

Noise Characterization: Keeping Reduction Based Perturbed Quantum Walk Search Optimal

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Abstract. In a recent work by Novo et al. (Sci. Rep. **5**, 13304, 2015), the invariant subspace method was applied to the study of continuous-time quantum walk (CTQW). In this work, we adopt the aforementioned method to investigate the optimality of a perturbed quantum walk search of a marked element in a noisy environment on various graphs. We formulate the necessary condition of the noise distribution in the system such that the invariant subspace method remains effective and efficient. Based on the noise, we further formulate how to set the appropriate coupling factor to preserve the optimality of the quantum walker.

1 Introduction

Quantum walks (QW) can be formulated in discrete time (DTQW) [1] and continuous time [2] (CTQW) versions. The connection between discrete time quantum walk and continuous time quantum walk has been well studied. It was shown that continuous-time quantum walk can be obtained as an appropriate limit of discrete-time quantum walks [3]. In this work, we adopt the invariant subspace method from [4], which allows us to perform a dimensionality reduction to simplify the analyses of CTQW on a uniform complete multi-partite graph. The invariant approach reduces the original complex graph to a much smaller graph. The reduced graph captures the dynamics of the original graph such that we can investigate, such as the optimality of a quantum walk search, and test by using current quantum technology. Furthermore, the analysis becomes more transparent and the dynamics of the walker can be more intuitively understood on an abstract level.

The contribution from this work is as follows. We extend reducible graphs from complete graphs, complete bipartite graphs and star graphs discussed in [4] to uniform complete multi-partite graphs. We further parameterize the coupling factor based on the configuration of a given uniform complete multi-partite graph to keep the CTQW search on uniform complete multi-partite graphs optimal. Finally we characterize the error patterns such that the perturbed Hamiltonian remains reducible and our parameterized coupling factor keeps

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the quantum walker search remain optimal.

In section 2, we describe the work that applies the Lanczos algorithm to perform dimensionality reduction to obtain the right form of the reduced adjacency matrix. We briefly show the theorems to show (a) how to choose the correct coupling factor based on the given parameters (configuration) on a reduced adjacency matrix and (b) the optimality is preserved once transformed back to the original adjacency matrix. In section 3, under three types of errors, systematic disorder, static diagonal disorder, and reducible non-diagonal disorder, we characterize the errors such that systematic dimensionality reduction to a small Hamiltonian is feasible. We draw our conclusion in section 4.

2 Background

Throughout the text we refer to a multi-partite graph as a P -partite with a slight twist on the standard notation. The difference is that the whole graph has actually $P + 1$ partitions where the extra one partition is the partition that contains the solution (marked vertex). A uniform complete P -partite graph (UCPG) can be denoted as $G(V_0, V_1, \dots, V_P)$. Let V be the union of all partitions. A UCPG is a graph with $P + 1$ partitions of vertices with the following properties: (1) each vertex v_i in vertex partition V_j connects to all other vertices in vertex partition V_k as long as $j \neq k$ (2) except vertex partition V_0 , each of the vertex partitions has the same size. Let the size of the vertex partition V_j be m_j , i.e. $m_j = |V_j|$. Since if a UCPG G has N vertices, then we know $P \times m_1 + m_0 = N$ since $m_1 = m_2 = \dots = m_P$.

Algorithm 1 Dimensionality Reduction and Coupling Factor Determination

Require: A UCPG G of N nodes with one marked element $|\omega\rangle$

Ensure: $|\omega\rangle$ can be found efficiently by CTQW

Start of process

- Dimensionality Reduction: Construct the reduced 3 by 3 Hamiltonian H_{ra} by use of Lanczos algorithm on the N by N adjacency matrix H_a based on a UCPG G
- Hamiltonian Construction: CTQW Hamiltonian $H_{seek} = -\gamma H_{ra} - |\omega\rangle\langle\omega|$
- Basis Change: Express $H_{seek} = H^{(0)} + H^{(1)}$ in the eigenbasis $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$ of $H^{(0)}$ by applying perturbation theory
- CTQW Initialization: Determine coupling factor γ to induce fast transport between two lowest eigenenergy states $|\omega\rangle$ and $|e_1\rangle$ in H_{seek}
- Existence of Constant Overlap: Demonstrate the initial starting state $|s\rangle$ and $|e_1\rangle$ have a non-exponentially small overlap such that $|s\rangle$ can reach $|\omega\rangle$ efficiently via $|e_1\rangle$. Therefore the optimality (quadratic speed-up) is thus preserved.

End of process

2.1 Dimensionality Reduction

Without loss of generality, let us assume the marked vertex $|\omega\rangle$ is in V_0 and $m_0 \gg 1$. The original adjacency matrix is

$$\begin{aligned}
 H_a &= \sum_{i \in V, i \neq V_0} |i\rangle\langle\omega| + \sum_{j \in V_0, j \neq \omega} \sum_{i \in V, i \neq V_0} |i\rangle\langle j| \\
 &+ \sum_{k=1}^P \sum_{j \in V_k} \sum_{i \in V, i \neq V_k} |i\rangle\langle j|.
 \end{aligned} \tag{1}$$

With renormalization, the subspaces are spanned by $|\omega\rangle, |S_{V_0-\omega}\rangle, |S_{V_1}\rangle, \dots, |S_{V_P}\rangle$ where

$$|S_{V_0-\omega}\rangle = \frac{1}{\sqrt{m_0-1}} \sum_{i \in V_0, i \neq \omega} |i\rangle, \quad |S_{V_i}\rangle = \frac{1}{\sqrt{m_i}} \sum_{j \in V_i, i \neq 0} |j\rangle. \quad (2)$$

Define $|S_{\bar{V}_0}\rangle = \frac{1}{\sqrt{N-m_0}} \sum_{i \in V, i \notin V_0} |i\rangle$, then by using Lanczos algorithm and the fact that non-solution partitions are of same size, the reduced adjacency Hamiltonian H_{ra} in $(|\omega\rangle, |S_{V_0-\omega}\rangle, |S_{\bar{V}_0}\rangle)$ basis is

$$H_{ra} = \begin{bmatrix} 0 & 0 & \sqrt{N-m_0} \\ 0 & 0 & \sqrt{(N-m_0)(m_0-1)} \\ \sqrt{N-m_0} & \sqrt{(N-m_0)(m_0-1)} & N-m_0-m_1 \end{bmatrix} \quad (3)$$

The entry $H_{ra}(3,3)$ can be easily verified because of the uniform size of non-solution partitions and $(P-1) \times m_1 = N - m_0 - m_1$.

2.2 CTQW Construction

Let $\alpha = \frac{m_0}{N}$ and $\alpha_1 = \frac{m_1}{N}$. Since H_{ra} expressed in the $(|\omega\rangle, |S_{V_0-\omega}\rangle, |S_{\bar{V}_0}\rangle)$ basis captures the same dynamics as H_a , the Hamiltonian of a CTQW [5] is

$$H_{seek} = -\gamma H_{ra} - |\omega\rangle\langle\omega| \quad (4)$$

where γ is the coupling parameter. Let

$$H_{seek} = H^{(0)} + H^{(1)}, \quad (5)$$

by Eqn.(3, 4) we know ^{1 2}

$$H^{(0)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\gamma N \sqrt{\alpha(1-\alpha)} \\ 0 & -\gamma N \sqrt{\alpha(1-\alpha)} & -\gamma N((1-\alpha) - \frac{P\alpha_1^2}{1-\alpha}) \end{bmatrix}, \quad (6)$$

$$H^{(1)} = \begin{bmatrix} 0 & 0 & -\gamma \sqrt{(1-\alpha)N} \\ 0 & 0 & 0 \\ -\gamma \sqrt{(1-\alpha)N} & 0 & 0 \end{bmatrix}. \quad (7)$$

It differs from the result in [4] because of the existence of a self-loop entry for the basis vector $S_{\bar{V}_0}$ in $H^{(0)}$. Because of this entry, it imposes a stronger constraint of equal size for non-solution partitions for feasible systematic dimensionality reduction. In Theorem 1, we show how to express a reduced Hamiltonian H_{seek} in the basis of its major matrix via perturbation theory.

Theorem 1 *For simplicity, let us name H_{seek} as H . Given a reduced Hamiltonian $H = H^{(0)} + H^{(1)}$ in the $(|\omega\rangle, |b_1\rangle, |b_2\rangle)$ basis where*

$$H^{(0)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & v_1 \\ 0 & v_1 & v_3 \end{bmatrix}, \quad H^{(1)} = \begin{bmatrix} 0 & 0 & v_2 \\ 0 & 0 & 0 \\ v_2 & 0 & 0 \end{bmatrix} \quad (8)$$

¹Clear $\alpha_1 = (1-\alpha)/P = m_1/N$

²Entry (3,3) at $H^{(0)}$ is thus $-\gamma N((1-\alpha) - (1-\alpha)/P) = -\gamma(N - m_0 - m_1)$

v_1 and v_2 are negative numbers and v_3 is a non-positive number where $v_1/v_2 = \sqrt{N\alpha} \geq 1$. Let the eigenvectors basis of $H^{(0)}$ be $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$. Let $\kappa = \frac{v_3}{v_1} > 0$ and $\beta_{\pm} = \frac{\kappa \pm \sqrt{\kappa^2 + 4}}{2}$, then eigenvector $|e_1\rangle = \frac{(|b_1\rangle + \beta_+ |b_2\rangle)}{\sqrt{1 + \beta_+^2}}$ and eigenvector $|e_2\rangle = \frac{(|b_1\rangle + \beta_- |b_2\rangle)}{\sqrt{1 + \beta_-^2}}$ with corresponding eigenvalues $\lambda_{\pm} = v_1 \beta_{\pm}$. H expressed in the $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$ eigenbasis is

$$H = \begin{bmatrix} -1 & v_2 \frac{\beta_+}{\sqrt{\beta_+^2 + 1}} & v_2 \frac{\beta_-}{\sqrt{\beta_-^2 + 1}} \\ v_2 \frac{\sqrt{\beta_+^2 + 1}}{\beta_+ - \beta_-} & \lambda_+ & 0 \\ \frac{-v_2 \sqrt{\beta_-^2 + 1}}{\beta_+ - \beta_-} & 0 & \lambda_- \end{bmatrix} \quad (9)$$

Proof: Let $|e'\rangle = |b_1\rangle + \beta |b_2\rangle$ be an eigenvector of $H^{(0)}$ with eigenvalue λ . After some calculation we obtain $\lambda = \beta v_1$ where where

$$\beta = \frac{\kappa \pm \sqrt{\kappa^2 + 4}}{2}, \quad \kappa = \frac{v_3}{v_1}. \quad (10)$$

Let eigenvector $|e'_1\rangle = |b_1\rangle + \beta_+ |b_2\rangle$ with eigenvalue λ_+ and eigenvector $|e'_2\rangle = |b_1\rangle + \beta_- |b_2\rangle$ with eigenvalue λ_- , by renormalization we have

$$|e_1\rangle = \frac{|e'_1\rangle}{\sqrt{\beta_+^2 + 1}}, \quad |e_2\rangle = \frac{|e'_2\rangle}{\sqrt{\beta_-^2 + 1}}. \quad (11)$$

By basis change and Eqn.(8) with $H^{(1)}$ in the $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$ eigenbasis, , we obtain $H^{(1)}|\omega\rangle$ as

$$v_2 |b_2\rangle = \frac{v_2 \sqrt{\beta_+^2 + 1}}{\beta_+ - \beta_-} |e_1\rangle + \frac{-v_2 \sqrt{\beta_-^2 + 1}}{\beta_+ - \beta_-} |e_2\rangle. \quad (12)$$

Hence, the Hamiltonian H can be expressed as shown in Eqn. (9).

Lemma 1 Given a derived reduced Hamiltonian H written in the $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$ basis as shown in Theorem 1, we then know that (a) Hamiltonian H is symmetric and (b) $\beta_+ > 0 > \beta_-$ and $\lambda_+ < 0, \lambda_- > 0$.

Proof: With the value of β_{\pm} as shown in Theorem 1, we know $\beta_+ \beta_- = -1$ and

$$\beta_+ (\beta_+ - \beta_-) = \beta_+^2 + 1, \quad (13)$$

$$\beta_- (\beta_+ - \beta_-) = -(1 + \beta_-^2). \quad (14)$$

With this observation, we can immediately conclude that

$$v_2 \frac{\beta_+}{\sqrt{\beta_+^2 + 1}} = \frac{v_2 \sqrt{\beta_+^2 + 1}}{\beta_+ - \beta_-}, \quad v_2 \frac{\beta_-}{\sqrt{\beta_-^2 + 1}} = \frac{-v_2 \sqrt{\beta_-^2 + 1}}{\beta_+ - \beta_-}. \quad (15)$$

Therefore, the property (a) that H is symmetric is proved. For property (b), it is immediate since $\sqrt{\kappa^2 + 4} > \kappa > 0$ and $v_1 < 0$.

For simplicity, let

$$\delta_1 = v_2 \frac{\beta_+}{\sqrt{\beta_+^2 + 1}}, \quad \delta_2 = v_2 \frac{\beta_-}{\sqrt{\beta_-^2 + 1}}. \quad (16)$$

By use of Lemma 1, H can be written in the $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$ basis as

$$H = \begin{bmatrix} -1 & \delta_1 & \delta_2 \\ \delta_1 & \lambda_+ & 0 \\ \delta_2 & 0 & \lambda_- \end{bmatrix} \tag{17}$$

where $|\omega\rangle$ and $|e_1\rangle$ can form the basis for the two states of the lowest eigenvalue.

Theorem 2 Given a Hamiltonian H in the form shown in Lemma 1, it is desirable to have $\lambda_+ = -1$ such that $|\omega\rangle$ and $|e_1\rangle$ form the basis for the two states of the lowest eigenvalue. Since $v_1 = -\gamma N(\sqrt{\alpha(1-\alpha)})$ then the degeneracy between site energies of $|\omega\rangle$ and $|e_1\rangle$ facilitates transport between these two low energy states, hence $\gamma = (N\sqrt{\alpha(1-\alpha)}\beta_+)^{-1}$. The transport between $|\omega\rangle$ and $|e_2\rangle$ is prohibited since δ_2 is much smaller than λ_- .

Proof: Since we desire to have faster transport between the lowest eigen energy states, we need to set

$$\lambda_+ = v_1\beta_+ = -1. \tag{18}$$

With the fact that $v_1 = -\gamma N(\sqrt{\alpha(1-\alpha)})$, we need to set

$$\gamma = (N\sqrt{\alpha(1-\alpha)}\beta_+)^{-1} \tag{19}$$

From Eqn.(16, 17) and λ_- , we know δ_2 is much smaller than λ_- because

$$\frac{\delta_2}{\lambda_-} = \frac{v_2}{v_1\sqrt{\beta_-^2 + 1}} = \frac{1}{\sqrt{\alpha N(\beta_-^2 + 1)}} < \frac{1}{\sqrt{\alpha N}}^3. \tag{20}$$

For a given UCPG G , by use of Eqn.(6-8), we can properly bound κ as

$$\kappa = \frac{v_3}{v_1} = \frac{\sqrt{(1-\alpha)(\frac{P-1}{P})}}{\sqrt{\alpha}}, \text{ then } 0 \leq \kappa < \sqrt{\frac{1-\alpha}{\alpha}}. \tag{21}$$

2.3 From Constant Overlap to Optimality Preserving

For a CTQW based on H_{seek} , we need to decide the value of coupling parameter γ to ensure the optimal performance of the underlying quantum walk is preserved. If the coupling parameter γ is wrongly chosen, the underlying CTQW search might not remain optimal, i.e. its quadratic speed-up might be lost. The determination process of correct γ is shown in Theorem 2. Theorem 3 is an extension of Theorem 2 to various cases with respect to the values of variable P and variable α .

Theorem 3 Given a UCPG $G = (V_0, V_1, \dots, V_P)$ and its adjacency matrix Hamiltonian H_a in the $(|\omega\rangle, |b_1\rangle, |b_2\rangle)$ basis where $N = \sum_{i=0}^P |V_i|$, we can obtain the reduced search Hamiltonian H_{seek} in a new eigenbasis $(|\omega\rangle, |e_1\rangle, |e_2\rangle)$ by use of Theorem 1 for constructing the underlying CTQW. We can then use Theorem 2 to determine the coupling factor $\gamma = (N\sqrt{\alpha(1-\alpha)}\beta_+)^{-1}$. The chosen γ ensures the underlying CTQW remains optimal.

Proof: There are two aspects that we need to address to show that the optimality $O(\sqrt{N})$ is preserved. One (1) is fast search speed and low escaping speed while the other is (2) the overlap between $|e_1\rangle$ and the initial system state $|s\rangle$ (a uniform superposition) as it determines

³since $m_0 \gg 1$, $\alpha = m_0/N$, then $\alpha N \gg 1$

how many times we need to repeat the experiment.

The search speed is determined by the dynamics between fast transport non-solution $|e_1\rangle$ and solution state $|\omega\rangle$, i.e. $|e_1\rangle \rightarrow |\omega\rangle$. The degenerate eigenspace formed by $|\omega\rangle$ and $|e_1\rangle$ captures the dynamics between those two states. The escape speed is from solution $|\omega\rangle$ to undesirable non-solution states $|e_2\rangle$. From Eqn.(17), we know that δ_1 is responsible for the search speed and δ_2 is responsible for escape speed. In Eqn.(20) We have shown that δ_2 is small with respect to λ_- , the escape speed is small. By use of Eqn.(16), we know that

$$|\delta_1| = \langle e_1 | H_{seek} | \omega \rangle = \left| \frac{v_2 \beta_+}{\sqrt{\beta_+^2 + 1}} \right| = \left| \frac{-1}{\sqrt{\alpha N (\beta_+^2 + 1)}} \right| \quad (22)$$

because

$$v_2 = -\gamma \sqrt{N(1 - \alpha)}, \gamma = (N \sqrt{\alpha(1 - \alpha)} \beta_+)^{-1}. \quad (23)$$

Hence, we obtain the running time

$$T_{run} = \frac{\pi}{2|\delta_1|} = \frac{\pi}{2} \sqrt{\alpha N (\beta_+^2 + 1)}. \quad (24)$$

Let us verify that the running time T_{run} remains optimal in different settings of UCPG G when the coupling factor γ is chosen based on Theorem 2. Briefly speaking, with a fixed search space of size N , the configuration of a UCPG G is controlled by variable P and variable α . We will discuss different settings based on those two variables.

Case 1: $P = 1$

This is a typical complete bipartite graph as seen in [4]. We immediately know that $\kappa = 0$ since $\alpha_1 = 1 - \alpha$ from Eqn.(21). This leads to $\beta_+ = 1$ from Eqn.(10). Because of that, no matter what value of α is, T_{run} at Eqn.(24) holds its quadratic speed-up.

Case 2: $2 \leq P \leq N - 1$ and $\alpha \propto \frac{1}{N}$

By Eqn.(21), we know that $\kappa \propto \sqrt{N - 1}$ and by Eqn.(10), we know $\beta_+ \propto \sqrt{N - 1}$. By plugging in the values of α and β_+ , T_{run} at Eqn.(24) still holds its quadratic speed-up.

Case 3: $2 \leq P \leq N - 1$ and $\alpha \propto 1$ (such as $\frac{N-1}{N}$)

By Eqn.(21), we know that $\kappa \propto 1/\sqrt{N - 1}$; by Eqn.(10), we know

$$\beta_+ \propto \frac{(1/\sqrt{N - 1}) + \sqrt{(1/N - 1) + 4}}{2} \simeq 1 \quad (25)$$

when N is large. By plugging in the values of α and β_+ , T_{run} at Eqn.(24) still holds its quadratic speed-up.

Case 4: $2 \leq P \leq N - 1$ and α is some constant (non-extreme values):

Immediately we know κ and β_+ are some constants that would not affect the complexity. Hence, T_{run} at Eqn.(24) still holds its quadratic speed-up.

However, the T_{run} above assumes that we start the search from eigenstate $|e_1\rangle$ to find $|\omega\rangle$, i.e. $|e_1\rangle \rightarrow |\omega\rangle$, which is not the case because we start from $|s\rangle$. Hence, at T_{run} the success

probability of observing $|\omega\rangle$ is the overlap between $|e_1\rangle$ and $|s\rangle$. The success probability is ⁴

$$P_O = |\langle e_1 | s \rangle|^2 = \left| \frac{\sqrt{\frac{\alpha}{\beta_+^2} - \frac{1}{\beta_+^2 N}} + \sqrt{1 - \alpha}}{\sqrt{1 + \frac{1}{\beta_+^2}}} \right|^2. \tag{26}$$

Therefore $1/P_O$ is the number of times we need to repeat the experiment. We need to show that P_O is some constant such that it would not affect the total complexity under the Big O notation. By examining the four cases listed in Theorem 3 and the corresponding values of α and β_+ into Eqn.(26), we know that P_O remains as some constant that is not exponentially small. The total runtime is $T_{run} \times \frac{1}{P_O}$ where T_{run} holds quadratic speed-up and $\frac{1}{P_O}$ is some constant that is not large (not scaling with N), the complexity still holds the quadratic speed-up. Therefore, we know that the chosen $\gamma = (N \sqrt{\alpha(1 - \alpha)}\beta_+)^{-1}$ ensures that the underlying CTQW remains optimal.

3 Noise Error Patterns and Optimality Preserving

Now we consider how to keep the dimensionality reduced quantum walk search optimal by characterizing the noise patterns in the system. The noises can be introduced due to the precision limitation and the noisy environment. For instance, not all numbers have a perfect binary representation and the approximated numbers would cause perturbation. Let matrix H'_a be the closest Hamiltonian to H_a that can be prepared by an available quantum system of limited precision. In the remaining of this section, we examine the effect of (1) systematic errors, (2) static errors, and (3) non-static errors on CQW and the coupling factor while the goal is to keep the search by CTQW optimal and to keep the feasibility of systematic dimensionality reduction. For simplicity, let us assume we are working on only complete graphs (CG) for the noise characterization.

3.1 Systematic Disorder

If the error is systematic, that is each adjacency matrix entry that connects two different sites in H'_a suffers an ϵ error in comparison to the original H_a . It is clear to see $H'_a = (1 - \epsilon)H_a$. This N by N H'_a matrix can be efficiently reduced to $(1 - \epsilon)H_{ra}$ smoothly by use of Lanczos algorithm. This can apply to all UCPG graphs, including CG, CBG and SG. The new coupling factor γ' scales by a factor of $\frac{\gamma}{1 - \epsilon}$ accordingly to keep the search optimal with

$$\gamma' = (N(\sqrt{\alpha(1 - \alpha)}\beta_+)(1 - \epsilon))^{-1}. \tag{27}$$

In reality, the distribution of errors is seldom perfectly systematic when environmental noise is considered. When an arbitrary noise distribution is introduced into a quantum system of a higher dimension, the task becomes daunting. The computation complexity exponentiates due to high dimensionality and the irregularity in noise distribution.

⁴Simply compute their inner product and we know that $|s\rangle = \frac{|\omega\rangle + \sqrt{m_0 - 1}|s_{V_0 - \omega}\rangle + \sqrt{N - m_0}|s_{V_0}\rangle}{\sqrt{N}}$

3.2 Static Diagonal Disorder

Environmental assisted quantum search [6] suggests that naturally occurring open quantum system dynamics can be advantageous for a quantum algorithm based on quantum walks affected by static diagonal errors. The scenario considered is diagonal disorder due to imperfect Oracle. An imperfect Oracle that marks each node of the graph erroneously that non-solution node j is marked with an energy ϵ_j while the solution node ω is marked with an energy $-1 + \epsilon_\omega$. With the state $|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle$, the perturbed unreduced search Hamiltonian with tuning factor $\gamma = 1/N$ is

$$H_{seek} = -|\omega\rangle\langle\omega| - |s\rangle\langle s| + \sum_{i=0}^{N-1} \epsilon_i |i\rangle\langle i| \quad (28)$$

where the static disorder ϵ_i are i.i.d random variables with mean 0 and standard deviation $\sigma \ll 1$. With degenerate perturbation theory, the reduced search Hamiltonian in the $\{|\omega\rangle, |s_\omega\rangle\}$ basis is

$$H_{seek} = \begin{bmatrix} -1 + \epsilon_\omega & -1/\sqrt{N} \\ -1/\sqrt{N} & -1 \end{bmatrix} \quad (29)$$

where $|s_\omega\rangle$ is the equal superposition of all nodes other than the solution node $|\omega\rangle$. The gap between the ground state and the first excited state of the perturbed Hamiltonian is

$$\Delta = \sqrt{\epsilon_\omega^2 + 4/N} \quad (30)$$

With $\gamma = 1/N$, the success probability of the algorithm is

$$P_\omega(t) \simeq \frac{\sin^2(\Delta t)}{1 + N\epsilon_\omega^2/4} \quad (31)$$

The maximum success probability is obtained when $t = \pi/\Delta \simeq O(\sqrt{N})$. From experimental aspect, this algorithm can always choose a fixed tuning factor, then the optimality is preserved by calculating the running time solely based on ϵ_ω that is some variable with value between 0 and 1. Interested readers can refer to this article [6] for technical details.

3.3 Reducible Non-Diagonal Noise

Suppose non-negative errors ϵ_{ij} , where $\forall i, j \in [1, N]$ and $i \neq j$, occur across the original adjacency matrix H_a . To make this scenario not be mixed with the static diagonal disorder, $e_{ii} = 0$, let the perturbed adjacency matrix be H'_a with $H'_{a_{ij}} = H_{a_{ij}} - \epsilon_{ij}$ and let the index for state $|\omega\rangle$ be 1 in the system. Without loss of generality, let $|\omega\rangle$ be the solution state and the first normal basis vector.

In the first iteration of Lanczos, we know $\langle\omega|\omega_1\rangle = 0$, where $|\omega_1\rangle = H'_a|\omega\rangle = \sum_{i=2}^N (1 - \epsilon_{i1})|i\rangle$. Let $|v_2\rangle = \frac{|\omega_1\rangle}{\| |\omega_1\rangle \|}$ be the second normal basis vector. In the second iteration, let $|\omega'_2\rangle = H'_a|v_2\rangle$. For the Lanczos to terminate at this stage, the first condition is that $|\omega'_2\rangle$ must be a linear combination of $|\omega\rangle$ and $|v_2\rangle$ that

$$|\omega'_2\rangle = c_1|\omega\rangle + c_2|v_2\rangle. \quad (32)$$

When the first condition is met, we have $\langle\omega'_2|v_2\rangle = c_2$ since $|v_2\rangle \perp |\omega\rangle$. By running Lanczos algorithm, it is necessary to have the following second condition

$$|\omega_2\rangle = |\omega'_2\rangle - \langle\omega'_2|v_2\rangle|v_2\rangle - \langle\omega'_2|\omega\rangle|\omega\rangle = 0 \quad (33)$$

such that Lanczos algorithm terminates. It is thus desired to have $c_1 = \|\omega_1\|$ due to Eqn. (32, 33). The corresponding reduced Hamiltonian $H_{ra'}$ in $(|\omega\rangle, |v_2\rangle)$ basis with noise is

$$H_{ra'} = \begin{bmatrix} 0 & c_1 \\ c_1 & c_2 \end{bmatrix} \quad (34)$$

This immediately implies

$$c_1 = \sqrt{\sum_{i=2}^N (1 - \epsilon_{i1})^2} \quad (35)$$

for the second condition to be satisfied. Next we need to examine the constraint on c_1, c_2 such that the first condition is satisfied. We need to show that

$$(H'_a |v_2\rangle) = \frac{1}{\|\omega_1\|} \left(\left(\sum_{i=2}^N (1 - \epsilon_{i1})(1 - \epsilon_{i1}) |\omega\rangle \right) + \sum_{k=2}^N \sum_{i=2, i \neq k}^N (1 - \epsilon_{ki})(1 - \epsilon_{i1}) |k\rangle \right) \quad (36)$$

is the same as Eqn. (32). This immediately gives us the requirement for c_2 the noise pattern that $\forall k \in [2, N]$

$$\sum_{i=2, i \neq k}^N (1 - \epsilon_{ki})(1 - \epsilon_{i1}) = c_2(1 - \epsilon_{k1}). \quad (37)$$

For c_1 , by use of Eqn. (35, 36) and the fact that $c_1 = \|\omega_1\|$, we know c_1^2 has

$$c_1^2 = \sum_{i=2}^N (1 - \epsilon_{i1})^2 = \sum_{i=2}^N (1 - \epsilon_{i1})(1 - \epsilon_{i1}). \quad (38)$$

4 Discussion

A high dimension Hamiltonian will not be feasible to implement using near-term quantum technology if we try to encode the given original configuration. With our reduction scheme and coupling factor determination approach, we can implement the dynamics of the high dimension Hamiltonian with very few quantum bits and the quantum walker will remain optimal while searching on this reduced system. Furthermore, since quantum system is susceptible to the noise, to keep reduction scheme possible we characterize the noise pattern for three types of error distribution: systematic disorder, static diagonal disorder, and reducible non-diagonal disorder. For the first two cases, it is apparent that no specific pattern is required while for the reducible non-diagonal errors, we know the pattern must satisfy Eqn.(37,38) such that the pattern guarantees systematic dimensionality reduction.

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