

Neutrino propagator in media: spin properties and spectral representation

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Abstract. We have found that in case of propagation of neutrino in the moving matter or external magnetic field there is a fixed 4-axis of polarization, which corresponds to spin projectors commute with a propagator. It means that, states with the definite spin projection on this axis in media have a definite dispersion law. The presence of this axis essentially simplifies the eigenvalue problem and allows one to build spectral representation of the propagator in media.

1 Introduction

It is known that, if neutrinos propagate in media, it can modify a standard picture of mixing and flavor oscillations. The most prominent effect in neutrinos passing through matter is related with resonance amplification of oscillations (MSW-effect) [1, 2], which solves the solar neutrino problem, see reviews [3]-[5]. To describe mixing and oscillations phenomena in neutrinos system, there is a simplified quantum-mechanical approach and more rigorous approach based on Quantum Field Theory [6]-[12]. The necessary element of QFT description is the neutrino propagator.

Here we discuss a spectral representation of neutrino propagator in matter moving with constant velocity or in constant homogeneous magnetic field. In this representation, based on the eigenvalue problem, propagator looks as a sum of single poles, accompanied by orthogonal matrix projectors. Spectral representation was discussed earlier for the dressed fermion propagator in theory with parity violation [13] and for the matrix propagator with mixing of few fermionic fields [14].

In the case of neutrino propagation in media one can obtain an answer for eigenvalues and eigenprojectors in a compact analytical form due to existence of spin projectors (with fixed polarization vector), commuting with propagator. The existence of these spin projectors is a new aspect and it allows one to reduce the algebraic problem for media to the vacuum case.

2 Propagator in moving matter and spin projectors

2.1 Axis of complete polarization and basis

Propagator neutrino in media contains two 4-vectors: momentum of particle p and matter velocity u , so altogether there are eight γ -matrix structures in decomposition of the inverse

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propagator

$$S(p, u) = s_1 I + s_2 \hat{p} + s_3 \hat{u} + s_4 \sigma^{\mu\nu} p_\mu u_\nu + s_5 i \varepsilon^{\mu\nu\lambda\rho} \sigma^{\mu\nu} u_\lambda p_\rho + s_6 \gamma^5 + s_7 \hat{p} \gamma^5 + s_8 \hat{u} \gamma^5, \quad (1)$$

where s_i are scalar function dependent on invariants.

Below we will solve the eigenvalue problem for inverse propagator and as a first step it is convenient to introduce γ -matrix basis with simple multiplicative properties. First of all, let us construct the 4-vector z^μ , which is a linear combination of p , u and has properties of fermion polarization vector: $z^\mu p_\mu = 0$, $z^2 = -1$.

$$z^\mu = b (p^\mu(u p) - u^\mu p^2), \quad b = [p^2((u p)^2 - p^2)]^{-1/2}. \quad (2)$$

Then, having vector z , one can build the generalized off-shell spin projectors¹:

$$\Sigma^\pm = \frac{1}{2}(1 \pm \gamma^5 \hat{z} \hat{n}), \quad n^\mu = p^\mu / W, \quad W = \sqrt{p^2}. \quad (3)$$

Important property of the spin projectors is that Σ^\pm commute with all γ -matrices in decomposition of inverse propagator (1). Multiplying the inverse propagator $S(p, u)$ (1) by unit

$$S = (\Sigma^+(z) + \Sigma^-(z))S \equiv S^+ + S^-, \quad (4)$$

one obtains two orthogonal contributions S^+ , S^- .

One more useful property of Σ^\pm is that “under observation” of the spin projectors (i.e. in S^+ , S^- terms) γ -matrix structures may be simplified. Namely: γ -matrices, which contain the matter velocity u^μ may be transformed to the set of four matrices without velocity: I , \hat{p} , γ^5 , $\hat{p} \gamma^5$. For example, one can rewrite the term \hat{u} in (1) as a linear combination \hat{p} and \hat{z} and use the projector property ($\Sigma^+ \cdot \gamma^5 \hat{z} \hat{n} = \Sigma^+$):

$$\Sigma^+ \hat{u} = \Sigma^+(a_1 \hat{p} + a_2 \hat{z}) = \Sigma^+(z)(a_1 \hat{p} - \frac{a_2}{W} \hat{p} \gamma^5). \quad (5)$$

After this simplification we have the vacuum set of Dirac matrices I , \hat{p} , γ^5 , $\hat{p} \gamma^5$ and it is convenient to use the off-shell momentum projections:

$$\Lambda^\pm = \frac{1}{2}(1 \pm \hat{n}), \quad n^\mu = \frac{p^\mu}{W},$$

which are orthogonal to each other.

Having the momentum Λ^\pm and spin projectors Σ^\pm , one can build the basis, which will be used below in the eigenvalue problem

$$\begin{aligned} R_1 &= \Sigma^- \Lambda^+, & R_2 &= \Sigma^- \Lambda^-, & R_3 &= \Sigma^- \Lambda^+ \gamma^5, & R_4 &= \Sigma^- \Lambda^- \gamma^5, \\ R_5 &= \Sigma^+ \Lambda^+, & R_6 &= \Sigma^+ \Lambda^-, & R_7 &= \Sigma^+ \Lambda^+ \gamma^5, & R_8 &= \Sigma^+ \Lambda^- \gamma^5. \end{aligned} \quad (6)$$

The inverse propagator (1) may be written as decomposition in this basis:

$$S(p, u) = \sum_{i=1}^4 R_i S_i(p^2, pu) + \sum_{i=5}^8 R_i S_i(p^2, pu), \quad (7)$$

where these two sums are orthogonal to each other.

So, by means of the basis (6) the eigenvalue problem for inverse propagator (7) is separated into two different problems: one for $R_1..R_4$ and the other for $R_5..R_8$. Every problem has two different eigenvalues and algebraically is similar to vacuum problem [13].

¹We call them as generalized because of appearance of additional factor \hat{n} . But in fact the Eq. (3) is the most general form of spin projectors for dressed fermion propagator in theories with γ^5 , see details in [14].

2.2 Spectral representation of propagator in matter

We would like to construct a spectral representation for the inverse propagator of the general form (1), (7), so we should solve the eigenvalue problem for inverse propagator S

$$S\Pi_i = \lambda_i\Pi_i, \quad \Pi_i\Pi_k = \delta_{ik}\Pi_k,$$

where λ_i are eigenvalues and Π_i are eigenprojectors. As a result, we obtain the spectral representation of inverse propagator S and propagator G :

$$S(p, u) = \sum_{i=1}^8 \lambda_i \Pi_i, \quad G(p, u) = \sum_{i=1}^8 \frac{1}{\lambda_i} \Pi_i. \quad (8)$$

The use of the matrix basis (6) essentially simplifies the solution of the eigenvalue problem. Repeating the algebraic operations from [13], one can write an answer for eigenvalue problem. Eigenvalues and eigenprojectors (for S^- in (7)) are:

$$\begin{aligned} \lambda_{1,2} &= \frac{S_1 + S_2}{2} \pm \sqrt{\left(\frac{S_1 - S_2}{2}\right)^2 + S_3 S_4}, \\ \Pi_1 &= \frac{1}{\lambda_2 - \lambda_1} \left((S_2 - \lambda_1) R_1 + (S_1 - \lambda_1) R_2 - S_3 R_3 - S_4 R_4 \right), \\ \Pi_2 &= \frac{1}{\lambda_1 - \lambda_2} \left((S_2 - \lambda_2) R_1 + (S_1 - \lambda_2) R_2 - S_3 R_3 - S_4 R_4 \right). \end{aligned} \quad (9)$$

Here S_i are the coefficients of decomposition of the inverse propagator in the basis (7). The eigenvalue problem for S^+ has similar solution.

In the case of Standard Model (SM) the inverse fermion propagator in matter looks like:

$$S(p, u) = \hat{p} - m - \alpha \hat{u} (1 - \gamma^5), \quad (10)$$

where α is some constant dependent on properties of media and flavour, see, e.g. [15]. Solutions of the eigenvalue problem (8) in this case are particular case of (9).

In case of SM it is easy to verify that the spin projection on the axis of complete polarization z is not a conservative value. The Hamiltonian is defined by Dirac operator (10)

$$H = p^0 - \gamma^0 S.$$

We can reduce the required commutator to the simpler one (recall that $[S, R] = 0$)

$$[R, H] = \gamma^0 [S, R] + [\gamma^0, R] S = [\gamma^0, R] S, \quad \text{where } R = \gamma^5 \hat{z} \hat{n}. \quad (11)$$

In the standard representation of γ -matrices we have

$$R = \begin{pmatrix} \sigma \mathbf{v} & -i\sigma \boldsymbol{\xi} \\ -i\sigma \boldsymbol{\xi} & \sigma \mathbf{v} \end{pmatrix}, \quad \mathbf{v} = n^0 \mathbf{z} - z^0 \mathbf{n}, \quad \boldsymbol{\xi} = [\mathbf{z} \times \mathbf{n}]. \quad (12)$$

If to require $[\gamma^0, R] = 0$, we come to condition $\boldsymbol{\xi} = 0$, i.e.

$$\boldsymbol{\xi} \equiv [\mathbf{z} \times \mathbf{n}] = bW[\mathbf{p} \times \mathbf{u}] = 0.$$

Thus, a spin projection on the axis z^μ is conserved only in the case $\mathbf{u}_\perp = 0$, when 3-momentum of the propagator coincides in direction (or opposite) with the matter velocity. In this case the found polarization vector z^μ (2) takes the following form:

$$z^\mu = \frac{1}{W} \left(|\mathbf{p}|, p^0 \frac{\mathbf{p}}{|\mathbf{p}|} \right), \quad (13)$$

which corresponds to the off-shell helicity state of fermion, since $W \neq m$. A particular case of $\mathbf{u}_\perp = 0$ is non-moving matter ($\mathbf{u} = 0, u_0 = 1$). One can see that the generalized spin projectors (3) in this case are helicity projectors. Thus, for the rest matter the well-known fact [16, 17] is that neutrino with definite helicity has a definite law of dispersion in matter.

In the general case, at arbitrary direction of the matter velocity, in spite of $[\Sigma^\pm, S] = 0$, the spin projection on the axis z^μ is not conserved: $[\Sigma^\pm, H] \neq 0$.

3 The propagation of neutrino in the external magnetic field

An inverse propagator of a neutral fermion with anomalous magnetic moment μ in a constant external electromagnetic field is:

$$S = \hat{p} - m - \frac{i}{2}\mu\sigma^{\alpha\beta}F_{\alpha\beta}, \quad \sigma^{\alpha\beta} = \frac{1}{2}[\gamma^\alpha, \gamma^\beta]. \quad (14)$$

In case of the magnetic field, it takes a more customary form:

$$S = \hat{p} - m + \mu\Sigma\mathbf{B}, \quad \Sigma = \gamma^0\boldsymbol{\gamma}\gamma^5.$$

Having the electromagnetic field tensor and 4-momentum, we can build a polarization vector z^μ ($z^2 = -1$ and $z_\mu p^\mu = 0$)² and the corresponding spin projectors

$$z^\mu = b\epsilon^{\mu\nu\lambda\rho}F_{\nu\lambda}p_\rho, \quad b = (p_0^2\mathbf{B}^2 - (\mathbf{p}\mathbf{B})^2)^{-1/2}, \quad (15)$$

$$\Sigma^\pm = \frac{1}{2}(1 \pm \gamma^5\hat{z}).$$

Vector z^μ in case of the magnetic field is as follows:

$$z^\mu = b((\mathbf{B}\mathbf{p}), p^0\mathbf{B}), \quad b = (p_0^2\mathbf{B}^2 - (\mathbf{B}\mathbf{p})^2)^{-1/2}$$

and matrix $\gamma^5\hat{z}$ can be rewritten as

$$R \equiv \gamma^5\hat{z} = b(\gamma^5\gamma^0(\mathbf{B}\mathbf{p}) + p^0\gamma^0(\Sigma\mathbf{B})), \quad R^2 = 1.$$

It allows one to see that spin projectors commute with the inverse propagator: $[S, \Sigma^\pm] = 0$.

Further we can apply the same trick that was used for the propagator in the matter to simplify the gamma-matrix structures “under observation” of the spin projector:

$$S = (\Sigma^+(z) + \Sigma^-(z))S \equiv S^+ + S^-. \quad (16)$$

Since two matrices commute $[S, R] = 0$, they have a common eigenvector:

$$S\Psi = \lambda\Psi, \quad \gamma^5\hat{z}\Psi = \sigma\Psi, \quad \sigma = \pm 1.$$

The eigenvector of the operator R is obvious: $\Psi^\pm = \Sigma^\pm\Psi_0$, so the system looks as follows:

$$S^\pm\Psi^\pm = \lambda\Psi^\pm, \quad \gamma^5\hat{z}\Psi^\pm = \pm\Psi^\pm.$$

²This vector arises in consideration of motion of a charged relativistic fermion in a constant and homogeneous magnetic field, see 4-th edition of textbook [18], §1.6. We consider another situation: neutral fermion with an anomalous magnetic moment in a magnetic field, but it turns out that in this case the constructed spin projector also commutes with the propagator.

The eigenvalues of R are equal to ± 1 and from (16) we can find the useful relation:

$$(\Sigma\mathbf{B})\Psi^\pm = \frac{1}{p^0}(\gamma^5(\mathbf{p}\mathbf{B}) \pm \gamma^0 \frac{1}{b})\Psi^\pm. \quad (17)$$

Then, in analogy with the case of matter, in the S^\pm contributions the γ -matrix structure can be transformed. Instead of (15) we get

$$S^\pm = \Sigma^\pm(z) \left[\hat{p} - m + \frac{\mu}{p^0}(\gamma^5(\mathbf{p}\mathbf{B}) \pm \gamma^0 \frac{1}{b}) \right]. \quad (18)$$

Let us recall that for a covariant matrix of the form

$$S = aI + b\hat{p} + c\gamma^5 + d\hat{p}\gamma^5, \quad (19)$$

the solutions of the matrix eigenvalue problem are known [13] and were used in the above (9).

The inverse propagator in the external field (15), (18) is non-covariant (in particular, it contains γ^0), but for algebraic problem this is not so important. Therefore, if we redefine the vector p^μ in S^\pm , we can get rid of γ^0 and use the ready answer for eigenvalues and eigenprojectors. To this end we can introduce “4-vector”

$$p_\pm^\mu = (p^0 \pm \frac{\mu}{bp_0}, \mathbf{p}) \quad (20)$$

and after this, the inverse propagator takes the form:

$$S^\pm = \hat{p}_\pm - m + \mu\gamma^5 \frac{(\mathbf{B}\mathbf{p})}{p_0}, \quad (21)$$

in which there are only I , \hat{p}_\pm and γ^5 matrix, and which is algebraically similar to the vacuum propagator. Therefore, we can use the formulae (9) for eigenvalues and eigenprojectors:

$$\begin{aligned} \lambda_1^\pm &= -m + \sqrt{W_\pm^2 + \frac{\mu^2}{p_0^2}(\mathbf{B}\mathbf{p})^2}, & \lambda_2^\pm &= -m - \sqrt{W_\pm^2 + \frac{\mu^2}{p_0^2}(\mathbf{B}\mathbf{p})^2}, \\ \Pi_1^\pm &= \frac{\Sigma^\pm}{2} \left(1 - \frac{1}{A^\pm} \left(\hat{p}_\pm + \frac{\mu(\mathbf{B}\mathbf{p})}{p_0} \gamma^5 \right) \right), & \Pi_2^\pm &= \frac{\Sigma^\pm}{2} \left(1 + \frac{1}{A^\pm} \left(\hat{p}_\pm + \frac{\mu(\mathbf{B}\mathbf{p})}{p_0} \gamma^5 \right) \right). \end{aligned} \quad (22)$$

Here $W_\pm = \sqrt{p_\pm^2}$, $A^\pm = \sqrt{W_\pm^2 + \mu^2(\mathbf{B}\mathbf{p})^2/p_0^2}$. If the eigenvalue is vanishing, we have the well-known dispersion law for the anomalous magnetic moment in the magnetic field [19]

$$E^2 = m^2 + \mathbf{p}^2 + \mu^2 \mathbf{B}^2 \pm 2\mu \sqrt{m^2 \mathbf{B}^2 + \mathbf{p}^2 \mathbf{B}_\perp^2}. \quad (23)$$

Here \pm corresponds to different signs in (21), i.e. to terms S^\pm in the propagator.

The inverse propagator (15) can be related with the Dirac Hamiltonian:

$$S = \gamma^0(p^0 - H_D), \quad H_D = \alpha\mathbf{p} + \beta m + \mu\gamma^0(\Sigma\mathbf{B}).$$

To verify conservation of spin projection on z , according to Eq. (11), we should calculate the commutator $[R, \gamma^0]$. In the standard representation of γ -matrices

$$[\gamma^5 \hat{z}, \gamma^0] = \begin{pmatrix} 0 & 2z^0 \\ 2z^0 & 0 \end{pmatrix}, \quad z^0 = b(\mathbf{B}\mathbf{p}).$$

So, we see that the projection of the spin on the axis of complete polarization (15) is conserved only in case of the transverse magnetic field.

4 Conclusions

We have built the spectral representation of the neutrino propagator both in the moving matter and a constant external magnetic field. This form of the propagator gives the simplest and most convenient algebraic construction and means in fact a complete diagonalization.

It turns out that both in the matter and the magnetic field there is the fixed 4-axis of complete polarization z'' , when all eigenvalues of the propagator (and, consequently, dispersion laws) are classified according to spin projection on this axis. In a particular case of the rest matter the operators Σ^\pm are projectors on the helicity states in correspondence with the results known earlier [16, 17]. Let us emphasize that for moving matter or magnetic field the vanishing of the commutator with the inverse propagator S [S, Σ^\pm] = 0 does not lead to conservation of spin projection on this axis, since the spin projectors Σ^\pm do not commute, generally speaking, with Hamiltonian.

The most evident development of this approach is related with neutrino oscillations in the matter, in particular, in astrophysical problems. The role of the fixed axis of complete polarization in spin dynamics may be also interesting.

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