One-loop Feynman integrals with Carlson hypergeometric functions

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Abstract. In this paper, we present analytic results for scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The calculations are considered in general space-time dimension D for two- and three-point functions and $D = 4$ for four-point functions. The analytic results are expressed in terms of the Carlson hypergeometric functions ($\mathcal{R}$-functions) and valid for both real and complex internal masses.

1 Introduction

In order to confront particle physics theory with high-precision of experimental data at future colliders, theoretical predictions including high-order corrections are required. In general framework for computing high-order corrections, detailed calculations for one-loop multi-leg and higher-loop are necessary for building blocks. When we compute scattering processes which Feynman diagrams involve internal unstable particles that can be on-shell, we have to resume Feynman propagators with a complex mass term in the denominator. In other words, one has to perform the perturbative renormalization in the Complex-Mass Scheme [1]. Therefore, the calculations for Feynman loop integrals with complex internal masses are of great interest. Furthermore, within the general framework for computing two-loop or higher-loop corrections scalar one-loop integrals in general space-time dimension play a crucial role for several reasons. Higher-terms in the ε-expansion from one-loop integrals are necessary for building blocks. In additional, one-loop integrals at higher space-time dimension $D > 4$ may be taken into account in the framework.

There have been available many calculations for scalar one-loop integrals in $D = 4 - 2\varepsilon$ dimensions at ε^0-expansion [2–11]. Scalar one-loop integrals in general dimension D have performed in [12–16]. However, not all of these calculations cover general dimension D with a general ε-expansion at general scale and complex internal masses. In this paper, based on the method in [5–8], we present analytic results for scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The calculations are considered in general space-time dimension D for two- and three-point functions and $D = 4$ for four-point functions. The analytic results are expressed in terms of the Carlson hypergeometric functions.

The layout of the paper is as follows: In section 2, we present in detail the method for evaluating scalar one-loop functions. In this section, analytic results for one-loop two-, three- and four-point functions are presented. Conclusions and outlooks are devoted in section 3. Several useful formulas used in this calculation can be found in the appendix.

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2 The calculations

Based on the method introduced in Refs. [5–7], we present the calculations for scalar one-loop functions with complex internal masses. Scalar one-loop N-point functions are defined

$$J_N = \int d^D l \frac{1}{\mathcal{P}_1 \mathcal{P}_2 \cdots \mathcal{P}_N}. \tag{1}$$

Where inverse Feynman propagators are given

$$\mathcal{P}_k \quad = \quad (l+q_k)^2 - m_k^2 + i\rho, \quad \text{with} \quad k = 1, 2, \cdots, N. \tag{2}$$

The Feynman prescription is $i\rho$. We use momenta $q_k = \sum_{j=1}^{k} p_j$, p_j are external momenta and they are inward as shown in Fig. 1. The internal masses in the Complex-Mass scheme are taken the form of

$$m_k^2 = m_{0k}^2 - i m_{0k}\,\Gamma_k, \quad \text{for} \quad \Gamma_k \geqslant 0. \tag{3}$$

The Γ_k are decay widths of unstable particles. The momenta q_k may take the following configuration

$$q_1 = q_1\ (q_{10}, q_{11}, 0, \cdots, 0, \overrightarrow{0}_{D-J}), \tag{4}$$
$$q_2 = q_2\ (q_{20}, q_{21}, 0, \cdots, 0, \overrightarrow{0}_{D-J}), \tag{5}$$
$$q_3 = q_3\ (q_{10}, q_{31}, q_{32}, 0, \cdots, 0, \overrightarrow{0}_{D-J}), \tag{6}$$
$$\cdots = \cdots,$$
$$q_{N-1} = q_{N-1}\ (q_{(N-1)0}, q_{(N-1)1}, \cdots, q_{(N-1)(J-1)}, \overrightarrow{0}_{D-J}) \tag{7}$$

which have J non-zero components. Here, $q_{10} = 0$ for $q_1^2 < 0$ and $q_{11} = 0$ for $q_1^2 > 0$. As a result, scalar product of external and internal momenta are obtained

$$q_k^2 \quad = \quad q_{k0}^2 - q_{k1}^2 - \cdots - q_{k(J-1)}^2, \tag{8}$$
$$l^2 \quad = \quad l_0^2 - l_1^2 - \cdots - l_{J-1}^2 - l_\perp^2, \tag{9}$$
$$l \cdot q_k \quad = \quad l_0 \cdot q_{k0} - l_1 \cdot q_{k1} \cdots - l_{J-1} \cdot q_{k(J-1)}. \tag{10}$$

In parallel space which is the linear span of the external momenta and its orthogonal space (POS) [5, 6], scalar one-loop N-point functions are taken the form of:

$$J_N \quad = \quad \frac{2\pi^{\frac{D-J}{2}}}{\Gamma(\frac{D-J}{2})} \int\limits_{-\infty}^{\infty} dl_0 dl_1 \cdots dl_{J-1} \int\limits_{0}^{\infty} dl_\perp \frac{l_\perp^{D-J-1}}{\mathcal{P}_1 \mathcal{P}_2 \cdots \mathcal{P}_N}. \tag{11}$$

The propagators now become

$$\mathcal{P}_k \quad = \quad (l_0 + q_{k0})^2 - (l_1 + q_{k1})^2 - \cdots - (l_{J-1} + q_{k(J-1)})^2 - l_\perp^2 - m_k^2 + i\rho, \tag{12}$$

for $k = 1, 2, \cdots, N$. The calculations can be summarized as follows. We first make the partition for the integrand of J_N as

$$\frac{1}{\mathcal{P}_1 \mathcal{P}_2 \cdots \mathcal{P}_N} \quad = \quad \sum_{k=1}^{N} \frac{1}{\mathcal{P}_k \prod\limits_{\substack{l=1 \\ k \neq l}}^{N} (\mathcal{P}_l - \mathcal{P}_k)}, \tag{13}$$

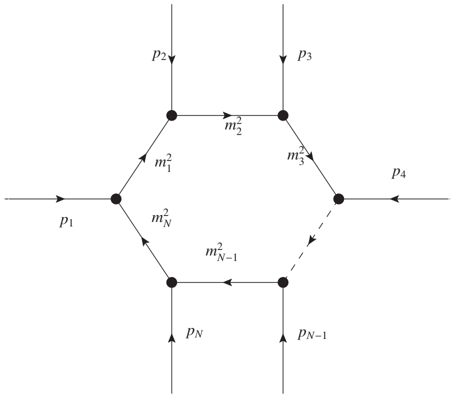

Figure 1. Generic Feynman diagrams at one-loop with N external lines. All external momenta are inward and follow momentum conservation $q_N = \sum_{j=1}^{N} p_j = 0$.

with

$$\begin{aligned}\mathcal{P}_l - \mathcal{P}_k &= (l_0 + q_{l0})^2 - (l_0 + q_{k0})^2 + (l_1 + q_{l1})^2 - (l_1 + q_{k1})^2 \\ &\quad + \cdots + (l_{J-1} + q_{l(J-1)})^2 - (l_{J-1} + q_{k(J-1)})^2 + m_k^2 - m_l^2 \qquad (14)\\ &= a_{lk} l_0 + b_{lk} l_1 + \cdots + c_{lk} l_{J-1} + \tilde{d}_{lk}. \qquad (15)\end{aligned}$$

Where we have introduced the following kinematic variables

$$a_{lk} = 2(q_{l0} - q_{k0}), \quad b_{lk} = -2(q_{l1} - q_{k1}), \quad \cdots, \qquad (16)$$

$$c_{lk} = -2(q_{l(J-1)} - q_{k(J-1)}), \quad \tilde{d}_{lk} = q_l^2 - q_k^2 + m_k^2 - m_l^2. \qquad (17)$$

Making a shift

$$l_0 \to l_0 + q_{k0}, \quad l_1 \to l_1 + q_{k1}, \cdots, \quad l_{J-1} \to l_{J-1} + q_{k(J-1)}, \qquad (18)$$

we convert all $\mathcal{P}_k$ in (13) to $\mathcal{P}_N$. As a matter of this fact, the $l_\perp$-integral then yields a simple form which can be taken easily as follows:

$$\int_0^\infty dl_\perp \frac{l_\perp^{D-J-1}}{[l_0^2 - l_1^2 - \cdots - l_{J-1}^2 - l_\perp^2 - m_k^2 + i\rho]} = \qquad (19)$$

$$= -\frac{\Gamma\left(\frac{D-J}{2}\right)\Gamma\left(\frac{J+2-D}{2}\right)}{2}\left(-l_0^2 + l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - i\rho\right)^{\frac{D-J-2}{2}}.$$

We then arrive at the $(J-1)$-fold integrals

$$\frac{J_N}{\Gamma\left(\frac{J+2-D}{2}\right)} = \pi^{\frac{D-J}{2}} \sum_{k=1}^{N} \int_{-\infty}^{\infty} dl_0 dl_1 \cdots dl_{J-1} \frac{\left(-l_0^2 + l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - i\rho\right)^{\frac{D-J-2}{2}}}{\prod_{\substack{l=1\\k\neq l}}^{N}\left[a_{lk} l_0 + b_{lk} l_1 + \cdots + c_{lk} l_{J-1} + d_{lk}\right]}. \qquad (20)$$

In this formula $a_{lk}, b_{lk}, \cdots, c_{lk} \in \mathbb{R}$ and $d_{lk} = (q_l - q_k)^2 - (m_l^2 - m_k^2) \in \mathbb{C}$ which is obtained from $\tilde{d}_{lk}$ after applying the shift (18). The integrals in (20) can be carried out with the help of residue theorem. For that purpose, one first linearizes the l_0 for example, .i.e $l_1' = l_1 + l_0$. The result reads

$$\frac{J_N}{\Gamma\left(\frac{J+2-D}{2}\right)} = \pi^{\frac{D-J}{2}} \sum_{k=1}^{N} \int_{-\infty}^{\infty} dl_0 dl_1 \cdots dl_{J-1} \frac{\left(-2l_0 l_1 + l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - i\rho\right)^{\frac{D-J-2}{2}}}{\prod_{\substack{l=1\\k\neq l}}^{N}\left[AB_{lk} l_0 + b_{lk} l_1 + \cdots + c_{lk} l_{J-1} + d_{lk}\right]} \qquad (21)$$

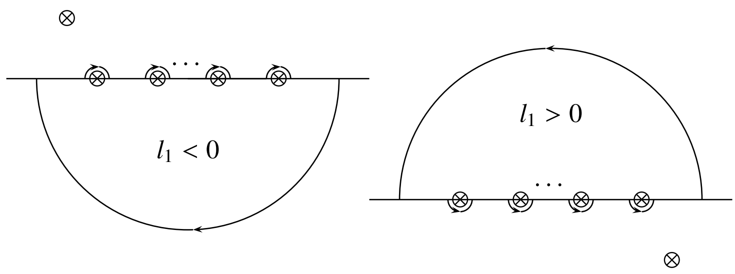

Figure 2. We close the contour integration for l_0 that the poles in (22) locate outside the contour.

with $AB_{lk} = a_{lk} - b_{lk}$. The singularity poles of the integrand in (21) are obtained:

$$l_0 = \frac{l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - i\rho}{2l_1}, \quad \mathrm{Im}(l_0) = -\frac{m_{0k}\Gamma_k + \rho}{2l_1}, \tag{22}$$

and

$$l_0^{(l)} = -\frac{b_{lk}l_1 + \cdots + c_{lk}l_{J-1} + d_{lk}}{AB_{lk}}, \quad \mathrm{Im}[l_0^{(l)}] = \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right). \tag{23}$$

The pole l_0 in (22) locates upper (lower) in l_0-complex plane if $l_1 < 0$ ($l_1 > 0$) respectively. We plan to close the contour integration for l_0 that l_0-poles in (22) locate outside the contour, seen Fig. 2 for more detail. As a result, the poles in (23) are only taken into account to the residue contributions for l_0-integration. The resulting reads

$$\frac{J_N}{\Gamma\left(\frac{J+2-D}{2}\right)} = \pi^{\frac{D-J}{2}} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ k\neq l}}^{N} \left\{ f_{lk}^{+} \int_0^\infty dl_1 + f_{lk}^{-} \int_{-\infty}^{0} dl_1 \right\} \cdots \int_{-\infty}^{\infty} dl_{J-1}\, [1 - \delta(AB_{lk})] \tag{24}$$
$$\times \frac{\left[\left(1 - 2\frac{b_{lk}}{AB_{lk}}\right) l_1^2 + \cdots + l_{J-1}^2 - 2\frac{c_{lk}}{AB_{lk}} l_1 l_{J-1} - 2\frac{d_{lk}}{AB_{lk}} l_1 + m_k^2 - i\rho\right]^{\frac{D-J-2}{2}}}{\prod\limits_{\substack{m=1 \\ m\neq k \\ m\neq l}}^{N} \left[\tilde{A}_{mlk} l_1 + \cdots + \tilde{C}_{mlk} l_{J-1} + \tilde{F}_{mlk}\right]}.$$

Where the δ-function is defined as

$$\delta(x) = \begin{cases} 0, & \text{if} \quad x \neq 0; \\ 1, & \text{if} \quad x = 0. \end{cases} \tag{25}$$

New kinematic variables $\tilde{A}_{mlk}, \cdots, \tilde{C}_{mlk} \in \mathbb{R}$ and $\tilde{F}_{mlk} \in \mathbb{C}$ are obtained from residue contributions of the poles in (23). The functions $f_{lk}^{\pm}$ indicate the location of the poles in (23) in the l_0 complex plane:

$$f_{lk}^{+} = \begin{cases} 0, & \text{if } \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right) < 0; \\ 1, & \text{if } \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right) = 0; \\ 2, & \text{if } \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right) > 0. \end{cases} \quad \text{and} \quad f_{lk}^{-} = \begin{cases} 0, & \text{if } \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right) > 0; \\ 1, & \text{if } \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right) = 0; \\ 2, & \text{if } \mathrm{Im}\left(-\frac{d_{lk}}{AB_{lk}}\right) < 0. \end{cases} \tag{26}$$

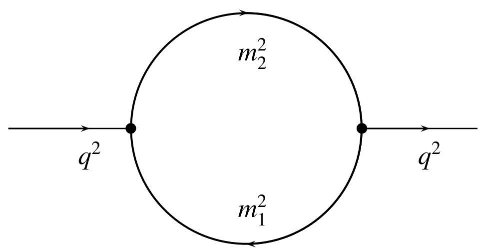

Figure 3. Bubble diagrams.

We continue to linearize l_1 in numerator of the integrand of (24) by applying a Euler shift $l_1 \to l_1 + \beta_{lk} l_2$. β_{lk} can be chosen in such a way of the disappearance of l_1^2-term. The residue theorem is applied against for l_1-integration. At the final stage, the resulting integrals can be expressed in terms of $\mathcal{R}$-functions [18] which is defined as

$$\int_r^\infty (x-r)^{\alpha-1} \prod_{i=1}^{k} (z_i + w_i x)^{-b_i} dx$$
$$= \mathcal{B}(\beta - \alpha, \alpha) \mathcal{R}_{\alpha-\beta}\left(b_1, \cdots, b_k, r + \frac{z_1}{w_1}, \cdots, r + \frac{z_k}{w_k}\right) \prod_{i=1}^{k} w_i^{-b_i}, \tag{27}$$

with $\beta = \sum_{i=1}^k b_i$. In next subsections, we present analytic results for scalar one-loop two-, three- and four-point functions. Detailed calculations for these functions have published in Ref. [17].

2.1 One-loop two-point functions

In POS, J_2 takes the form of [5, 6]

$$J_2 = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \int_{-\infty}^{\infty} dl_0 \int_0^\infty dl_\perp \frac{l_\perp^{D-2}}{[(l_0+q_{10})^2 - l_\perp^2 - m_1^2 + i\rho][l_0^2 - l_\perp^2 - m_2^2 + i\rho]}. \tag{28}$$

Here $q = q(q_{10}, \overrightarrow{0}_{D-1})$ for $q^2 > 0$. If $q^2 < 0$, we refer [17] for detailed evaluations. The results in [17] have shown that the below formulas for J_2 are valid for both above cases. The $\mathcal{R}$-function representation for two-point integrals is as follows [17]:

$$\frac{J_2}{\Gamma\left(3-\frac{D}{2}\right)} = \frac{\pi^{(D-1)/2} e^{i\pi(3-D)/2}}{2} \mathcal{B}\left(\frac{4-D}{2}, \frac{1}{2}\right) \tag{29}$$
$$\times \left\{ \left(\frac{q^2 + m_1^2 - m_2^2}{2q^2}\right) \mathcal{R}_{\frac{D-4}{2}}\left(\frac{3-D}{2}, 1; -m_1^2 + i\rho, -\frac{(q^2+m_1^2-m_2^2)^2}{4q^2}\right) \right.$$
$$\left. + \left(\frac{q^2 - m_1^2 + m_2^2}{2q^2}\right) \mathcal{R}_{\frac{D-4}{2}}\left(\frac{3-D}{2}, 1; -m_2^2 + i\rho, -\frac{(q^2-m_1^2+m_2^2)^2}{4q^2}\right) \right\}.$$

We can derive other representations for J_2 by employing the transformations in appendix for $\mathcal{R}$-functions from (46) to (51). For example, using Euler's transformation (50) for $\mathcal{R}$-functions

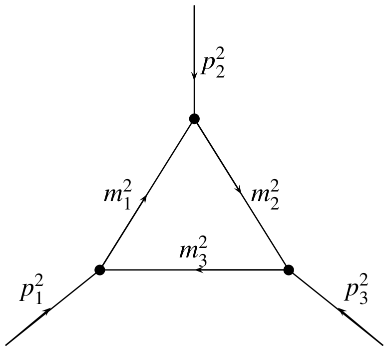

Figure 4. Triangle diagrams.

(50), Eq. (29) becomes

$$\frac{J_2}{\Gamma\left(3-\frac{D}{2}\right)} = -\pi^{(D-1)/2}\mathcal{B}\left(\frac{4-D}{2},\frac{1}{2}\right) \times\left\{\frac{(m_1^2-i\rho)^{\frac{D-3}{2}}}{q^2+m_1^2-m_2^2}\mathcal{R}_{-\frac{1}{2}}\left(\frac{5-D}{2},2;\frac{-1}{m_1^2-i\rho},\frac{-4q^2}{(q^2+m_1^2-m_2^2)^2}\right) +\frac{(m_2^2-i\rho)^{\frac{D-3}{2}}}{q^2-m_1^2+m_2^2}\mathcal{R}_{-\frac{1}{2}}\left(\frac{5-D}{2},2;\frac{-1}{m_2^2-i\rho},\frac{-4q^2}{(q^2-m_1^2+m_2^2)^2}\right)\right\}. \tag{30}$$

It can be seen that the right hand sides of Eqs. (29,30) are symmetric under the interchange of $m_1^2 \leftrightarrow m_2^2$. From Eqs. (29,30) we can take the limits of $m_1^2 = m_2^2 \to 0$ and $q^2 \to 0$ respectively, seen Ref. [17] for more detail.

2.2 One-loop three-point functions

The momenta q_1, q_2 take the following configuration $q_1 = q_1(q_{10}, q_{11}, \overrightarrow{0}_{D-2})$, $q_2 = q_2(q_{20}, q_{21}, \overrightarrow{0}_{D-2})$. Here $q_{10} = 0$ for $q_1^2 < 0$ and $q_{11} = 0$ for $q_1^2 > 0$. The results for J_3 in this paper cover both the above cases. The integral J_3 in POS takes the form of [5, 6]

$$J_3 = \frac{\pi^{\frac{D-2}{2}}}{\Gamma\left(\frac{D-2}{2}\right)}\int_{-\infty}^{\infty}dl_0\int_{-\infty}^{\infty}dl_1\int_{-\infty}^{\infty} l_\perp^{D-3}\, dl_\perp \frac{1}{[(l_0+q_{10})^2-(l_1+q_{11})^2-l_\perp^2-m_1^2+i\rho]} \times\frac{1}{[(l_0+q_{20})^2-(l_1+q_{21})^2-l_\perp^2-m_2^2+i\rho][l_0^2-l_1^2-l_\perp^2-m_3^2+i\rho]}. \tag{31}$$

Scalar one-loop three-point functions are also expressed in terms of $\mathcal{R}$-functions [18] as follows [17]

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D}{2}} i\, \mathcal{B}(4-D,1) \sum_{k=1}^{3}\sum_{\substack{l=1\\k\neq l}}^{3} \frac{[1-\delta(AB_{lk})]}{A_{mlk}} (\alpha_{lk}-i\rho)^{\frac{D-4}{2}} \times\Bigg\{\mathcal{S}^{+}_{lk}\, f^{+}_{lk}\, \mathcal{R}_{D-4}\left(\frac{4-D}{2},\frac{4-D}{2},1;+Z^{(1)}_{lk},+Z^{(2)}_{lk},+F_{mlk}\right) +\mathcal{S}^{-}_{lk}\, f^{-}_{lk}\, \mathcal{R}_{D-4}\left(\frac{4-D}{2},\frac{4-D}{2},1;-Z^{(1)}_{lk},-Z^{(2)}_{lk},-F_{mlk}\right)\Bigg\}, \tag{32}$$

for $m \neq l$. When all internal masses are real, $f^{+}_{lk} = f^{-}_{lk} = 1$ and $\mathcal{S}^{\pm}_{lk} = 1$, Eq. (32) confirms the results of, for instance, J_3 in the Eq. (11) of [6]. We can derive other represents for J_3 by applying several transformations for $\mathcal{R}$-functions, as shown in appendix. For example, with the help of (50), one obtains

$$\frac{J_3}{\Gamma\left(2-\frac{D}{2}\right)} = -\pi^{\frac{D}{2}} i\, \mathcal{B}(4-D,1) \sum_{k=1}^{3}\sum_{\substack{l=1\\k\neq l}}^{3} \frac{[1-\delta(AB_{lk})]}{C_{mlk}} (m_k^2)^{(D-4)/2} \times\Bigg\{f^{+}_{lk}\, \mathcal{R}_{-1}\left(\frac{6-D}{2},\frac{6-D}{2},2;+\frac{1}{Z^{(1)}_{lk}},+\frac{1}{Z^{(2)}_{lk}},+\frac{1}{F_{mlk}}\right) -f^{-}_{lk}\, \mathcal{R}_{-1}\left(\frac{6-D}{2},\frac{6-D}{2},2;-\frac{1}{Z^{(1)}_{lk}},-\frac{1}{Z^{(2)}_{lk}},-\frac{1}{F_{mlk}}\right)\Bigg\}, \tag{33}$$

for $m \neq l$. The kinematic variables appear in subsection are listed:

$$\begin{aligned}
a_{lk} &= 2(q_{l0}-q_{k0}), & b_{lk} &= -2(q_{l1}-q_{k1}),\\
AB_{lk} &= a_{lk}-b_{lk}, & c_{lk} &= (q_k-q_l)^2+m_k^2-m_l^2,\\
A_{mlk} &= -AB_{km}\, b_{lk}+AB_{lk}\, b_{km}, & C_{mlk} &= -AB_{km}\, c_{lk}+AB_{lk}\, c_{km},\\
F_{mlk} &= C_{mlk}/A_{mlk}, & Z^{(1,2)}_{lk} &= \tfrac{c_{lk}}{a_{lk}+b_{lk}} \pm \sqrt{\left(\tfrac{c_{lk}}{a_{lk}+b_{lk}}\right)^2-\tfrac{m_k^2-i\rho}{\alpha_{lk}}}.
\end{aligned}$$

The factor $\mathcal{S}^{\pm}_{lk}$ is given

$$\begin{aligned}
\mathcal{S}^{\pm}_{lk} &= \mathrm{Exp}\left[\pi i\theta(-\alpha_{lk})\,\theta[\mp\mathrm{Im}(Z^{(1)}_{lk})]\theta[\mp\mathrm{Im}(Z^{(2)}_{lk})]\,(D-4)\right]\\
&\times\mathrm{Exp}\left[-\pi i\theta(\alpha_{lk})\,\theta[\pm\mathrm{Im}(Z^{(1)}_{lk})]\theta[\pm\mathrm{Im}(Z^{(2)}_{lk})]\,(D-4)\right].
\end{aligned} \tag{34}$$

We turn our attention into the analytic results for scalar one-loop four-point functions in next subsection.

2.3 One-loop four-point functions

At present, the calculations for four-point functions are performed in $D = 4$. We set configuration of external momenta as follows $q_1 = (q_{10}, q_{11}, 0, 0)$, $q_2 = (q_{20}, q_{21}, 0, 0)$, $q_3 = (q_{30}, q_{31}, q_{32}, 0)$. Where $q_{10} = 0$ for $q_1^2 < 0$ and $q_{11} = 0$ for $q_1^2 > 0$. Our result presented in this paper cover all the above cases. In POS, J_4 takes the form of

$$J_4 = 2\int_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int_0^{\infty} dl_\perp \frac{1}{\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4}, \tag{35}$$

with $\mathcal{P}_k = (l+q_k)^2 - m_k^2 + i\rho$ for $k = 1, 2, \cdots, 4$. Scalar one-loop four-point functions are

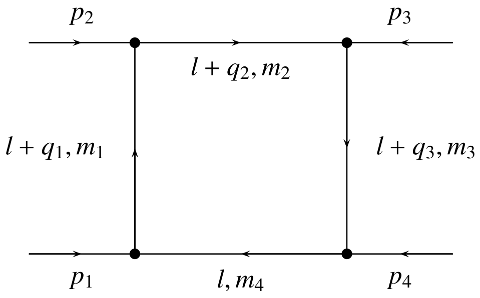

Figure 5. Box diagrams.

written as one-fold integrals [17] as follows

$$
\begin{aligned}
\frac{J_4}{i\pi^2} = &\sum_{k=1}^{4}\sum_{\substack{l=1\\k\neq l}}^{4}\sum_{\substack{m=1\\m\neq l\\m\neq k}}^{4}\frac{\Big(1-\delta(AC_{lk})\Big)\Big(1-\delta(B_{mlk})\Big)}{AC_{lk}(B_{mlk}A_{nlk}-B_{nlk}A_{mlk})}\times \\
&\times\Bigg\{\int_0^{\infty} dz\, G(z)\Bigg[(f^{+}_{lk}g^{+}_{mlk}+f^{-}_{lk}g^{+}_{mlk})\ln\left(\frac{F_{nmlk}}{\beta_{mlk}}\right)-f^{+}_{lk}g^{+}_{mlk}\ln\left(\frac{z+F_{nmlk}}{\beta_{mlk}}\right) \\
&\quad -f^{+}_{lk}g^{-}_{mlk}\ln\left(-\frac{z+F_{nmlk}}{\beta_{mlk}}\right)-(f^{-}_{lk}g^{+}_{mlk}+f^{+}_{lk}g^{+}_{mlk})\ln\left(\frac{S(\sigma_{mlk},z)}{P_{mlk}z+Q_{mlk}}\right) \\
&\quad +f^{+}_{lk}g^{+}_{mlk}\ln\left(\frac{S(\sigma_{mlk}=0,z)}{P_{mlk}z+Q_{mlk}}\right)+f^{+}_{lk}g^{-}_{mlk}\ln\left(-\frac{S(\sigma_{mlk}=0,z)}{P_{mlk}z+Q_{mlk}}\right)\Bigg] \\
&+\int_{-\infty}^{0} dz\, G(z)\Bigg[-f^{+}_{lk}g^{-}_{mlk}\ln\left(-\frac{F_{nmlk}}{\beta_{mlk}}\right)+(f^{-}_{lk}g^{-}_{mlk}+f^{-}_{lk}g^{+}_{mlk})\ln\left(\frac{z+F_{nmlk}}{\beta_{mlk}}\right) \\
&\quad -f^{-}_{lk}g^{-}_{mlk}\ln\left(\frac{F_{nmlk}}{\beta_{mlk}}\right)-(f^{-}_{lk}g^{+}_{mlk}+f^{-}_{lk}g^{-}_{mlk})\ln\left(\frac{S(\sigma_{mlk}=0,z)}{P_{mlk}z+Q_{mlk}}\right) \\
&\quad +f^{-}_{lk}g^{-}_{mlk}\ln\left(\frac{S(\sigma_{mlk},z)}{P_{mlk}z+Q_{mlk}}\right)+f^{+}_{lk}g^{-}_{mlk}\ln\left(-\frac{S(\sigma_{mlk},z)}{P_{mlk}z+Q_{mlk}}\right)\Bigg]\Bigg\}
\end{aligned}
\tag{36}
$$

Where the related kinematic variables are given:

$$
\begin{aligned}
a_{lk} &= 2(q_{l0}-q_{k0}), & b_{lk} &= -2(q_{l1}-q_{k1}), \\
c_{lk} &= -2(q_{l2}-q_{k2}), & d_{lk} &= (q_l-q_k)^2-(m_l^2-m_k^2), \\
AC_{lk} &= a_{lk}+c_{lk}, & \alpha_{lk} &= b_{lk}/AC_{lk}, \\
A_{mlk} &= a_{mk}-\tfrac{a_{lk}}{AC_{lk}}AC_{mk}, & B_{mlk} &= b_{mk}-\tfrac{b_{lk}}{AC_{lk}}AC_{mk}, \\
C_{mlk} &= d_{mk}-\tfrac{d_{lk}}{AC_{lk}}AC_{mk}, & D_{mlk} &= -4(q_l-q_k)^2/AC_{lk}^2, \\
F_{nmlk} &= \tfrac{C_{nlk}B_{mlk}-B_{nlk}C_{mlk}}{A_{nlk}B_{mlk}-B_{nlk}A_{mlk}}\pm i\rho', & \beta^{(1,2)}_{mlk} &= \frac{\left(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\right)\pm\sqrt{\left(\frac{A_{mlk}}{B_{mlk}}-\alpha_{lk}\right)^2-D_{mlk}}}{D_{mlk}}, \\
Q_{mlk} &= -2\left(\tfrac{C_{mlk}}{B_{mlk}}\right)-2\left(\tfrac{d_{lk}}{AC_{lk}}\right)\beta_{mlk}, & P_{mlk} &= -2\left(\tfrac{A_{mlk}}{B_{mlk}}-\alpha_{lk}-\beta_{mlk}D_{mlk}\right), \\
E_{mlk} &= -2d_{lk}/AC_{lk}, & S^{(\sigma)}_{mlk} &= D_{mlk}+P_{mlk}\sigma_{mlk},
\end{aligned}
$$

with $\sigma_{mlk} = 0, -11/\beta mlk$. The $S(\sigma_{mlk}, z)$ and $G(z)$ are obtained:

$$S(\sigma_{mlk}, z) = S^{(\sigma)}_{mlk} z^2 + (E_{mlk} + Q_{mlk}\sigma_{mlk})z - m_k^2 + i\rho, \tag{37}$$

$$G^{-1}(z) = Z_{mlk}\, z^2 + K_{mlk} z - \beta_{mlk}(m_k^2 - i\rho) - F_{nmlk} Q_{mlk}, \tag{38}$$

with $Z_{mlk} = D_{mlk}\beta_{mlk} - P_{mlk}$ and $K_{mlk} = E_{mlk}\beta_{mlk} - Q_{mlk} - P_{mlk}F_{nmlk}$. The functions $f^{\pm}_{lk}$ (and $g^{\pm}_{mlk}$) are defined as in (26) with replacing c_{lk}/AB_{lk} by d_{lk}/AC_{lk} (and C_{mlk}/B_{mlk}) respectively. The J_4 in (36) is decomposed into two basic integrals as follows:

$$\mathcal{I}_1 = \int_0^\infty \frac{1}{(z+T_1)(z+T_2)} dz = \mathcal{R}_{-1}(1, 1; T_1, T_2), \tag{39}$$

$$\mathcal{I}_2 = \int_0^\infty \frac{\ln(1+z/T_3)}{(z+T_1)(z+T_2)} dz = \tag{40}$$

$$= \lim_{\omega\to 0} \frac{1}{\omega} \left\{ \int_0^\infty \frac{1}{(z+T_1)(z+T_2)} dz - \int_0^\infty \frac{(1+z/T_3)^{-\omega}}{(z+T_1)(z+T_2)} dz \right\} \tag{41}$$

$$= \lim_{\omega\to 0} \frac{1}{\omega} \left\{ \mathcal{R}_{-1}(1, 1; T_1, T_2) - \frac{\mathcal{B}(1+\omega, 1)}{T_3^{-\omega}} \mathcal{R}_{-1-\omega}(1, 1, \omega; T_1, T_2, T_3) \right\}. \tag{42}$$

The ε-expansions for all $\mathcal{R}$-functions appear in this paper have devoted in Ref. [17]. The numerical checks for all analytic formulas in this paper and applications of this work to compute Feynman diagrams in real scattering processes have shown in [17].

3 Conclusions

We have presented the analytic results for scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The analytic results in this paper are valid for both real and complex internal masses. The calculations have carried out in general space-time dimension for two- and three-point functions. At present work, the four-point functions have performed in $D = 4$. The analytic formulas have expressed in terms of the $\mathcal{R}$-functions. In future work, we will extend this work to tensor one-loop integrals (to be published).

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Appendix: Useful relations for $\mathcal{R}$-functions

Useful relations for $\mathcal{R}$-functions are also listed in this appendix. The formulas shown here are collected from Ref. [18]. We denote that b, z and e_i are k-tuple

$$b = (b_1, b_2, \cdots, b_k), \tag{43}$$

$$z = (z_1, z_2, \cdots, z_k), \tag{44}$$

$$e_i = (0, 0, \cdots, 1, 0, \cdots, 0) \quad \text{where the 1 is located at the } i\text{th entry.} \tag{45}$$

The relations are presented as follows

$$\mathcal{R}_t(b,z) = \sum_{i=1}^{k} \frac{b_i}{\beta} \mathcal{R}_t(b+e_i,z), \tag{46}$$

$$\mathcal{R}_{t+1}(b,z) = \sum_{i=1}^{k} \frac{b_i}{\beta}\, z_i\, \mathcal{R}_t(b+e_i,z), \tag{47}$$

$$\beta\mathcal{R}_t(b,z) = (\beta+t)\mathcal{R}_t(b+e_i,z) - t z_i \mathcal{R}_{t-1}(b+e_i,z), \tag{48}$$

$$\partial_{z_i}\mathcal{R}_t(b,z) = \frac{b_i}{\beta} t \mathcal{R}_{t-1}(b+e_i,z), \tag{49}$$

$$\mathcal{R}_t(b,z) = \prod_{i=1}^{k} z_i^{-b_i} \mathcal{R}_{-\beta-t}(b+e_i,z^{-1}), \quad \text{Euler's transformation} \tag{50}$$

$$\mathcal{R}_t(b,\lambda z) = \lambda^t \mathcal{R}_t(b,z) \quad \text{scaling law.} \tag{51}$$

References

[1] A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Nucl. Phys. B **724** (2005) 247.
[2] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.
[3] D. T. Nhung and L. D. Ninh, Comput. Phys. Commun. **180** (2009) 2258.
[4] A. Denner and S. Dittmaier, Nucl. Phys. B **844** (2011) 199.
[5] D. Kreimer, Z. Phys. C **54** (1992) 667.
[6] D. Kreimer, Int. J. Mod. Phys. A **8** (1993) 1797.
[7] J. Franzkowski, *Dissertation*, Mainz 1997.
[8] K. H. Phan, PTEP **2017** (2017) no.6, 063B06.
[9] G. Cullen, J. P. Guillet, G. Heinrich, T. Kleinschmidt, E. Pilon, T. Reiter and M. Rodgers, Comput. Phys. Commun. **182** (2011) 2276
[10] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.
[11] G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0803** (2008) 042.
[12] E. E. Boos and A. I. Davydychev, Theor. Math. Phys. **89** (1991) 1052 [Teor. Mat. Fiz. **89** (1991) 56].
[13] A. I. Davydychev, J. Math. Phys. **33** (1992) 358.
[14] A. I. Davydychev and R. Delbourgo, J. Math. Phys. **39** (1998) 4299
[15] J. Fleischer, F. Jegerlehner and O. V. Tarasov, Nucl. Phys. B **672** (2003) 303
[16] J. Blümlein, K. H. Phan and T. Riemann, Acta Phys. Polon. B **48** (2017) 2313.
[17] Khiem Hong Phan and Thinh Nguyen Hoang Pham, arXiv:1710.11358 [hep-ph].
[18] B. C. Carlson, Special Functions of Applied Mathematics. New York: Academic Press, 1977.