Parametrizations of three-body hadronic $B$- and $D$-decay amplitudes

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Abstract. A short review of our recent work on amplitude parametrizations of three-body hadronic weak $B$ and $D$ decays is presented. The final states are here composed of three light mesons, namely the various charge $\pi\pi\pi$, $K\pi\pi$ and $KK\bar{K}$ states. These parametrizations are derived from previous calculations based on a quasi-two-body factorization approach where the two-body hadronic final state interactions are fully taken into account in terms of unitary $S$- and $P$-wave $\pi\pi$, $\pi K$ and $K \bar{K}$ form factors. They are an alternative to the isobar-model description and can be useful in the interpretation of CP asymmetries.

1 Introduction

1.1 Motivations: why study three-body hadronic $B$ and $D$ decays? 

Three-body hadronic $B$ and $D$ decays provide a rich tool to study not only the Standard Model, QCD, CP violation [1] but also hadron physics. The hadron physics, often characterized by two-body resonances and their interferences, affect weak observables and any reliable determination of the later will require a good knowledge of the final state meson-meson interactions. This can be realized by introducing theoretical constraints such as unitarity, analyticity, chiral symmetry and the use of data from reactions other than $B$ and $D$ decays. Basic Dalitz-plot analyzes rely on sums of relativistic Breit-Wigner amplitudes representing the different possible implied resonances to which some non resonant background amplitude is added. The $S$-wave resonance contributions are often difficult to fit. Can one go beyond this isobar model approach?

One can replace the sums of relativistic Breit-Wigner components by parametrizations [2] in terms of unitary two-meson form factors keeping the weak-interaction dynamics governing the flavor-changing process via $W$-meson exchange. These parametrizations are based on published results and motivated by analyzes of high-statistics present and forthcoming data at BES III, LHCb, Belle II, Super c-tau factory .... Up to now there is no three-body decay factorization theorem but major contributions arise from intermediate resonances such

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as $\rho(770)$, $K^*(892)$, $\phi(1020)$ which allows to describe three-body decays as quasi-two-body ones. For instance, for the three-meson final state of the $D^0 \to K_S^0 \pi^- \pi^+$ decay, one can introduce quasi-two-body pairs, $[K_S^0 \pi^+]_L$, $[K_S^0 \pi^-]_L$, $K_S^0 [\pi^+ \pi^-]_L$, two of the three mesons forming a state of angular momentum 0 or 1 with $L = S$ or $P$, respectively.

### 1.2 QCD quasi-two-body factorization

Decays are mediated by local four-quark operators $O_i(\mu)$ forming the weak effective nonrenormalizable Hamiltonian $\mathcal{H}_{\text{eff}}$. Schematically for $B \to M_1 M_2^* (M_2^* \to M_3 M_4)$ one has

$$
\langle M_1 M_2^* | \mathcal{H}_{\text{eff}} | B \rangle = \frac{G_F}{\sqrt{2}} V_{\text{CKM}} \sum_i C_i(\mu) \langle M_1 M_2^* | O_i(\mu) | B \rangle,
$$

(1)

where $G_F$ is the Fermi decay constant, $V_{\text{CKM}}$ the product of Cabibbo-Kobayashi-Maskawa matrix elements and $C_i(\mu)$ Wilson coefficients renormalized at scale $\mu \sim m_b$ (or $m_c$ in $D$ decays). In the factorization approach [3] with the strong coupling $\alpha_s(\mu)$, i.e. at scale $\mu$,

$$
\langle M_1 M_2^* | O_i(\mu) | B \rangle = \left( \langle M_1 | J_i^0 | B \rangle \langle M_2^* | J_{2v} | 0 \rangle \right)
+ \langle M_1 | J_i^0 | 0 \rangle \langle M_2^* | J_{4v} | B \rangle \left[ 1 + \sum_n r_n \alpha_s^n(\mu) + O \left( \frac{\Lambda_{\text{QCD}}}{m_b} \right) \right],
$$

(2)

where $r_n$ are strong interaction constant factors and $| 0 \rangle$ the vacuum state. For the leading order the factorization takes place with either weak quark currents $J_1$, $J_2$ or $J_3$, $J_4$. The radiative corrections can be evaluated to a given order $\alpha_s^n(\mu)$. The nonperturbative corrections to the heavy-quark limit $O(\frac{\Lambda_{\text{QCD}}}{m_b})$ are less reliable for $D$ decays as $m_c \sim m_b/3$; therefore, even though the factorization is more phenomenological for charged mesons, it can still represent a good starting point.

The amplitude $\langle M_1 | J_i^0 | B \rangle = \langle M_1 | \bar{B} | J_i^0 | 0 \rangle$ is a heavy-to-light transition form factor which can be evaluated within light-front and relativistic constituent quark models, light-cone sum rules, continuum functional QCD and lattice QCD (see Appendix A4 of Ref. [2]). Semi-leptonic decay measurements like $D^0 \to \pi^- e^+ \nu_e$ can also allow a phenomenological determination of these form factors.

The matrix element $\langle M_2^* | J_{2v} | 0 \rangle \propto \langle M_3 M_4 | J_{2v} | 0 \rangle$, where the $M_3 M_4$ resonance, $M_2^*$, originates from a $\bar{q}q$ pair, corresponds to the $M_3 M_4$ form factor. It has been shown, in Ref. [4], that, using dispersion relations and field theory, this form factor can be fully determined, if the $M_3 M_4$ strong interaction is known at all energies. These form factors are calculated from Muskeshilvili-Omnès equations [5] using two-body data, unitarity, asymptotic QCD and chiral symmetry constraints at low energies.

The term $\langle M_1 | J_i^0 | 0 \rangle$, related to the $M_1$ weak decay constant, is known from experiment, e.g. the pion decay constant, $f_\pi$ or that of the kaon, $f_K$. It can also be evaluated with lattice-regularized QCD and other nonperturbative approaches.

The matrix element $\langle M_2^* | J_{4v} | B \rangle \propto \langle M_3 M_4 | J_{4v} | B \rangle$ corresponding to $B$ meson transitions to two-meson pairs via the $M_2^*$ resonance is the biggest uncertainty in our approach. It could be evaluated from semi-leptonic processes: like $B^0 \to K^- \pi^- \mu^+ \mu^-$ or $D^0 \to K^- \pi^+ \mu^+ \mu^-$. In the derivation of the amplitude presented here it will be related to the $\langle M_2^* | \to M_3 M_4 | J_{2v} | 0 \rangle$ form factor. Within the soft-collinear effective theory, the amplitude can be factorized in terms of generalized $B$-to-two-body form factor and two-hadron light-cone distribution amplitude [6].


1.3 Application to the $D^+ \to K^-\pi^+\pi^+$ decay

In this process, studied in Ref. [7], the final state $\pi^+\pi^+$ interaction can be neglected and the quasi-two-body $[K^-\pi^+]_{S,P} \pi^+$ can be introduced. There is no penguin contribution (loop with $W$ meson) and only the effective Wilson-coefficients $a_{1(2)}$ appear in the quasi-two-body factorized amplitude,

$$
\langle [K^-\pi^+]_{S,P} \pi^+ | H_{\text{eff}} | D^+ \rangle = \frac{G_F}{\sqrt{2}} \cos^2 \theta_C \left[ a_1 \langle [K^-\pi^+]_{S,P} | \bar{s} \gamma^\nu(1-\gamma_5) c | D^+ \rangle \langle \pi^+_2 | \bar{u} \gamma_\nu(1-\gamma_5) c | D^+ \rangle \right] + a_2 \langle [K^-\pi^+]_{S,P} | \bar{s} \gamma^\nu(1-\gamma_5) d | 0 \rangle \langle \pi^+_2 | \bar{u} \gamma_\nu(1-\gamma_5) c | D^+ \rangle \right] + \langle \pi^-_1 \leftrightarrow \pi^+_2 \rangle,
$$

(3)

$\theta_C$ being the Cabbibo angle. The matrix element $\langle [K^-\pi^+]_{S,P} | \bar{s} \gamma^\nu(1-\gamma_5) d | 0 \rangle$ is given by the $K\pi$ form factors. The term $\langle [K^-\pi^+]_{S,P} | \bar{s} \gamma^\nu(1-\gamma_5) c | D^+ \rangle$ is less straightforward to evaluate. Assuming a dominant intermediate resonance $R$, it can be written as being proportional to the $D \to R [R \to K\pi]$ transition form factor multiplied by the $K\pi$ form factors. This description is a feature of crucial importance to our proposed parametrizations. In Eq. (3), $\langle \pi^-_1(p) | \bar{u} \gamma_\nu(1-\gamma_5) c | D^+ \rangle$ is the $D\pi$ transition form factor.

Parametrized amplitudes based on quasi-two-body factorization have been given in Ref. [2] in terms of analytic and unitary meson-meson form factors for final states composed of three light mesons, namely the various charge $\pi\pi\pi, K\pi\pi$ and $KK\bar{K}$ states. For these hadronic three-body decays we have shown, in previous studies, that this approach is phenomenologically successful. Below, we illustrate these parametrizations for the $B \to K\pi^+\pi^-$ [8–10] and $D^0 \to K^+_S K^- K^-$ [11] for meson-meson final states in $S$ wave. Formulae for meson-meson final states in $P$ wave are given in Ref. [2].

2 Parametrized amplitudes for the $B \to K\pi^+\pi^-$ decays

2.1 Parametrization of the $B \to K[\pi^+\pi^-]_S$ amplitudes

Let us label the momenta as $B(p_B) \to K(p_1)\pi^+(p_2)\pi^-(p_3)$ with $s_{12} = (p_1 + p_2)^2, s_{13} = (p_1 + p_3)^2, s_{23} = (p_2 + p_3)^2$ and $s_{12} + s_{13} + s_{23} = m_B^2 + m_K^2 + 2m_\pi^2$. As can be seen from Eq. (1) of Ref. [8] the $B \to K\pi^+\pi^-_S$ amplitude can be parametrized in terms of three complex parameters, $b_i^S, i = 1, 2, 3$, for the different charged states $B = B^+, K = K^\pm$ and $B = B^0(\bar{B}^0), K = K^0(\bar{K}^0)$ or $K^0_S$. For the $B^-$ decays one has

$$
\mathcal{A}_S(s_{23}) \equiv \langle K^- [\pi^+\pi^-]_S | H_{\text{eff}} | B^- \rangle = b_1^S \left( m_B^2 - s_{23} \right) F_{00}^\pi(s_{23}) + \left( b_2^S F_{0}^B(s_{23}) + b_3^S \right) F_{00}^\pi(s_{23}),
$$

(4)

where $F_{00}^{BK}(s)$ is the $B$ to $K$ transition form factor (see Refs. [2, 8]). The non-strange scalar form factor $F_{00}^\pi(s)$ contains the contributions of $f_0(500), f_0^*(980)$ and $f_0^*(1400)$. Several models are compared in Fig. 8 of Ref. [12]. Although there are large differences, it has been checked by the authors that, with the fitted form factor to obtain the lowest $\chi^2$ for $D^0 \to K^0_S\pi^+\pi^-$, the main conclusions achieved for the $B^+ \to \pi^+\pi^-\pi^+$ in Ref. [13] were unchanged (see Ref. [12] for explanations). The modulus of the Mouchallam pion scalar form factor [14], calculated by solving the Muskhelishvili-Omnès equation [5], is close to that of the form factor obtained in Ref. [12], notably below 1 GeV. A plot of the strange scalar form factor $F_{00}^\pi(s)$, which receives the contribution of the $f_0^*(980)$ and $f_0^*(1400)$, can be found in Fig. 6 of Ref. [15]. It has been calculated using the Muskhelishvili-Omnès approach.
In terms of the original amplitude [8] one has

\[ b_1^S = \frac{G_F}{\sqrt{2}} \left[ \chi f_K F_0^{B \to f_0(980)}(m_K^2) U - C \right], \tag{5} \]

where \( C = f_\pi F_\pi \left( \lambda_u P_4 G_{1L} + \lambda_i P_1 \right) \), \( \lambda_u = V_{ub} V_{us}^* \), \( \lambda_i = V_{tb} V_{ts}^* \), \( F_\pi \) is the \( B\pi \) form factor at \( m_\pi^2 = 0 \), \( P_4 G_{1L} \), \( P_1 \) complex charming penguin parameters and \( U \) is a short-distance contribution given in terms of CKM matrix element multiplied by effective Wilson coefficients. The fitted parameter \( \chi \) represents the strength of the non-strange pion form factor contribution, furthermore. Its value can be estimated from the \( f_0(980) \) decay properties [8]. A summary of the models for the scalar-isoscalar pion form factor can be found in Appendix A4 of Ref. [2] and, as just noted above, see also the recent determination of Ref. [15] and the review talk [16].

### 2.2 Parametrization of the \( B \to [K\pi^\pm]_S \pi^\mp \) amplitudes

In terms of the two complex parameters \( c_1^S \), \( c_2^S \) (see Eq. (68) of Ref. [10]) one has

\[ \mathcal{A}_S(s_{12}) \equiv \langle \pi^- [K^+\pi^-]_S | \mathcal{H}_{\text{eff}} | B^- \rangle = \left( c_1^S + c_2^S s_{12} \right) \frac{F_0^{Br}(s_{12}) F_0^{K\pi}(s_{12})}{s_{12}}, \tag{6} \]

where \( F_0^{K\pi}(s) \) (contribution of \( K_0^0(800) \) or \( \kappa \) and of \( K_0^0(1430) \), see e.g. Fig. 7 of Ref. [12]) and \( F_0^{Br}(s) \) are the \( K\pi \) and \( Br \) scalar form factors, respectively. This parametrization has been used with success in the amplitude analysis [17] of the Dalitz-plot distribution of the LHCb \( B \to K_0^0 \pi^+ \pi^- \) data. One has [10]

\[ c_1^S = \frac{G_F}{\sqrt{2}} (M_B^2 - m_\pi^2)(m_K^2 - m_\pi^2) \left[ \lambda_u \left( d_4^S(S) - \frac{a_1^d(S)}{2} + c_4^u \right) + \lambda_c \left( d_4^S(S) - \frac{a_1^d(S)}{2} + c_4^u \right) \right], \tag{7} \]

where \( \lambda_c = V_{cb} V_{cs}^* \). The \( a_1^d(S), \) \( i = 4, 10 \) are the leading order effective Wilson coefficients including vertex and penguin corrections. The \( c_4^u \) are free fitted parameters simulating non-perturbative and higher order contributions to the penguin diagrams. Models for the \( F_0^{K\pi}(s) \) form factor are described in Ref. [2], see also some complementary aspects in Ref. [16].

### 3 \( D^0 \to K_S^0[K^+K^-]_S \) and \( D^0 \to [K_S^0 K^\pm]_S K^\mp \) parametrized amplitudes

The \([K^+K^-] \) pairs can have isospin 0 or 1 but the \([K_S^0 K^\pm] \) ones have isospin 1. The \( f_0(980), f_0(1400), a_0(980) \) and \( a_0(1450) \) contribute to the following parametrized amplitude

\[ \mathcal{A}_{S,0}(s_{23}) = h_4^S \left( m_{D^0}^2 - s_{23} \right) F_{0,0}(s_{23}) + h_5^S \left( m_{K^0}^2 - s_{23} \right) F_{0,0}(s_{23}) + h_3^S \left( m_{D^0}^2 - s_{23} \right) G_0^{KK}(s_{23}). \tag{8} \]

where \( s_{23} \) is the energy squared of the \( K^+K^- \) pair while \( s_{12} \) is associated to the \( K_S^0 K^- \) pair and \( s_{13} \) to the \( K_S^0 K^+ \) one. The decay amplitude associated with the \( a_0(980)^- \) and \( a_0(1450)^- \) resonances, can be parametrized as:

\[ \mathcal{A}_{S,-}(s_{12}) \equiv \left( h_3^S + h_5^S s_{12} \right) G_0^{KK}(s_{12}). \tag{9} \]

The amplitude carrying contributions from \( a_0(980)^+ \) and \( a_0(1450)^+ \) reads

\[ \text{[The interested reader will find, in Appendix B of Ref. [2], the corresponding relations for the other parameters.]} \]
Models for the $F_{(DK)_{\eta\pi}(s)}^{KK}(s)$ form factors entering Eq. (8) have been derived in Ref. [18, 19] (see their Figs. 1) solving three coupled channels viz. $\pi\pi, KK$ and $4\pi$ (effective $2\pi$-$2\pi$ or $\sigma\sigma$ or $\rho\rho$ ... ) and imposing chiral symmetry constraints. The $F_{(SK)_{\eta\pi}(s)}^{KK}(s)$ form factor has also been calculated in a dispersive approach in Ref. [15] (see their Fig. 7).

In Eqs. (9) and (10), the scalar-isovector $G_{0}^{KK}(s)$ form factor, built in Ref. [20] from a unitary $S$-wave coupled channel ($\eta\pi, K\bar{K}$) model, is plotted in their Fig. 7. This model, derived from the Muskhelishvili-Omnès equation [5], imposes the presence of the $a_0(980)$ and $a_0(1450)$ and includes asymptotic QCD and chiral symmetry constraints. Models for the transition form factor $F_{0}^{DK}(s)$ in Eq. (10) can be found in Ref. [2]. The above complex $h_7^0$ coefficients are given in terms of the original amplitudes in Appendix B of Ref. [2].

4 Concluding remarks

Alternatives to isobar Dalitz-plot model for weak $D, B$ decays into various $\pi\pi\pi, K\pi\pi$ and $KK\bar{K}$ charge states have been presented in Ref. [2]. Let us recall that isobar parametrizations do not respect unitarity and extraction of strong CP phases should be taken with caution. Furthermore $S$-wave resonance contributions are hard to fit.

Our parametrizations, although not fully three-body unitary, are based on a sound theoretical application of QCD factorization to a hadronic quasi-two-body decay. They assume that final three-meson state are preceded by intermediate resonant states which is justified by phenomenological and experimental evidence. Analyticity, unitarity, chiral symmetry plus correct asymptotic behavior of the two-meson scattering amplitude in $S$ and $P$ waves are implemented via analytical and unitary $S$- and $P$-wave $\pi\pi, \pi K$ and $K\bar{K}$ form factors entering in hadronic final states of our amplitude parametrizations.

These parametrized amplitudes can be readily used adjusting parameters in a least-square fit to the Dalitz plot for a given decay channel and employing tabulated form factors as functions of momentum squared or energy. The reproduction of the Dalitz-plot data might require some adjustment of the meson-meson form factors. The addition of phenomenological amplitudes (contributions of higher interacting waves, in particular $D$ waves or $J=2$ resonances), and possible three-body rescattering effects may be necessary.

We have exemplified here expressions for the $B \rightarrow K\pi^+\pi^-$ [8–10] and $D^0 \rightarrow K_S^0 K^+ K^-$ [11] for meson-meson final states in $S$ wave. In Ref. [2] one can find other explicit amplitude expressions for meson-meson final states in $S$ and $P$ wave for $B^\pm \rightarrow \pi^+\pi^-\pi^\pm, B \rightarrow K^0\pi^+\pi^-, B^\pm \rightarrow K^+K^-\pi^\pm, D^+ \rightarrow \pi^-\pi^+\pi^+, D^+ \rightarrow K^-\pi^+\pi^+, D^0 \rightarrow K_S^0 \pi^+\pi^-$. Previous studies have shown that this approach is successful. In addition, expressions for $D^0 \rightarrow K_S^0 K^+ K^-$ are also given in Ref. [2]. We have derived preliminary parametrized amplitudes for the $B^\pm \rightarrow K^+K^-\pi^\pm$ decays [1, 21] and for the $B^0 \rightarrow K_S^0 K^+ K^-$ process presently analyzed by the LHCb collaboration.

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