EPJ Web of Conferences **191**, 07004 (2018) https://doi.org/10.1051/epjconf/201819107004
QUARKS-2018

Failure of mean-field approximation in weakly coupled dilaton gravity

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Abstract. We investigate black hole evaporation in a weakly coupled model of two-dimensional dilaton gravity paying a particular attention to the validity of the semiclassical mean-field approximation. Our model is obtained by adding a reflecting boundary to the celebrated RST model describing N gravitating massless scalar fields to one-loop level. The boundary cuts off the region of strong coupling. Although our model is explicitly weakly coupled, we find that the mean field approximation inevitably fails at the end of black hole evaporation. We propose an alternative semiclassical method aiming at direct calculation of S-matrix elements and illustrate it in a simple shell model.

1 Introduction

Mean-field theory points at a non-unitary evolution in quantum gravity [1]. It implies that quantum entanglement between the black hole interior and the rest of the Universe is destroyed by evaporation process, so that initially pure states transform into mixed states of Hawking quanta. An intuition based on AdS/CFT correspondence makes us believe that quantum gravity is unitary, and, to the contrary, mean field theory may break near the black hole horizon [2]. Nevertheless no mechanism explaining restoration of quantum coherence was proposed.

Therefore, we still need simple solvable models to address the black hole information puzzle. One may hope to resolve it in the simplified context of two-dimensional dilaton gravity. This class of models has been thoroughly investigated in 1991-96 but now it is mostly forgotten because the information loss problem there still persists. In this article we argue that difficulties of these models are not related to their fundamental inconsistency, but to the failure of the standard mean field approach. We propose a new semiclassical method to compute directly the S-matrix elements for processes of gravitational scattering to test unitarity explicitly.

Recently we proposed the weakly coupled model of two-dimensional dilaton gravity with a dynamical boundary [3], [4]. We showed that this model possesses an infinite number of exact analytic solutions. The model used in this report is obtained by adding one-loop quantum corrections, both in the bulk and at the boundary.

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This report is based on the forthcoming publications [5], [6]. In Section 2 we propose a modified model describing evaporating black holes and consider semiclassical solutions. In Section 3 we argue that the mean field theory inevitably breaks down at the last stage of black hole evaporation. In Section 4 we compute the semiclassical amplitude of point particle gravitational scattering off the boundary. We pay particular attention to the high-energy processes which produce black holes at classics.

2 Weakly coupled dilaton gravity

For a start we consider the CGHS model of gravity with a boundary [7],

$$S_{CGHS} = \int_{\mathcal{M}} d^2x\,\sqrt{-g}\,e^{-2\phi}\left(R + 4(\nabla\phi)^2 + 4\lambda^2\right) + \int_{\partial\mathcal{M}} d\tau\,e^{-2\phi}(K + 2\lambda) \tag{1}$$

with appropriate choice of the boundary terms ensuring self-consistency. The spacetime boundary $\partial\mathcal{M}$ in Eq. (1) is situated on the line $\phi = \phi_0$, where ϕ is the dilaton. We introduced the proper time τ along the boundary and its extrinsic curvature $K = \nabla_\mu n^\mu$ with outer-directed normal $n^\mu \propto \partial^\mu\phi$. The gravitational coupling parameter $e^\phi < e^{\phi_0}$, it makes the model weakly coupled. Note that black holes in the model (1) have masses larger than $M_{cr} = 2\lambda e^{-2\phi_0}$ [4].

As a matter content we add the massless scalar field f,

$$S_f = -\frac{1}{2}\int_{\mathcal{M}} d^2x\,\sqrt{-g}(\nabla f)^2\ . \tag{2}$$

This field is Weyl-invariant, it decouples from the metric and splits into two non-interacting chiral sectors: $f(u,v) = f_{out}(u) + f_{in}(v)$, where u, v are the null coordinates. However, the spacetime boundary $\partial\mathcal{M} : u = U(v)$ mixes left- and right-moving components according to the reflection law $f_{out}(U(v)) = f_{in}(v)$.

Next, we incorporate Hawking evaporation and its backreaction on the metric by adding one-loop corrections to the action,

$$S_Q = \int_{\mathcal{M}} d^2x\,\sqrt{-g}\left[-\frac{1}{2}(\nabla\chi)^2 + Q\chi R - Q^2\phi R\right] + \int_{\partial\mathcal{M}} d\tau\left[(Q\chi - Q^2\phi)K + \lambda Q^2\right]\ . \tag{3}$$

Action for the field χ encodes one-loop contributions from $N = 24Q^2$ massless scalar fields similar to (2). The choice of additional boundary terms is unique and fixed by the Wess-Zumino consistency condition.

The last bulk term $\propto Q^2$ makes our model equivalent to the RST model [8]. This model admits exact bulk solutions in the conformal gauge $ds^2 = -e^{2\phi(u,v)}dvdu$,

$$\chi(u,v) = 2Q\phi(u,v) + \tilde{\chi}_{out}(u) + \tilde{\chi}_{in}(v)\ , \qquad e^{-2\phi} + Q^2\phi = -\lambda^2 vu - \mathcal{T}(v) - \mathcal{H}(u)\ , \tag{4}$$

$$\partial_v^2\mathcal{T} = \frac{(\partial_v f_{in})^2}{2} + \frac{(\partial_v\tilde{\chi}_{in})^2}{2} + Q\partial_v^2\tilde{\chi}_{in}\ , \qquad \partial_u^2\mathcal{H} = \frac{(\partial_u f_{out})^2}{2} + \frac{(\partial_u\tilde{\chi}_{out})^2}{2} + Q\partial_u^2\tilde{\chi}_{out}\ , \tag{5}$$

where the functions $\tilde{\chi}_{in}$, $\tilde{\chi}_{out}$ are fixed by the initial conditions at $u \to -\infty$ and $v \to +\infty$. Vacuum configuration is

$$\tilde{\chi}_{in}(v) = Q\ln(\lambda v)\ , \qquad \tilde{\chi}_{out}(u) = Q\ln(-\lambda u)\ .$$

The boundary conditions at $\phi = \phi_0$ give differential equation for the form of the boundary $u = U(v)$,

$$\partial_v U = \frac{e^{-2\phi_0}}{\lambda^2(e^{-2\phi_0} + Q^2/2)^2}\left(\partial_v\mathcal{T} + \lambda^2 U + \frac{Q^2}{2}\frac{\partial_v^2 U}{\partial_v U}\right)^2\ , \tag{6}$$

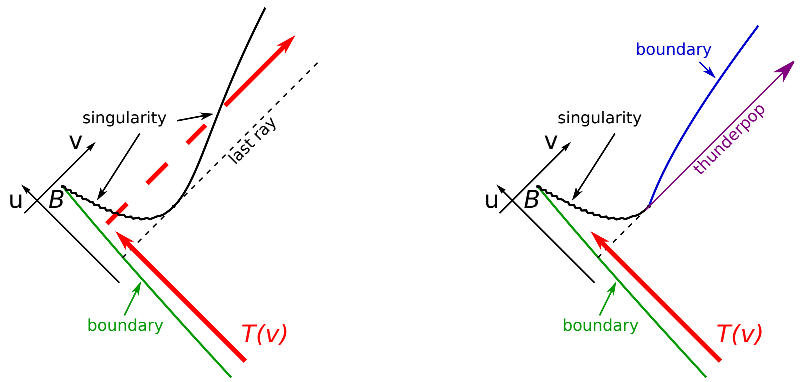

Figure 1. Left panel: the spacetime for the solution of Eq. (6). Red arrows schematically depict the wave packets reflecting from the regular timelike boundary (green). It crosses the singularity through unphysical region $\phi > \phi_0$ after reflection off the boundary. Right panel: glued spacetime where the solution above the last ray is replaced with a regular boundary (blue) satisfying Eq. (6). The thunderpop is emited from the point where the singularity becomes null and the blach hole ceases to exsist.

where the incoming wave packet $\mathcal{T}(v)$ is specified by initial conditions. We solved Eq. (6) with asymptotic behaviour $U(v) \sim -e^{-2\phi_0}/(\lambda^2 v)$ at $v \to 0$ corresponding to the flat spacetime at the beginning of the scattering process, see Fig. 1. Using the inverse function $v = V(u)$ we obtained $\mathcal{H}(u)$ from Eq. (4).

We observe that at large energies the line $\phi(v, u) = \phi_0$ has a bifurcation point B in Fig. 1. The second branch of $\phi = \phi_0 \ : \bar{U}(v) \neq U(v)$ is spacelike. It can be interpreted as a black hole singularity since the boundary conditions for the fields are violated there. The singularity touches the last ray in Fig. 1 and eventually becomes timelike. Fields to the future of the last ray are in causal contact with the black hole interior. Clearly, one cannot propagate through the unphysical region $\phi > \phi_0$. Rather, we have to glue a new solution with a timelike boundary satisfying Eq. (6), see the right panel of Fig. 1.

The new glued solution is continuous, it describes a flat spacetime in the asymptotic future and the total energy is conserved. Unfortunately, new difficulties appear. The solution is not smooth at the last ray and it has a delta-functional singularity in energy density usually called the thunderpop [8]. This is a signal of large quantum fluctuations near the last ray. Clearly, the thunderpop is an artefact of the mean field theory and it should be smeared out in the full quantum treatment. However, the only way to avoid the thunderpop within the mean field theory is to introduce a naked singularity which we initially tried to avoid.

3 Failure of the mean field theory

Note that the above violation of the mean field theory is general, not specific to our model. Namely, consider [9] the correlation function of the scalar field in a vicinity of the last ray $\bar{u} = \bar{u}_e$, see Fig. 2,

$$G(\bar{u}', \bar{u}'') = \langle f_{out}(\bar{u}') f_{out}(\bar{u}'') \rangle = \langle f_{in}(\bar{v}(\bar{u}')) f_{in}(\bar{v}(\bar{u}'')) \rangle = G_{vac}(\bar{v}(\bar{u}'), \bar{v}(\bar{u}'')) \ ,$$

where $\bar{u}'' = \bar{u}_e + \varepsilon$, $\bar{u}' = \bar{u}_e - \varepsilon$, we used the reflection law, and introduced asymptotically flat null coordinates $\bar{v}$, $\bar{u}$ at the boundary $\bar{v} = \bar{v}(\bar{u})$. This correlation function differs from the vacuum one

$$G_{vac}(\bar{u}', \bar{u}'') \propto \ln |\bar{u}' - \bar{u}''|$$

within an arbitrary small interval $\sim \varepsilon$ near the last ray because $\bar{v}(\bar{u}_e + 0) \neq \bar{v}(\bar{u}_e - 0)$. This implies presence of particles moving along the last ray with arbitrary large momenta $\sim \varepsilon^{-1}$. Therefore, the thunderbolt as a general unphysical feature which may be blurred by introducing quantum region of size $\Delta\bar{u} \sim \lambda^{-1}$. Direct calculations show that instant change of boundary conditions in quantum field theory indeed produces singular waves with infinite energy [10].

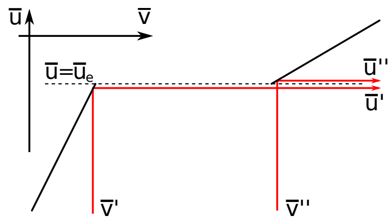

Figure 2. Thunderbolt arises due to sudden jump of the boundary across the null ray $\bar{u} = \bar{u}_e$.

The thunderbolt argument alongside with analysis of evaporating solutions in Section 2 implies that the mean field theory inevitably breaks down. The true reason is that considered glued solutions are singular and cannot be used as correct saddle points of the path integral,

$$\langle \Psi_{out} | \hat{S} | \Psi_{in} \rangle = \int \mathcal{D}\Phi \, \Psi_{in}[\Phi] \Psi^*_{out}[\Phi] e^{iS[\Phi]} \approx \int \mathcal{D}\Phi \, e^{iS_{eff}[\Phi_c]} , \qquad \Leftarrow \quad \frac{\delta}{\delta\Phi} S = 0 , \quad (7)$$

where $\Phi = \{g_{\mu\nu}, \phi, f\}$ represents all fields. Therefore, the semiclassical expansion fails at zero order [1].

To overcome this obstacle we propose a modified semiclassical approach based on analytic continuation of gravitational S-matrix elements. The gravitational scattering amplitude is saturated by configurations which satisfy asymptotically flat boundary conditions for initial and final states. If the black hole is formed a real analytic solution with correct behaviour in future does not exist. Therefore, the integral (7) is saturated by complex-valued saddle point solutions. Since there are many complex solutions to the field equations we need a criterion to pick up the branch giving the leading physical contribution to the path integral.

The criterion was derived in the simple models describing spherically-symmetric gravitationally-interacting shells of matter [11], [12]. The prescription is to add an imaginary contribution to the shell's total energy,

$$M \mapsto M + i\epsilon , \qquad \epsilon \to +0 . \quad (8)$$

The corresponding complex solution gives correct semiclassical amplitude (7) which we calculate in the next Section for the dilaton gravity with the boundary (1) interacting with a massive pointlike particle.

[1] Alternative reasoning relies on observation that the Dirichlet boundary condition $\phi = \phi_0$ is in fact an infinitely stiff interaction of the pointlike particle with dilaton field and may be inconsistent itself on the quantum level. The closest analogy is the Klein paradox: it is known that a relativistic electron scattering off the potential $V \sim 2m_e$ produces negative probabilities. The Klein paradox is resolved in the secondly quantized theories.

4 Semiclassical S-matrix elements

In this Section we investigate a toy model of the massive particle interacting with the CGHS gravity with the boundary. Definition of the S-matrix

$$\hat{S} = \lim_{\substack{t_{out}\to+\infty \\ t_{in}\to-\infty}} \hat{U}_0(0,t_{out})\hat{U}(t_{out},t_{in})\hat{U}_0(t_{in},0)$$

includes interacting and free evolution operators $\hat{U},\ \hat{U}_0$. Thus, the path integral (7) contains

$$S_{eff}[\Phi] = S_{CGHS} + S_m + S_0(0_-,t_{in}) + S_0(t_{out},0_+) - i\ln\Psi_{in} - i\ln\Psi^*_{out}\ . \tag{9}$$

The dilaton gravity action S_{CGHS} contains the boundary terms on the surfaces of the constant Schwarzschild time $t = t_{in,out}$ and at large distance $r = r_\infty$. The second term is the action of the massive particle,

$$S_m = -m\int ds\,\sqrt{-g_{\mu\nu}\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau}}\ , \tag{10}$$

where $y^\mu(\tau)$ parametrizes the worldline and τ is the corresponding proper time. Free actions S_0 emerge from the free evolution operators in the definition of S-matrix and

$$\Psi_{in}\approx\exp(ip_{in}r)\ ,\qquad\qquad \Psi_{out}\approx\exp(ip_{out}r)$$

are the wave functions of the particle at the beginning and at the end of the scattering process.

A covariant junction condition at the worldline $y^\mu(\tau)$ is

$$4e^{-2\phi}[u^\mu\partial_\mu\phi] = m\ , \tag{11}$$

where u^μ is the right-directed normal to the worldline and square brackets denote jump of the derivative inside.

The saddle-point solutions describe flat and Schwarzschild spacetimes,

$$\begin{aligned} &\phi = -\lambda r\ ,\\ &ds^2 = -dT^2 + dr^2\ , && r < r(\tau)\ ,\\ &ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}\ , \qquad f(r) = 1 - \frac{M}{2\lambda}e^{-2\lambda r} && r > r(\tau)\ , \end{aligned}$$

to the left and to the right of the particle's worldline $r(\tau)$, where T and t are the respective times.

Using Eq. (11) one rewrites this condition as

$$\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) = 0\ ,\qquad\qquad V_{eff}(r) = 1 - \left(\frac{M}{m} + \frac{m}{8\lambda}e^{-2\lambda r}\right)^2 . \tag{12}$$

At small energies the particle classically reflects from the boundary: $\dot{r}\mapsto-\dot{r}$. At large energy when the black hole classically forms the event horizon $r_h = \frac{1}{2\lambda}\ln\frac{M}{2\lambda}$ appears at real axis of r and one should deform r-contour avoiding coordinate singularity. The prescription (8) implies that r_h is bypassed from below, see Fig. 3.

Using the outer Schwarzschild metric, one finds the time contour corresponding to the trajectory in Fig. 3, see the right panel. The half-residue contributes twice at the event horizon into the imaginary part of the outer time,

$$\Im m\,(t_{out} - t_{in}) = 2\pi\,\mathrm{Res}(f(r))^{-1}\Big|_{r=r_h} = \frac{\pi}{\lambda}\ . \tag{13}$$

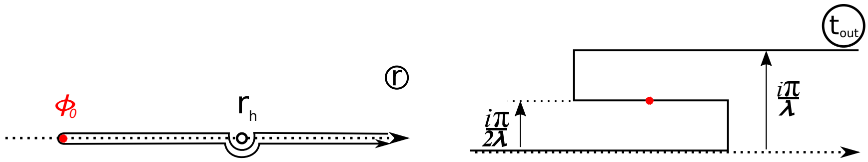

Figure 3. Left panel: the real contour C_r bypassing the event horizon r_h by a tiny excursion into the complex plane. Right panel: the Schwarzschild time contour corresponding to the solution with regularigation (8).

Now we evaluate the total action (9). First, the gravity action becomes

$$S_{CGHS} = 2e^{-2\phi_0}\int ds\, K - 2\int_{t=t_i} d\sigma\, e^{-2\phi}K - 2\int_{t=t_f} d\sigma\, e^{-2\phi}K + O(e^{-2\lambda r_\infty})\,. \qquad (14)$$

Discontinuity of the normal n^μ across the particle's worldline gives rise to non-zero contributions from intersection points with surfaces $t = t_{in,out}$ and the boundary $\phi = \phi_0$. Following Ref. [12] we evaluated

$$S_{CGHS} = 4e^{-2\phi_0}\left(\operatorname{arsh}\sqrt{1 - V_{eff}(r_0)} - \operatorname{arsh}\sqrt{1 - \frac{V_{eff}(r_0)}{1-\frac{M}{M_{cr}}}}\right) + \frac{\sqrt{M^2 - m^2}}{\lambda} + O(e^{-2\lambda r_\infty})\,.$$

We rewrite the particle action using Eq. (12) as

$$S_m = -m\int_{r_0}^{r_i}\frac{dr}{\sqrt{-V_{eff}(r)}} - m\int_{r_0}^{r_f}\frac{dr}{\sqrt{-V_{eff}(r)}}\,, \qquad (15)$$

where we took into account change of sign $\sqrt{-V_{eff}(r)}$ along the contour C_r.

The free actions of the particle with total energy M are,

$$S_0(0_-, t_i) = \sqrt{M^2 - m^2}(r_- - r_i) - Mt_i + O(e^{-2\lambda r_i})\,,$$
$$S_0(t_f, 0_+) = \sqrt{M^2 - m^2}(r_+ - r_f) + Mt_f + O(e^{-2\lambda r_f})\,, \qquad (16)$$

where $r_\pm = r(0_\pm)$ for a free motion of the particle.

The initial and final wave functions give rise to contributions,

$$-i\ln\Psi_i(r_-) = -\sqrt{M^2 - m^2}\,r_-\,, \qquad -i\ln\Psi_f^*(r_+) = -\sqrt{M^2 - m^2}\,r_+\,. \qquad (17)$$

Combining Eqs. (15), (16), (17), and Eq. (14) one obtains in the limit $r_{f,i} \to +\infty$,

$$S_{eff} = \frac{\sqrt{M^2 - m^2}}{\lambda} + \frac{2M_{cr}}{\lambda}\left[\operatorname{arsh}\left(\frac{M}{m} + \frac{m}{4M_{cr}}\right) - \operatorname{arsh}\left(\frac{\left|\frac{M}{m} - \frac{m}{4M_{cr}}\right|}{\sqrt{1 - \frac{M}{M_{cr}}}}\right)\right] +$$
$$-\frac{\sqrt{M^2 - m^2}}{\lambda}\ln\left(\frac{M_{cr}}{4\lambda}\left(1 + \frac{m}{\sqrt{M^2 - m^2}}\sqrt{\left(\frac{M}{m} + \frac{m}{4M_{cr}}\right)^2 - 1}\right) + \frac{m^2 M}{16\lambda(M^2 - m^2)}\right) +$$
$$-\frac{M}{\lambda}\ln\left[\frac{M(4M^2 - 3m^2) - m(4M^2 - m^2)\sqrt{\left(\frac{M}{m} + \frac{m}{4M_{cr}}\right)^2 - 1} + \frac{m^2}{M_{cr}}\left(M^2 + \frac{m^2}{4}\right)}{\left(1 - \frac{M}{M_{cr}}\right)\left(M(4M^2 - 3m^2) - \sqrt{M^2 - m^2}(4M^2 - m^2)\right)}\right]\,. \qquad (18)$$

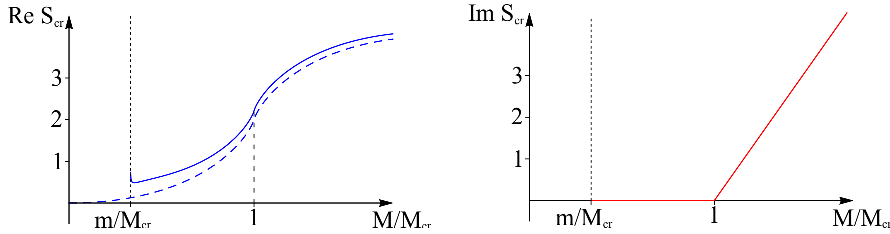

Figure 4. Real (blue) and imaginary (red) parts of the total action (18) at $m = M_{cr}/3$ as functions of M. Dashed blue plot corresponds to the case $m = 0$, Eq. (19). For the real part we took $M_{cr} = 2\lambda$. The interval $M < m$ is kinematically forbidden.

One notes that all divergences cancel out as expected according to S-matrix definition.

In the massless case Eq. (18) simplifies,

$$\lim_{m\to 0} S_{eff} = -\frac{M - M_{cr}}{\lambda} \ln\left(1 - \frac{M}{M_{cr}}\right) + \frac{M}{\lambda}\left(1 - \ln\frac{M_{cr}}{2\lambda}\right) . \tag{19}$$

Equation (19) can be compared with particular S-matrix elements in field theory where initial and final states describe δ–like wave packets.

The semiclassical exponent of the amplitude is shown in in Fig. 4 as a function of the total energy M. Its imaginary part is zero if the reflection is subcritical ($m < M < M_{cr}$) and linearly grows with M at higher energy,

$$\Im m\, S_{eff} = \frac{\pi}{\lambda}(M - M_{cr}) \ , \qquad\qquad M > M_{cr} \ .$$

Positive imaginary part of S_{eff} at $M > M_{cr}$ implies that the reflection probability is exponentially supressed,

$$\mathcal{P} \approx \exp\left(-2\Im m\, S_{eff}\right) = \exp\left(-S_{BH} + S_{cr}\right) \ , \tag{20}$$

where $S_{BH} = \frac{2\pi}{\lambda}M$ is the black hole entropy and S_{cr} is the entropy of the black hole with critical mass $M_{cr} = 2\lambda e^{-2\phi_0}$. Appearance of the critical entropy in the transition probability (20) is physically reasonable since the suppression is expected to vanish on the verge of classical black hole formation. It does not agree with suppression by the number of black hole states $e^{S_{BH}}$ which may signal the inconsistency of the model.

5 Conclusions

In this report we considered the weakly coupled dilaton gravity and reaffirmed that the apparent non-unitarity remains in the models with boundaries. The usual mean field approximation produces singular solutions which cannot be used as the true saddle points of the path integral for S-matrix elements.

We applied the semiclassical method of complex trajectories for calculation of the S-matrix elements. The method gives saddle point solutions with correct properties. We illustrated the method with a toy model describing interaction of a massive particle with gravity by calculating the semiclassical scattering amplitude. Notably, our result is inconsistent with entropic suppression which may point at fundamental inconsistency of the model under consideration.

We are going to use this result as a keystone in findings of the correct complex-valued classical solution in the model with scalar field.

Acknowledgments. We are grateful to D. Levkov, Y. Zenkevich and S. Sibiryakov for collaboration in this research. We thank D. Eremeev for help at early stages of this project. The work is supported by the RSF grant 14-22-00161.

References

[1] S. W. Hawking, Phys. Rev. D **14** 2460 (1976).

[2] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, JHEP **1302** 062 (2013).

[3] M. Fitkevich, EPJ Web Conf. **125** 05004 (2016).

[4] M. Fitkevich, D. Levkov and Y. Zenkevich, JHEP **1704** 108 (2017).

[5] M. Fitkevich, D. Levkov and Y. Zenkevich, "Dilaton gravity with a boundary: from unitarity to black hole evaporation," *(in preparation).*

[6] M. Fitkevich, D. Levkov and S. Sibiryakov, *(in preparation).*

[7] C. G. Callan, Jr., S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D **45** no.4, R1005 (1992).

[8] J. G. Russo, L. Susskind and L. Thorlacius, Phys. Rev. D **46** 3444 (1992).

[9] A. Strominger, "Les Houches lectures on black holes," (1994).

[10] A. Ishibashi and A. Hosoya, Phys. Rev. D **66** 104016 (2002).

[11] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85** 5042 (2000).

[12] F. Bezrukov, D. Levkov and S. Sibiryakov, JHEP **1512** 002 (2015).