NLO and NNLO corrections to polarized top quark decays

Stefan Groote[1],* and *Jürgen G.* Körner[2],**

[1]Füüsika Instituut, Tartu Ülikool, W. Ostwaldi 1, EE-50411 Tartu, Estonia
[2]PRISMA Cluster of Excellence, Institut für Physik, Johannes-Gutenberg-Universität, D-55099 Mainz, Germany

Abstract. We present partial results on NLO and NNLO QCD, and NLO electroweak corrections to polarized top quark decays. In parallel we derive positivity bounds for the polarized structure functions in polarized top quark decays and check them against the perturbative corrections to the structure functions.

1 Introduction

In the limited space available to us in this write-up of a talk given at the International Workshop on QCD Theory and Experiment (QCD@Work 2018) in Matera, Italy, we cannot review the subject of polarized top quark decays in any depth. Instead we take the opportunity to report on results on radiative corrections to polarized top quark decays obtained by our group in the last few years. We will share our insights into the problem, why we did the calculations and how we did them. We also take the opportunity to specify which perturbative calculations have been done and which remain to be done.

The motivation for studying polarized top quark decays is provided by the huge sample of singly produced polarized quarks at the LHC. The dominant source of polarized top quarks is from weak *t*–channel production with an average polarization of ≈ 90 % for the produced top quarks. Up to date the LHC detectors have seen $\sim 10^7$ singly produced top quarks. The projected overall luminosity of the future high luminosity HL-LHC is 3 ab^{-1} which corresponds to $\sim 10^9$ singly produced top quarks. Top quarks retain their polarization at birth when they decay since the life time of the top quark is so short.

2 The polarized top quark three-body decay $t(\uparrow) \to X_b + \ell^+ + \nu_\ell$

The full angular decay distribution for polarized top quark decay $t(\uparrow) \to X_b + \ell^+ + \nu_\ell$ in the top quark rest frame can be written in terms of the four structure functions A, B, C and D. The decay distributuion reads [1]

$$\frac{d\Gamma}{d\cos\theta d\phi} = A + B\,P_t \cos\theta_P + C\,P_t \sin\theta_P \cos\phi + D\,P_t \sin\theta_P \sin\phi$$

$$= A\left(1 + \frac{B}{A}\,P_t \cos\theta_P + \frac{C}{A}\,P_t \sin\theta_P \cos\phi + \frac{D}{A}\,P_t \sin\theta_P \sin\phi\right), \tag{1}$$

where the angles θ_p and ϕ are defined in Fig. 1. In the classification of Ref. [2] this is the helicity system Ib.

---

*e-mail: groote@ut.ee
**e-mail: jukoerne@uni-mainz.de
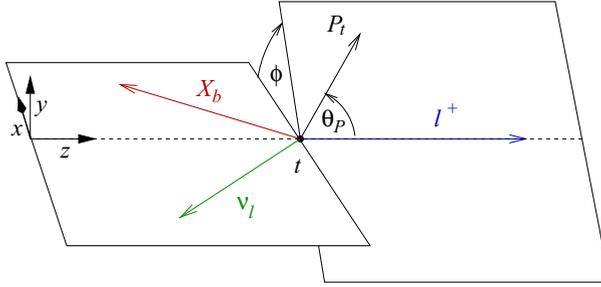
Figure 1. Definition of the polar angle θ_P and the azimuthal angle ϕ in the three-body decay of a polarized top quark.

In the usual classification the structure functions A, B and C are T–even structure functions and D is a T–odd structure function as can be seen by rewriting the angular factor multiplying D in the form

$$\sin \theta_P \sin \phi \propto \vec{p}_\nu \cdot (\vec{p}_\ell \times \vec{s}_t). \qquad (2)$$

The T–odd structure function D can be fed by final-state interactions (also called rescattering corrections) or by CP–violating interactions. We will present examples of both contributions further on.

It is clear that the angular decay distribution must remain positive definite over all of phase space. The positivity of the rate will be a recurring theme in this write-up.

2.1 The LO angular distribution

For the leading order (LO) Born term contribution one obtains the amplitude

$$
\begin{aligned}
M &= \bar{u}(b)\gamma^\mu(1 - \gamma_5)u(t)\,\bar{u}(\ell)\gamma_\mu(1 - \gamma_5)v(\nu) \\
&= 2\bar{u}(b)(1 + \gamma_5)v(\nu)\,\bar{u}(\ell)(1 - \gamma_5)u(t), \qquad (3)
\end{aligned}
$$

where we have used a Fierz transformation of the second kind to convert the $(V - A)^\mu(V - A)_\mu$ form to a $(S + P)(S - P)$ form (see e.g. [3]). The polarized angular decay distribution then reads (we set $P_t = |\vec{P}_t| = 1$)

$$
\begin{aligned}
W^P(\cos \theta_P) &= \sum_{spins} |M|^2 = 8\, tr\{\slashed{p}_b \slashed{p}_\nu\}\, tr\{(\slashed{p}_t + m_t)\tfrac{1}{2}(1 + \gamma_5 \slashed{s}_t)(1 + \gamma_5)\slashed{p}_\ell\} \\
&= 16 m_t^4\, x_\ell(1 - x_\ell)(1 + \cos \theta_P), \qquad (4)
\end{aligned}
$$

where $x_\ell = 2E_\ell/m_t$. Quite naturally, the same result is obtained more tediously if one uses the $(V - A)^\mu(V - A)_\mu$ form of the amplitude (3). At LO there are no azimuthal correlations, i.e. $C = D = 0$! It is not difficult to see that the abscence of LO azimuthal correlations is in line with the postulate of positivity for the LO rate.

2.2 The NLO angular distribution

The NLO QCD contribution to the structure functions A, B and C have been calculated in [3]. We denote the NLO contribution by $A^{NLO} = A^{(0)} + A^{(1)}$ etc.. Setting $\phi = 0$ the angular decay distribution at NLO reads

$$W(\theta_P) = A^{(0)}\left(\left(1 + \frac{A^{(1)}}{A^{(0)}}\right) + \left(1 + \frac{B^{(1)}}{B^{(0)}}\right)\cos \theta_P + \frac{C^{(1)}}{A^{(0)}}\sin \theta_P\right) \qquad (5)$$

where $A^{(1)}/A^{(0)} = -0.0846955$, $B^{(1)}/A^{(0)} = -0.0863048$ and $C^{(1)}/A^{(0)} = -0.0024$. The above values of the coefficient functions represent average values of the respective functions averaged over x_ℓ in the interval $[x, 1]$ where $x = m_W/m_t$. As concerns the $\cos\theta_P$ dependence, the NLO rate remains barely positive for $\cos\theta_P = -1$ and $P_t = 1$ as can be seen from

$$W(\theta_P) \sim 1 + \frac{1 + B^{(1)}/A^{(0)}}{1 + A^{(1)}/A^{(0)}} \cos\theta_P = 1 + 0.99824 \cos\theta_P. \tag{6}$$

Next we analyze positivity including the T–even azimuthal correlation proportional to the strucrture function C. We expand the angular rate around the point of risk $\theta_P = \pi$. One has $\cos(\pi - \delta) = -1 + \frac{1}{2}\delta$ and $\sin\delta = \delta$. We then obtain ($\Delta = (A^{NLO} - B^{NLO})/A^{NLO} = 0.001758$)

$$W(\theta_P) = \Delta - \frac{C^{(1)}}{A^{(0)}} \frac{1}{1 + A^{(1)}/A^{(0)}} \delta + \frac{(1 - \Delta)}{2} \delta^2. \tag{7}$$

A lower bound on the positivity of the rate is obtained when the discriminant of the quadratic equation vanishes. One obtains

$$\left|\frac{C^{(1)}}{A^{(0)}}\right| \leq \underbrace{\sqrt{2\Delta(1 - \Delta)}(1 + \frac{A^{(1)}}{A^{(0)}})}_{0.05422}. \tag{8}$$

It is apparent that the NLO contributions listed above ($|C^{(1)}/A^{(0)}| = 0.0024$) easily satisfy the bound. The lower bound occurs at $\delta = -8.36 \times 10^{-4}\,\pi$ which shows that the small-δ expansion is well justified. This calculation provides the setting for deriving a bound on the size of the T–odd structure function D to be discussed in the next subsection.

2.3 NLO positivity bounds for the T–odd structure function D

There is a CP–conserving Standard Model contribution to the T–odd structure function D coming from the NLO electroweak rescattering correction as shown as absorptive parts in Fig. 2. One can check that there are no NLO absorptive QCD contributions.
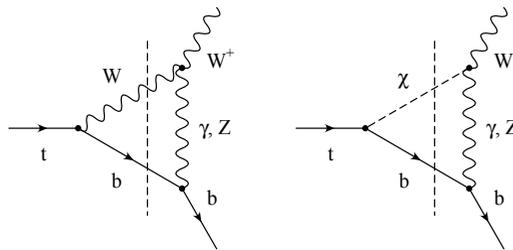
Figure 2. Electroweak absorptive parts of the four Feynman diagrams that contribute to T-odd correlations in polarized top quark decays

We present the result of calculating the absorptive parts of the two diagrams in terms of the imaginary part of the effective coupling constant g_R in the effective Lagrangian

$$\mathcal{J}^\mu_{\text{eff}} = -\frac{g_w}{\sqrt{2}} \bar{b} \left\{\gamma_\mu(V_L P_L + V_R P_R) + \frac{i\sigma^{\mu\nu}q_\nu}{m_W}(g_L P_L + g_R P_R)\right\} t \tag{9}$$

where $P_{L,R} = (1 \mp \gamma_5)/2$. The SM structure of the tbW^+ vertex is obtained by dropping all terms except for the contribution proportional to $V_{tb}^* \sim 1$. $\mathrm{Im}\, g_R$ is contributed to by *CP*–conserving rescattering effects and by *CP*–violating New Physics effects.

The electroweak rescattering contribution to $\mathrm{Im}\, g_R$ can be calculated to be [4]

$$\mathrm{Im}\, g_R = -2.175 \times 10^{-3} . \tag{10}$$

We agree with the result in [5] but disagree with [6].

A bound on $\mathrm{Im}\, g_R$ can be obtained by first calculating the contribution of $\mathrm{Im}\, g_R$ to the structure function D. The result is

$$\frac{D}{A^{(0)}} = -\frac{3\pi}{4} \frac{(1-x^2)}{(1+2x^2)} \mathrm{Im}\, g_R \tag{11}$$

A positivity bound on $\mathrm{Im}\, g_R$ can then be derived in analogy to the bound on the T–even structure function C in Eq. (8) but now setting $\sin\phi = 1$ and replacing C by D. One then obtains the $O(\alpha_s)$ bound [1]

$$-0.0420 \le \mathrm{Im}\, g_R \le 0.0420 . \tag{12}$$

For once, the *CP*–conserving electroweak absorptive contribution to $\mathrm{Im}\, g_R = -2.175 \times 10^{-3}$ can be seen to easily satisfy the bound.

As concerns experiment the bound is tighter than the experimental bound obtained by the ATLAS collaboration [7]
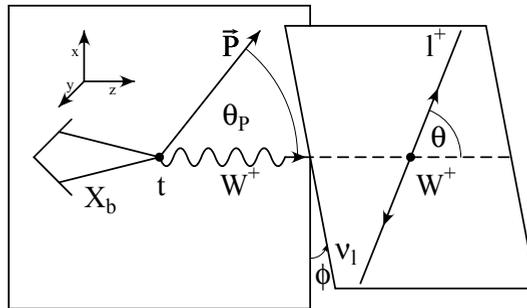
$$-0.18 \le \mathrm{Im}\, g_R \le 0.06 . \tag{13}$$

We mention that the NLO electroweak corrections to the two polarized T–even structure functions B and C have not been done.

3 The two-stage sequential polarized top quark decay
$t(\uparrow) \to b + W^+(\to \ell^+ + \nu_\ell)$

The two-stage sequential two-body decay process $t(\uparrow) \to b + W^+(\to \ell^+ + \nu_\ell)$ is described by two polar angles θ and θ_P, and the azimuthal angle ϕ as defined in Fig. 3.

Figure 3. Definition of the polar angles θ and θ_P, and the azimuthal angle ϕ in the two-stage sequential two-body decay $t(\uparrow) \to X_b + W^+(\to \ell^+ + \nu_\ell)$.

The count of the independent structure functions is best done by considering the independent double spin density matrix elements $H^{\lambda_t\, \lambda_t'}_{\lambda_W\, \lambda_W'}$ of the W^+ which form a hermitian (3×3) matrix

$$\left(H^{\lambda_t\, \lambda_t'}_{\lambda_W\, \lambda_W'} \right)^\dagger = \left(H^{\lambda_t'\, \lambda_t}_{\lambda_W'\, \lambda_W} \right). \tag{14}$$

There are altogether ten independent double spin density matrix elements

$$H_{++}^{++}, \; H_{++}^{--}, \; H_{--}^{++}, \; H_{--}^{--}, \; H_{00}^{++}, \; H_{00}^{--}, \; \text{Re}\,H_{+0}^{+-}, \; \text{Im}\,H_{+0}^{+-}, \; \text{Re}\,H_{-0}^{-+}, \; \text{Im}\,H_{-0}^{-+} \qquad (15)$$

out of which eight are T-even and two are T-odd structure functions. Compare this to the three T-even and one T-odd structure functions that describe the direct three-body decays of polarized top quarks discussed in Sec. 2.

Let us concentrate on the polar angle distribution which is obtained by integrating over the azimuthal angle ϕ. One obtains

$$
\begin{aligned}
W(\theta, \theta_P) \;=\;& \frac{3}{8}\big(1 + \cos\theta\big)^2\big(T_+ + T_+^P P_t \cos\theta_P\big) + \frac{3}{8}\big(1 - \cos\theta\big)^2\big(T_- + T_-^P P_t \cos\theta_P\big) \\
& + \frac{3}{4}\sin^2\theta\big(L + L^P P_t \cos\theta_P\big)
\end{aligned}
\qquad (16)
$$

where $T_+, T_+^P, T_-, T_-^P, L, L^P$ are linear combinations of $H_{++}^{++}, H_{++}^{--}, H_{--}^{++}, H_{--}^{--}, H_{00}^{++}, H_{00}^{--}$.

At LO and for $m_b = 0$ one has $T_- = -T_-^P$, $L = +L^P$, $T_+ = T_+^P = 0$, i.e. the $\cos\theta_P$ dependence of the longitudinal and transverse-minus rates are given by $L : (1 + P_t \cos\theta_P)$ and $T_- : (1 - P_t \cos\theta_P)$. Similar to the discussion in Sec. 2.2 one is precariously close to a violation of positivity when $P_t = 1$. Using the $O(\alpha_s)$ results of [8, 9] we have checked that positivity is not spoiled at NLO QCD. In the same vein one can derive NLO bounds for the T–even and T–odd structure functions $\text{Re}\,H_{+0}^{+-}, \text{Im}\,H_{+0}^{+-}, \text{Re}\,H_{-0}^{-+}, \text{Im}\,H_{-0}^{-+}$ which again are satisfied by the NLO QCD and the NLO electroweak results.

We mention that we are in the process of calculating the NLO electroweak corrections to the eight T–even structure functions of sequential polarized top quark decay [10].

3.1 NNLO QCD corrections to sequential polarized top quark decay

As a last topic of this presentation we discuss the calculation of NNLO QCD corrections to the eight T–even structure functions in sequential polarized and unpolarized top quark decays for which we have obtained some partial results [11, 12]. The main idea behind our approach has been laid down in the NNLO calculation of the total rate in [13]. One converts a two-scale problem $\Gamma(m_t, m_W)$ to a one scale problem $\Gamma(m_t)$ bei expanding in the ratio $x = m_W/m_t$ such that

$$\Gamma(m_t, \, m_W) \to \Gamma\Big(m_t, \sum a_i \, x^i\Big) \qquad (17)$$

In practice we terminate the expansion at $i = 10$. We found very satisfactory convergence of the expansion.

The NNLO results are obtained from the absorptive parts of 36 $O(\alpha_s^2)$ three-loop top quark self-energy diagrams which we denote by Σ. The unpolarized and polarized rates are then obtained from the trace

$$\Gamma + \Gamma^P = \frac{1}{m_t} \, \text{Im}\, tr\left\{(\slashed{p}_t + m_t)(1 + \gamma_5 \slashed{s}_t^\ell)\Sigma\right\} \qquad (18)$$

where s_t^ℓ is the longitudinal polarization four-vector of the top quark. One needs to avail of a covariant representation of s_t^ℓ which is given by

$$s_t^{l,\mu} = \frac{1}{|\vec{q}|}\Big(q^\mu - \frac{p_t \cdot q}{m_t^2}p_t^\mu\Big), \qquad (19)$$

The unwieldy denominator factor $|\vec{q}|$ comes in through the normalization condition $s_t \cdot s_t = -1$. Express $|\vec{q}| = \sqrt{q_0^2 - q^2} = \sqrt{(pq/m_t)^2 - q^2}$ through the (unphysical) inverse propagator

of the top quark $N = (p_t + q)^2 - m_t^2$. Then expand in terms of inverse powers of N up to the desired order

$$\frac{1}{|\vec{q}|} = \frac{2m_t}{N} \sum_{i=0}^{\infty} \binom{2i}{i} \left(\frac{2q^2N - q^4 + 4m_t^2q^2}{4N^2} \right)^i . \tag{20}$$

One can replace q^2 by m_W^2 everywhere since one is cutting through the W^+–line when taking the absorptive parts of the three-loop diagrams.

In this way we have calculated NNLO QCD results for the three helicity fractions $\mathcal{F}_{T_+}$, $\mathcal{F}_{T_-}$ and $\mathcal{F}_L$ [11] and the polarized rate $\Gamma^P_{T_++T_-+L}$ [12]. One finds that the perturbation series' are well-behaved. There is no real obstacle but hard work and the handling of huge computer codes to calculate the NNLO corrections to the remaining structure functions.

We have checked elements of our three-loop calculation by doing the corresponding NLO calculation involving the absorptive parts of four NLO two-loop top quark self energy diagrams. The results of the $x = m_W/m_t$ expansion agrees with the corresponding expansion of the known closed-form NLO results [8, 9].

For example, when one expands up to $O(x^{10}, x^{10} \ln x)$ the NLO expansion reads

$$\hat{\Gamma}^{(1)}_{(T_++T_-+L)} = C_F \left[\frac{5}{4} + \frac{3}{2}x^2 - 6x^4 + \frac{46}{9}x^6 - \frac{7}{4}x^8 - \frac{49}{300}x^{10} + \right.$$
$$\left. - 2(1-x^2)^2(1+2x^2)\zeta(2) + \left(3 - \frac{4}{3}x^2 + \frac{3}{2}x^4 + \frac{2}{5}x^6 \right) x^4 \ln x \right],$$

$$\hat{\Gamma}^{(1)}_{(T_++T_-+L)^P} = C_F \left[-\frac{15}{4} - \frac{17}{8}x^4 - \frac{1324}{225}x^5 - \frac{31}{36}x^6 + \right.$$
$$\left. + \frac{48868}{11025}x^7 - \frac{23}{288}x^8 + \frac{884}{6615}x^9 - \frac{3}{100}x^{10} + (1+4x^2)\zeta(2) \right]. \tag{21}$$

Note that the expansion of the parity-conserving (*pc*) rate $(T_+ + T_- + L)$ involves only even powers of x while the expansion of the parity-violating (*pv*) polarized rate $(T_+ + T_- + L)^P$ involves even and odd powers of x. This pattern holds true for all *pc* and *pv* $O(\alpha_s)$ and the known $O(\alpha_s^2)$ T-even structure functions for which we lack a deep understanding. Our hope is that some chance reader of this presentation can provide us with a solution to this empirical paradigma.

Acknowledgement

References

[1] S. Groote and J. G. Körner, Phys. Rev. D **96** (2017) no.11, 111301 [arXiv:1711.06432 [hep-ph]].

[2] J. G. Körner and D. Pirjol, Phys. Rev. D **60** (1999) 014021 [hep-ph/9810511].

[3] S. Groote, W. S. Huo, A. Kadeer and J. G. Körner, Phys. Rev. D **76** (2007) 014012 [hep-ph/0602026].

[4] M. Fischer, S. Groote and J. G. Körner, Phys. Rev. D **97** (2018) no.9, 093001 [arXiv:1802.02492 [hep-ph]].

[5] A. Arhrib and A. Jueid, JHEP **1608** (2016) 082 [arXiv:1606.05270 [hep-ph]].

[6] G.A. Gonzalez-Sprinberg, R. Martinez and J. Vidal, JHEP **1107** (2011) 094; Erratum: [JHEP **1305** (2013) 117] [arXiv:1105.5601 [hep-ph]].

[7] M. Aaboud *et al.* [ATLAS Collaboration], JHEP **1704** (2017) 124 [arXiv:1702.08309 [hep-ex]].

[8] M. Fischer, S. Groote, J. G. Körner, M. C. Mauser and B. Lampe, Phys. Lett. B **451** (1999) 406 [hep-ph/9811482].

[9] M. Fischer, S. Groote, J. G. Körner and M. C. Mauser, Phys. Rev. D **65** (2002) 054036 [hep-ph/0101322].

[10] S. H. Do, M. Fischer, S. Groote and J. G. Körner, to be published

[11] A. Czarnecki, J. G. Körner and J. H. Piclum, Phys. Rev. D **81** (2010) 111503 [arXiv:1005.2625 [hep-ph]].

[12] A. Czarnecki, S. Groote, J. G. Körner and J. H. Piclum, Phys. Rev. D **97** (2018) no.9, 094008 [arXiv:1803.03658 [hep-ph]].

[13] I. R. Blokland, A. Czarnecki, M. Slusarczyk and F. Tkachov, Phys. Rev. Lett. **93** (2004) 062001 [hep-ph/0403221].