F} (X, Y, w^-) = \gamma X - Y + \beta \text{ just to $\text{HP}^n$ action.}$$

So the target geometry associated with $S_{\text{loc}}^{(q, \mathcal{Q})}$ is expected to be a kind of QKT, i.e. “Quaternion-Kähler with torsion” [25]. To date, not too much known about such geometries...

## 5 Summary and Outlook

Two different deformations of $\mathcal{N} = 8$ supersymmetric mechanics based on the supergroups $SU(2|2)$ and $SU(4|1)$ as a generalization of the $SU(2|1)$ mechanics were sketched.

$\mathcal{N} = 4, d = 1$ harmonic superspace methods were used to construct a new class of $\mathcal{N} = 4$ supersymmetric mechanics models, those with $d = 1$ Quaternion-Kähler sigma models as the bosonic core.

A few generalizations of QK mechanics were proposed, in particular “Quaternion-Kähler with torsion” (QKT) models.

Some further lines of study:

(a) To construct the Hamiltonian formalism for the new class of mechanical systems, including $\mathcal{N} = 4$ supercharges. To perform quantization, at least for the simplest case of $\text{HP}^n$ mechanics, to find the energy spectrum.
(b). To explicitly construct some other particular $N = 4$ SQM models, e.g., those associated with symmetric QK manifolds ("Wolf spaces").

(c). To construct locally supersymmetric versions of other off-shell $N = 4, d = 1$ multiplets (such as $(3, 4, 1)$, $(1, 4, 3)$, etc) and the associated SQM systems.

(d). To establish links between the two types of SQM deformations described in this talk.

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References