Fractal structure of hadrons and non-extensive statistics [*]

Eugenio Megías[1,2,**], *Airton* Deppman[3,***], *Tobias* Frederico[4,****], and *Débora P.* Menezes[5,†]

[1]Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Avenida de Fuente Nueva s/n, 18071 Granada, Spain

[2]Departamento de Física Teórica, Universidad del País Vasco UPV/EHU, Apartado 644, 48080 Bilbao, Spain

[3]Instituto de Física, Universidade de São Paulo, Rua do Matão Travessa R Nr. 187, Cidade Universitária, CEP 05508-090 São Paulo, Brazil

[4]Instituto Tecnológico da Aeronáutica, 12228-900 São José dos Campos, Brazil

[5]Departamento de Física, CFM, Universidade Federal de Santa Catarina, CP 476, CEP 88040-900 Florianópolis, Brazil

Abstract. The role played by non-extensive thermodynamics in physical systems has been under intense debate for the last decades. Some possible mechanisms that could give rise to non-extensive statistics have been formulated along the last few years, in particular the existence of a fractal structure in thermodynamic functions for hadronic systems. We investigate the properties of such fractal thermodynamical systems, in particular the fractal scale invariance is discussed in terms of the Callan-Symanzik equation. Finally, we propose a diagrammatic method for calculations of relevant quantities.

1 Introduction

Entropy emerges as an important quantity in different areas that try to describe systems of increasing complexity. The formulation of new entropic forms that generalize the one proposed by Boltzmann constitutes an important research line. In particular, the non-additive entropy introduced by Tsallis [1] has found wide applicability in the last few years, see e.g. [2, 3]. However, the full understanding of the non-extensive statistics formulated by Tsallis has not been accomplished yet. Several connections between Boltzmann and Tsallis statistics have been proposed so far, see e.g. [4–6], but it seems that the physical meaning of the entropic index q is not understood in the general case. In the present work, we make a detailed analysis of the connection based on a system featuring fractal structure in its thermodynamic properties. Fractals are conceived as objects with an internal structure that can be considered as an ideal gas of a specific number of subsystems, which, in turn, are also fractals of the same kind. The self-similarity between fractals at different levels of the internal structure reveals the typical scale invariance. The results obtained in the present work offer a new perspective in the analysis of hadron structure.

---

[*]Presented by E. Megías at the QCD@Work: International Workshop on QCD, 25-28 June 2018, Matera, Italy.

[**]e-mail: emegias@ugr.es

[***]e-mail: deppman@if.usp.br

[****]e-mail: tobias@ita.br

[†]e-mail: debora.p.m.26@gmail.com

2 Tsallis statistics and QCD thermodynamics

Tsallis statistics constitutes a generalization of Boltzmann-Gibbs (BG) statistics, under the assumption that the entropy of the system is non-additive. For two independent systems A and B

$$S_{A+B} = S_A + S_B + (1-q)S_A S_B, \qquad (1)$$

where the entropic index q measures the degree of non-extensivity [1]. Let us define the q-exponential $e_q^{(\pm)}(x) = [1 \pm (q-1)x]^{\pm 1/(q-1)}$, with $e_q^{(+)}(x)$ defined for $x \geq 0$ and $e_q^{(-)}(x)$ for $x < 0$, and the q-log function $\log_q^{(\pm)}(x) = \pm(x^{\pm(q-1)} - 1)/(q-1)$. Then the grand-canonical partition function for a non-extensive ideal quantum gas at finite chemical potential is [7]

$$\log \Xi_q(V,T,\mu) = -\xi V \int \frac{d^3 p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \log_q^{(-r)}\left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)}\right), \qquad (2)$$

where $x = (E_p - \mu)/(kT)$, the particle energy is $E_p = \sqrt{p^2 + m^2}$ with m the mass and μ the chemical potential, $\xi = \pm 1$ for bosons and fermions respectively, and Θ is the step function. Eq. (2) reduces to the Bose-Einstein and Fermi-Dirac partition functions in the limit $q \to 1$.

The thermodynamics of Quantum Chromodynamics (QCD) in the confined phase can be studied within the Hadron Resonance Gas (HRG) approach, which is based on the assumption that physical observables in this phase admit a representation in terms of hadronic states which are treated as non-interacting and point-like particles [8]. These states are taken as the conventional hadrons listed in the review by the Particle Data Group. Within this approach the partition function is then given by [7, 9]

$$\log \Xi_q(V,T,\{\mu\}) = \sum_i \log \Xi_q(V,T,\mu_i), \qquad (3)$$

where μ_i refers to the chemical potential for the *i-th* hadron. The thermodynamic functions
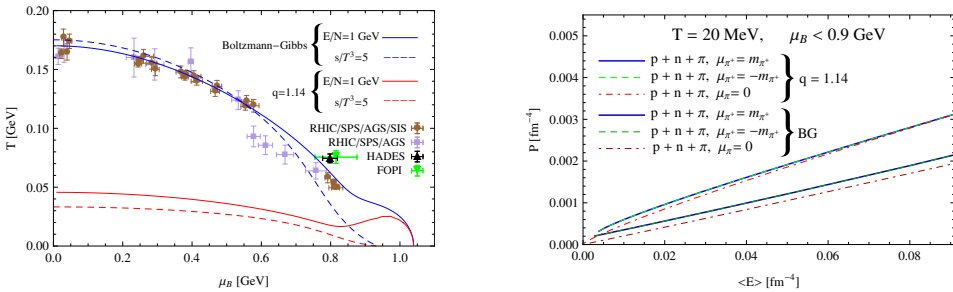

Figure 1. Left panel: Chemical freeze-out line $T = T(\mu_B)$. Right panel: EoS of hadronic matter.

can be obtained from Eq. (3) by using the standard thermodynamic relations. There are different proposals for the conditions determining the transition line between the confined and the deconfined regimes of QCD. Using the arguments of [10], the phase transition line in the $T \times \mu_B$ diagram can be determined by the condition $\langle E \rangle / \langle N \rangle = 1\,\text{GeV}$. A different method based on the entropy density has been proposed in [11]. The result for the chemical freeze-out line is displayed in Fig. 1 (left). We also display in Fig. 1 (right) the Equation of State (EoS) for hadronic matter at finite baryonic chemical potential. It it remarkable that $P(E)$ becomes larger in Tsallis statistics as compared to BG statistics (see a discussion in [7, 9]).

3 Thermofractals and Tsallis statistics

The emergence of the non-extensive behavior has been attributed to different causes: long-range interactions, correlations and memory effects [4]; temperature fluctuations [5]; and finite size of the system. In this work we will study a natural derivation of non-extensive statistics in terms of thermofractals. These are systems in thermodynamical equilibrium presenting the following properties [6, 12]:

1. The total energy is given by

$$U = F + E,\qquad\qquad(4)$$

   where F is the kinetic energy of N constituent subsystems, and E is the internal energy of those subsystems, which behave as particles with an internal structure.

2. The constituent particles are thermofractals. The energy distribution $P_{\text{TF}}(E)$ is self-similar or self-affine, so that at some level n of the hierarchy of subsystems, $P_{\text{TF}(n)}(E)$ is equal to the distribution in the other levels.

3. At level n in the hierarchy of subsystems the phase space is so narrow that one can consider $P_{\text{TF}(n)}(E_n)dE_n = \rho dE_n$, with ρ being independent of the energy E_n.

The energy distribution of a thermodynamical system is given, according to BG statistics, by

$$P_{\text{BG}}(U)dU = A\exp(-U/kT)dU,\qquad\qquad(5)$$

where A is a normalization constant. However, in the case of thermofractals the phase space must include the momentum degrees of freedom of free particles as well as their internal degrees of freedom. According to property 2 of self-similar thermofractals, the internal energy is given by [6]

$$dE = \frac{F}{kT}\left[P_{\text{TF}}(\varepsilon)\right]^\nu d\varepsilon,\qquad\qquad(6)$$

where $\varepsilon/(kT) = E/F$ and ν is to be determined. Then, the total energy distribution is given by

$$P_{\text{TF}(0)}(U)dU = A'F^{\frac{3N}{2}-1}\exp\left(-\frac{\alpha F}{kT}\right)dF\left[P_{\text{TF}(1)}(\varepsilon)\right]^\nu d\varepsilon,\qquad \alpha = 1 + \frac{\varepsilon}{kT}.\qquad(7)$$

This expression relates the distributions at level 0 and level 1 of the subsystem hierarchy. After integration in F, one finds

$$\Omega = \int P_{\text{TF}(0)}(U)dU = \int_0^\infty A\cdot\left[1 + \frac{\varepsilon}{kT}\right]^{-3N/2}\left[P_{\text{TF}(1)}(\varepsilon)\right]^\nu d\varepsilon,\qquad A = \Gamma\left[\frac{3}{2}N\right](kT)^{\frac{3}{2}N}A',\qquad(8)$$

where $\Gamma[z]$ is the Euler gamma function, and obviously $\Omega = 1$. At this point it is possible to impose the identity

$$P_{\text{TF}(0)}(U) \propto P_{\text{TF}(1)}(\varepsilon),\qquad\qquad(9)$$

corresponding to a self-similar solution for the thermofractal probability distribution. Then, the simultaneous solution of Eqs. (8) and (9) is obtained with [12, 13]

$$P_{\text{TF}(n)}(\varepsilon) = A_n\cdot\left[1 + (q-1)\frac{\varepsilon}{k\tau}\right]^{-\frac{1}{q-1}},\qquad\qquad(10)$$

where $q - 1 = 2(1-\nu)/(3N)$ and $\tau = N(q-1)T$. The energy distribution of thermofractals then obeys Tsallis statistics, and this constitutes one of the most important results of the present work. [1]

---

[1]In the following we will define $N' = N + 2/3$ as the effective number of particles taking into account their internal degrees of freedom (see [13] for details).

4 Callan-Symanzik equation and diagrammatic representation

Due to the evident similarities between hadron structure and thermofractal structure [12, 14–17], it is possible to show that the thermofractal description has close connections to quantum field theory as far as scaling properties are concerned. This will be done in a future work [18], but we will advance some aspects as follows.

4.1 Scale invariance and Callan-Symanzik equation

Thermofractals are scale invariant. Formally, this property should be accomplished with the scale invariance of the distribution of kinetic and internal energy. Using $F^{(0)}/T^{(0)} = F^{(n)}/T^{(n)}$ and $\varepsilon/(kT) = E^{(n)}/F^{(n)}$, one finds [13]

$$\lambda_n := \frac{E^{(n)}}{E^{(0)}} = \left(\frac{1}{N}\right)^{\frac{n}{1-D}} ,\tag{11}$$

where D is the fractal dimension. Using thermofractals it can be obtained the fractal dimension of hadrons, resulting in $D = 0.69$ [6], a value close to that resulting from intermittence analysis [19]. Using these scaling properties, the vertex function of thermofractals writes [13]

$$\Gamma(E,\varepsilon,T) \propto (kT')^{-(1-D)} g \prod_{i=1}^{N'} \left(2\pi \frac{E_i}{kT'}\right)^{-3/2} [P_{\text{TF}}(\varepsilon_i)]^{\nu} ,\tag{12}$$

where $T' = T^{(n)}$. Introducing $M = kT'$ as the scale parameter that determines the fractal structure of the subsystem at a certain level, then one can derive the Callan-Symanzik equation for thermofractals, which writes

$$\left[M\frac{\partial}{\partial M} + \sum_{i=1}^{N'} E_i \frac{\partial}{\partial E_i} + d\right]\Gamma = 0,\tag{13}$$

where $d = 1 - D$ is the anomalous dimension for thermofractals. This result is equivalent to the one obtained in [20].

4.2 Diagrammatic representation

Finally, let us mention that by using the formalism presented in Sec. 3, it is possible to have a diagrammatic representation of the probability densities of thermofractals that can facilitate the calculation of the partition function and other relevant quantities [13]. The basic diagrams are summarized in Fig. 2 (left). An example of diagram which gives the probability to find a constituent subsystem fractal f at the third level of the initial fractal i is displayed in Fig. 2 (right) (see [13] for further details).

5 Conclusions

We have introduced the non-extensive statistics in the form of Tsallis statistics of a quantum gas at finite temperature and chemical potential, and we have shown its application to study the EoS of QCD. In a second step we have investigated the structure of a thermodynamical system presenting fractal structure. It has been shown that the fractal structure in thermodynamics naturally leads to non-extensive statistics in the form of Tsallis. Using this formalism, we have introduced a diagrammatic formulation for practical calculations. Finally, based on the scale invariance of thermofractals, the Callan-Symanzik equation has been obtained. These results open the opportunity to develop a *'field theoretical approach'* for thermofractals, leading to a possible theoretical understanding of the non-extensive properties of hadronic systems. This study will be addressed in details in [18].
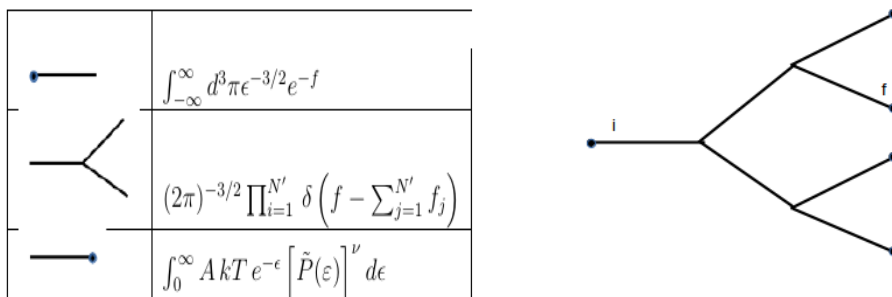
Figure 2. Left panel: Basic diagrams for thermofractals and their mathematical expressions. Right panel: Example of a tree graph representing different levels of a thermofractal.

Acknowledgement

References

[1] C. Tsallis, J. Statist. Phys. **52**, 479 (1988).
[2] P. Tempesta, Phys. Rev. **A84**, 021121 (2011).
[3] N. Kalogeropoulos, Int. J. Mod. Phys. **B28**, 1450162 (2014).
[4] L. Borland, Phys. Lett. **A245**, 67 (1998).
[5] G. Wilk, Z. Wlodarczyk, Phys. Rev. **C79**, 054903 (2009).
[6] A. Deppman, Phys. Rev. **D93**, 054001 (2016).
[7] E. Megias, D.P. Menezes, A. Deppman, Physica **A421**, 15 (2015).
[8] R. Hagedorn, Lect. Notes Phys. **221**, 53 (1985).
[9] D.P. Menezes, A. Deppman, E. Megias, L.B. Castro, Eur. Phys. J. **A51**, 155 (2015).
[10] J. Cleymans, K. Redlich, Phys. Rev. **C60**, 054908 (1999).
[11] A. Tawfik, Nucl. Phys. **A764**, 387 (2006).
[12] A. Deppman, E. Megias, EPJ Web Conf. **141**, 01011 (2017).
[13] A. Deppman, T. Frederico, E. Megias, D.P. Menezes, Entropy **20**, 633 (2018).
[14] A. Deppman, Physica **A391**, 6380 (2012).
[15] E. Megias, D.P. Menezes, A. Deppman, EPJ Web Conf. **80**, 00040 (2014).
[16] A. Deppman, E. Megias, D. Menezes, J. Phys. Conf. Ser. **607**, 012007 (2015).
[17] A. Deppman, Universe **3**, 62 (2017).
[18] A. Deppman, T. Frederiro, E. Megias, D.P. Menezes, in progress (2018).
[19] D. Ghosh, A. Deb, R. Chattopadhyay, S. Sarkar, A.K. Jafry, M. Lahiri, S. Das, K. Purkait, B. Biswas, J. Roychoudhury, Phys. Rev. **C58**, 3553 (1998).
[20] A. Deppman, Adv. High Energy Phys. **2018**, 9141249 (2018).