On a Probable Anisotropy of the Unruh Radiation: the Case of a Massless Scalar Field in (1+1)D Space-Time

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Abstract. The Unruh effect is considered for the case of a massless scalar field in an (1+1)D space-time. It is shown that under some natural assumptions like finitness of the integration volume or finitness of the interaction propagation speed the effect should be anisotropic.

1 Introduction

More than forty years ago Hawking [1, 2] and Unruh [3] showed that an observer set into a gravitational field of local strength **a** (or uniformly accelerated in a Minkowski space-time with acceleration **a**) will register thermal emission with the temperature T_{HU} (the Hawking-Unruh temperature) defined as

$$T_{\mathrm{HU}} = \frac{\hbar a}{2\pi c k_B},\tag{1}$$

where $\hbar$ is the Planck constant, c is the speed of light, k_B is the Boltzmann constant.

Despite the obvious presence of the preferred direction $\mathbf{a}/a$, the Hawking-Unruh radiation is usually believed to be isotropic, while some similar effects (e.g., the Schwinger effect and the Doppler effect) demonstrate significant anisotropy. The problem of isotropy of the Unruh effect was considered in a number of papers [4–10] where some contradictory results were obtained. It seems that no consensus on isotropy of this radiation has been achieved as of yet. The currently dominating opinion is that the Unruh radiation is isotropic (see, e.g., [11]).

Here we consider the most simple case of a field – a massless scalar field in an (1+1)D space-time[1]. Within such a simple setup, certain corrections to the standard treatment of the Unruh effect are discussed below to reveal its possible anisotropy. In this investigation we focus on the properties of the distribution function of the Unruh radiation without consideration of any specific detector (except one of its general properties – the locality) or its interaction with the quantum field.

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[1]A detailed description and generalization of the obtained results for the case of massless and massive scalar particles in a (3+1)D space-time will be presented in a separate publication.

2 The Unruh effect for a massless scalar field in an (1+1)D Rindler space-time

Let us consider the Unruh effect for a massless scalar field in an (1+1)D Rindler space-time. The main quantity – the average number of Rindler particles in the Minkowski vacuum $|0_M\rangle$ – has the following value:

$$\langle 0_M | \hat{N}_\kappa^R | 0_M \rangle = 2 \int_{-\infty}^{+\infty} |\beta_{\kappa k}|^2 dk, \tag{2}$$

where $\hat{N}_\kappa^R$ is the operator of number of Rindler particles in mode κ, factor 2 acounts for particles and antiparticles, $\beta_{\kappa k} = \left(\Psi_k^{M*}, \Psi_\kappa^R \right)$ is the Bogoliubov coefficient between the complex-conjugate eigen-mode Ψ_k^{M*} in the parent Minkowski space-time and the eigen-mode Ψ_κ^R in the Rindler space-time. Here the round brackets denote the KFG scalar product in the Rindler space-time:

$$(\psi, \phi) = \frac{i}{2} \int_0^\infty (\psi^* \partial_\tau \phi - \phi \partial_\tau \psi^*) \frac{d\rho}{\rho}, \tag{3}$$

where ρ and τ are the Rindler coordinate and Rindler time related to the coordinates (t, x) in the parent Minkowski space-time via following equations:

$$\tau = \frac{c}{a} \text{arcth} \frac{ct}{x}, \quad \rho = \sqrt{x^2 - c^2 t^2}. \tag{4}$$

The inverse relations are:

$$t = \frac{\rho}{c} \sinh \frac{a\tau}{c}, \quad x = \rho \cosh \frac{a\tau}{c}. \tag{5}$$

The eigen-modes in the Rindler space-time are given as

$$\Psi_\kappa^R (\tau, \rho) = C_\kappa^R \exp \left(-i \left[\omega\tau - \kappa\rho_0 \ln (\rho/\rho_0) \right] \right), \tag{6}$$

where C_κ^R is the normalization constant, κ is the wave-vector (such that quantum frequency is $\omega = c|\kappa|$), $\rho_0 = c^2/a$ is the constant Rindler coordinate of the observer in the Rindler space-time.

The eigen-modes in the Minkowski space-time are given as

$$\Psi_k^M (t, x) = C_k^M \exp \left(-i \left[\epsilon t - kx \right] \right), \tag{7}$$

where C_k^M is the normalization constant, $\epsilon = c|k|$ is the quantum frequency, k is the wave-vector.

To calculate the Bogoliubov coefficients $\beta_{\kappa k}$ one should represent the modes $\Psi_k^M (t, x)$ as functions of Rindler coordinates (τ, ρ). With an account of (5), formula (7) can be rewritten as

$$\Psi_k^M (\tau, \rho) = C_k^M \exp \left(ik\rho \exp \left[-\frac{a\tau}{c} \text{sgn} (k) \right] \right). \tag{8}$$

One of the key points of the present derivation is the use of an explicit form of the upper limit of the improper integral (3) in accordance with its mathematical definition:

$$\int_0^\infty f(\rho) d\rho = \lim_{\rho_1 \to \infty} \int_0^{\rho_1} f(\rho) d\rho, \tag{9}$$

where $f(\rho)$ is the integrand of (3). The necessity and physical sense of this operation along with the reasons of its disregard at the singular lower limit of the integral will be described

elsewhere. To employ the definition (9) one should preliminarily define the rules of relative behaviour of the parameters tending to the singular points (one of well-known examples of such a rule is the Cauchy p.v. integration). In the considered case a rule should relate ρ_1 and τ.

In the spirit of definition (9) let us define the incomplete Bogoliubov coefficient:

$$\beta_{\kappa k}(\tau,\rho_1) = \int_0^{\rho_1} \frac{d\rho}{\rho} \left[\Psi_k^M(\tau,\rho) \, \partial_\tau \Psi_\kappa^R(\tau,\rho) - \Psi_\kappa^R(\tau,\rho) \, \partial_\tau \Psi_k^M(\tau,\rho) \right]. \tag{10}$$

Substituting (6) and (8) into (10) after some transformations one obtains

$$\beta_{\kappa k}(\tau,\rho_1) = \frac{C_k^M C_\kappa^R}{2} \exp\left(ic\kappa\tau \left[\operatorname{sgn}(k) - \operatorname{sgn}(\kappa)\right]\right) (|k|\rho_0)^{-i\mu\operatorname{sgn}(\kappa)+1} \frac{a}{c} \times$$
$$\times \int_0^{|k|\rho_1 \exp\left(-\frac{a\tau}{c}\operatorname{sgn}(k)\right)} [\mu - \lambda] \exp\left(i\lambda\operatorname{sgn}(k)\right) \lambda^{i\mu\operatorname{sgn}(\kappa)-1} d\lambda. \tag{11}$$

As the upper limit of the integral in (11) tends to infinity, one obtains the well-known result of Unruh [3], which we will further denote as $\beta_{\kappa k}^{\text{Unruh}}$.

However, at independent behaviour of the limits ρ_1 and τ the expression (11) leads to two distinct cases:

1) For $k < 0$ (quanta coming from the forward hemisphere)

$$\lim_{\tau\to\infty}\lim_{\rho_1\to\infty} \beta_{\kappa k}(\tau,\rho_1) = \lim_{\rho_1\to\infty}\lim_{\tau\to\infty} \beta_{\kappa k}(\tau,\rho_1) = \beta_{\kappa k}^{\text{Unruh}}. \tag{12}$$

2) For $k > 0$ (quanta coming from the backward hemisphere)

$$\lim_{\tau\to\infty}\lim_{\rho_1\to\infty} \beta_{\kappa k}(\tau,\rho_1) = \lim_{\rho_1\to\infty} \beta_{\kappa k}(\tau,\rho_1) = \beta_{\kappa k}^{\text{Unruh}}, \tag{13}$$

but

$$\lim_{\rho_1\to\infty}\lim_{\tau\to\infty} \beta_{\kappa k}(\tau,\rho_1) = 0. \tag{14}$$

One may characterize case 1) as a well-defined case, where the result is clear. On the contrary, case 2) is not well-defined. The very fact of such an ambiguity for $k > 0$ shows that one should calculate the Bogoliubov coefficients with special attention, and the contrast between the two cases may indicate a significant physical difference.

Expression (11) shows that any rule of the relative behaviour of ρ_1 and τ, which provides increasing ρ_1 slower than $\exp(a\tau/c)$, will lead to the zero value of Bogoliubov coefficient. Note, that such behaviour of the integration limit corresponds to the law of motion of massless particles (quanta, e.g., photons) emitted at the observer position in the forward direction at $\tau = 0+$ (i.e., just after the beginning of the accelerated motion).

One may argue that the actual $\beta_{\kappa k}$ should not depend on τ (the scalar product (3) provides independence on time) and (13) is valid at any finite time τ and hence valid at $\tau \to \infty$. But this does not allow us to ignore the fact that at $\tau \to \infty$ one has a significant difficulty with calculation of Bogoliubov coefficient $\beta_{\kappa k}$ (in other words, if the case were clear, no obstacles should appear at $\tau \to \infty$). Also it should be emphasized that the limit $\tau \to \infty$ is much more appropriate than consideration at finite values of τ since it corresponds to the stationary solution, which is implied by the traditional consideration of the Unruh effect.

Now let us estimate the typical scales needed to obtain the classical Unruh result [3]. Consider a body which is accelerated at 1 g ($\simeq$ 981 cm·s^{-2}) during 20 years of its proper time. In accordance with the "thermal" behaviour of the expected radiation, its spectrum will peak at wavelengths comparable with ρ_0 (distance from accelerated body to its Rindler

horizon $\rho = 0$), which is about 1 light year (l.y.) in the considered case. For such wavelengths the appropriate limit of integration can be choosen as several units of ρ_1 in the Rindler space (and several units of x_1 in the Minkowski space). After 20 proper years of acceleration (the proper time equals to τ at the observer position $\rho = \rho_0$), which correspond to about 2.4×10^8 years in the primary Minkowski space-time, one would obtain the following.

1) For quanta propagating in the forward direction $k > 0$

$$\rho_1^{\rightarrow} = \rho_0 \exp\left(\frac{a\tau}{c}\right) \simeq 5 \cdot 10^8 \text{ l.y.} \tag{15}$$

which in accordance with (5) yields

$$x_1^{\rightarrow} = \rho_1^{\rightarrow} \cosh\left(\frac{a\tau}{c}\right) \simeq \frac{\rho_0}{2} \exp\left(\frac{2a\tau}{c}\right) \simeq 1.2 \cdot 10^{17} \text{ l.y.} \tag{16}$$

The latter value is about 10^7 times larger than the size of the observable Universe! This means that to give classical Unruh result [3] the quanta from the back hemisphere should "feel" the space in front of the observer at the corresponding distance before registration. In any conception of physical reality (and interpretation of quantum mechanics) it is difficult to imagine that such a vast space needed for integration and interaction may be physically motivated.

2) For quanta propagating in the backward direction $k < 0$

$$\rho_1^{\leftarrow} = \rho_0 \exp\left(-\frac{a\tau}{c}\right) \simeq 2 \cdot 10^{-9} \text{ l.y.} \tag{17}$$

which in accordance with (5) yields

$$x_1^{\leftarrow} = \rho_1^{\leftarrow} \cosh\left(\frac{a\tau}{c}\right) \rightarrow const \simeq \rho_0/2 \simeq 0.5 \text{ l.y.} \tag{18}$$

At least this value remains well within the observable part of the Universe.

3 Discussion

The behaviour of the incomplete Bogoliubov coefficient $\beta_{\kappa k}(\tau, \rho_1)$ in expression (14) has a clear physical explanation. To reveal it, let us consider the integrand of expression (3) and the properties of the eigen-modes (6) in detail. It follows from (6) that the modes in the Rindler space are the eigen-modes of the following operator:

$$\hat{\kappa} = -i \frac{\rho}{\rho_0} \frac{\partial}{\partial \rho} = -i\rho_0^{-1} \frac{\partial}{\partial \xi_\rho}, \tag{19}$$

where $\xi_\rho = \ln \rho/\rho_0$. We emphasize here the role of the spatial variable ξ_ρ, since modes (6) for massless particles represent plane-waves in the ξ_ρ-space. The eigen-values of operator $\hat{\kappa}$ are the wave-vector values κ in the Rindler space, and we will refer to $\hat{\kappa}$ as Rindler wave-vector operator. It satisfies the commutation relation $[\rho, \hat{\kappa}] = i\rho/\rho_0$, and in the vicinity of the observer position this operator can be directly associated with momentum operator via $\hat{p} \simeq \hbar\hat{\kappa}$.

Formula (8) shows Minkowski plane waves in Rindler coordinates. This can be rewritten via ξ_ρ:

$$\Psi_k^M(\tau, \xi_\rho) = C_k^M \exp\left(ik\rho_0 \exp\left[\xi_\rho - \frac{a\tau}{c} \text{sgn}(k)\right]\right). \tag{20}$$

Minkowski plane waves Ψ_k^M are not the eigen-functions of the Rindler wave-vector operator $\hat{\kappa}$, but one can define Rindler quasi-wave-vector[2] κ^q of mode Ψ_k^M by means of:

$$\kappa^q = \left(\hat{\kappa}\Psi_k^M\right)/\Psi_k^M = k\exp\left[\xi_\rho - \frac{a\tau}{c}\mathrm{sgn}\,(k)\right] = k\frac{\rho}{\rho_0}\exp\left[-\frac{a\tau}{c}\mathrm{sgn}\,(k)\right]. \qquad (21)$$

The time-dependent exponential factor in (21) is just a Doppler factor expressed via observer rapidity $(a\tau/c)$. The spatial factor (ρ/ρ_0) may be interpreted as a consequence of inhomogeneity of the Rindler space-time.

Formula (21) shows that in any finite volume of the Rindler space the modes (8) propagating in the forward direction (i.e. with $k > 0$) become frozen waves $\Psi_k^M\left(\tau, \xi_\rho\right) \to const$ from the point of view of a Rindler observer, i.e. their Rindler quasi-wave-vector tends to zero exponentially with time τ. At the same time, in any finite volume of the Rindler space the modes (8) propagating in the backward direction (with $k < 0$) become more and more energetic from the point of view of a Rindler observer, i.e. their quasi-wave-vector tends to infinity exponentially with time τ at any $\rho > 0$. Both these properties are natural consequencies of the Doppler effect.

It is evident that frozen waves cannot provide an adequate content of any mode (6). The modes (8) with $k > 0$ come out of "freezing" at large values of distance $\rho \sim \rho_0\exp\left(a\tau/c\right)$, i.e. their quasi-wave-vector becomes noticeable $\kappa^q \sim k$ and they become to oscillate fast enough. From this distance they can make adequate (non-zero) contribution to modes (6). This result is in full accordance with consideration in section 2, where it was shown that these energetic "parts" of eigen-modes may appear outside of the observable Universe. The doubts presented here partially reflect the following concern of Fulling [13]: "The quantum theory usually deals with phenomena that happen on a microscopic scale. It is hard to understand how the global structure of the Universe can affect the physics inside a small Cauchy-complete region. Nevertheless, a decomposition of a field into modes appears unavoidably to involve global integral transformations [like scalar product (3) – auth.]."

The one of the deep reasons of problems with correct unambigous calculation of the Unruh effect is the disregard of the time dependence of gravitational and quantum fields and attempt to use the Rindler metric and eigen-modes (6) (as stationary limits) from the very beginning of consideration to obtain a stationary limit for Unruh effect. Let us imagine that the Rindler metric does not cover the whole space immediately but propagates with some finite speed. Then the KFG equation of this intermediate space will significantly differ from KFG equations in the Minkowski and Rindler spaces. It will contain time-dependent coefficients (since the metric will be time-dependent) and there will be no stationary states (hence, separation of variables will not be possible). Solutions of this correct KFG equation will take into account the time-dependent influence of the apparent gravitational field (i.e. arising from the accelerated motion of the observer) and retardation inherent to all solutions of the wave equations. Here it should be noted that the scalar product (3) provides independence on time only for the functions, which are solutions of the KFG equation, and both [(7) and (6)] modes are the solutions of both (in the Minkowski and Rindler spaces) KFG equations. At the same time, the modes in the intermediate time-dependent metric will not satisfy the KFG equations in the Minkowski and Rindler spaces. In this situation it is impossible to construct constant (time-independent) Bogoliubov coefficients between Minkowski and intermediate modes. Then it is natural to expect some differential equation (on time) for the Bogoliubov coefficients. In the limiting case $\tau \to \infty$ the intermediate metric will tend to the Rindler metric, the intermediate modes will tend to modes (6), but the Bogoliubov coefficients $\beta_{\kappa k}$

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[2]Quasi-wave-vector is the local coordinate-dependent value which is similar to classical momentum $p(x)$ in the semi-classical approximation of quantum mechanics (see, e.g., [12])

may tend to zero for $k > 0$ and will not correspond to the Bogoliubov coefficients $\beta_{\kappa k}^{\text{Unruh}}$ between the modes (7) and (6). This shows that an important point for a correct consideration of the Unruh effect is to use the time-dependent KFG equation taking the finite speed of the propagation of changes of the metric tensor into account and to find corresponding time-dependent solutions (even in the case of an apparent gravitational field).

For the finiteness of speed of metric propagation the following explanation may be suggested. Most of the mathematical models are more or less far from physical reality. Applicability of a particular mathematical model can be determined in two ways. First of them (most reliable) is to control the correspondence of theoretical results and experimental data. When an experiment is impossible one should use the second, theoretical, way: if no improvement leading to significant change of theoretical result can be found, the model may be considered correct. Considering the Unruh effect, one deals with the second case. Hence it is relevant to emphasize that usage of Rindler coordinates with immediate covering of the whole space is a pure mathematical abstraction. The general relativity demands realization of coordinate systems with multiple physical bodies (which can serve as some marker points of the coordinate system) and means (instruments) to determine distances between these points. Let us consider Minkowski space-time with two identical absolutely rigid bars (of course, in the sense allowed by special relativity) "R" and "M" of length L with measuring ticks placed according to some reference (etalon) scale. A similar realization of an accelerated coordinate system was considered by Ginzburg and Frolov [14]. To complete the picture the ticks may be sources of radio signals which are unique for every tick and periodic with some reference (etalon) time interval. When the bars are at rest, they represent Minkowski coordinates. Further, let us imagine that at the moment $t = 0$ the point $x_R = 0$ of the bar "R" begins to move with acceleration a. First, it is evident that the point $x_R = L$ will begin to accelerate only after the period L/c. At time $t < L/c$ the coordinate x_R in the vicinity of $x_R = L$ corresponds to Minkowski coordinate x_M and describes the Minkowski space. Second, coordinate x_R in the vicinity of $x_R = 0$ becomes close to the Rindler coordinate ρ and describes the Rindler space at some reasonable level of accuracy, i.e. the difference between the measurable coordinate x_R and the ideal coordinate ρ is small enough. This transition between the Rindler and Minkowski coordinates is especially markable when $L \gg c^2/a$. Note that under considered conditions the bar "R" seems one of the most acceptable physical realizations of coordinate system of an accelerated observer placed at the point $x_R = 0$. This example also emphasizes the unavoidable role of the observer (detector) and its locality in consideration of the Unruh effect. One may say that a reference observer "generates" a specific Rindler space-time in expanding (but finite-size) region around itself.

Another explanation can be suggested, which is based on the equivalence principle. Let us imagine that transformation of quantum states occurs under the influence of a real physical potential. Then the amplitude of the transformation will be characterized by the elements of the S-matrix instead of Bogoliubov coefficients. In accordance with the equivalence principle (which, in particular, allows us to set correspondence between the Unruh and Hawking effects) one may consider a resting observer in a uniform gravitational field instead of the accelerated observer in the Minkowski space and obtain the same result for distribution of vacuum radiation. As the first step of this thought experiment one may imply immediate turning-on of the gravitational field in the whole space. But if one aims to achieve a more complete analogy between these setups, the next step should be made: turning-on of the gravitational field at the moment $t = 0$ ($\tau = 0$) (which corresponds to the moment of the beginning of motion of the accelerated observer in the Minkowski space) at the resting observer position (for example, by appearance of a corresponding mass distribution which is possible in a thought experiment). It is evident in this case that a real gravitational field (i.e. that arising due to the presence of a mass) will propagate with the speed of light in the primary space

and change its metric correspondingly. Returning to consideration of an accelerated observer in the Minkowski space, one may conclude that the Rindler metric will cover the primary Minkowski space with a finite speed.

It is possible, that a more appropriate (more close to the perfect case) physical realization of a Rindler coordinate system can be suggested than the considered above rigid bar "R". But in any case, both considered examples show that it is difficult to expect that the speed of propagation of metric may exceed the speed of light (whatever gravitational field is real or apparent). The cumulative effect of the finite speed of metric propagation and retardation in the eigen-modes will appear as some effective time-dependent cut-off in the integral (3).

However, an effective cut-off may appear in the integral (3) for more trivial reasons. For example, even very weak interactions of the considered quantum field with other fields existing in the ambient vacuum may lead to destruction of the ideal eigen-state (such as plane-wave modes (7)) on large but finite spatial scales.

4 An approximate integration and the resulting number of Rindler particles

However a time-dependent KFG equation will be considered in the following publications, we believe that the final result of such a model (correct stationary limits of the Bogoliubov coefficient and the average number of particles) may be well approximated by calculation of the integral (3) over large but finit Rindler space region $[0; R]$ around the observer position ρ_0. The full size of this region, R, should be much larger than the Rindler horizon size and the wavelengths of considered quanta: $R \gg \max\{\rho_0, 2\pi k^{-1}, 2\pi\kappa^{-1}\}$.

Then the average number of particles can be estimated by the following formula:

$$\langle 0_M | \hat{N}_\kappa^R | 0_M \rangle = \frac{1}{2} \lim_{\tau \to \infty} \int_{-\infty}^{+\infty} \left| \int_0^R \left(\Psi_k^M \partial_\tau \Psi_\kappa^R - \Psi_\kappa^R \partial_\tau \Psi_k^M \right) \frac{d\rho}{\rho} \right|^2 dk, \tag{22}$$

which yields the following final result:

$$\langle 0_M | \hat{N}_\kappa^R | 0_M \rangle = \begin{cases} 2\left(\exp\left(\frac{2\pi c^2 |\kappa|}{a} \right) - 1 \right)^{-1}, & \kappa < 0 \\ 0, & \kappa > 0 \end{cases} \tag{23}$$

5 Conclusion

Under assumptions of finiteness of the integration volume or finiteness of the interaction propagation speed, the Unruh effect should be significantly anisotropic. This property does not rely on any specific detector conception, but is fundamental like the Unruh effect itself.

The suggested consideration can be naturally extended on the (3+1)D Rindler space-time. The obtained result and its further development may be important for specific applications of the Hawking-Unruh effect such as description of black hole evaporation and motion dynamics of accelerated bodies.

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