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Gravitational lensing in conformal and emergent gravity

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Abstract. The gravitational bending of light in the framework of conformal gravity is considered where an exact solution for null geodesics in the Mannheim-Kazanas is obtained. The linear coefficient γ characteristic to conformal gravity is shown to contribute enhanced deflection compared to the angle predicted by General Relativity for small γ. We also briefly consider gravitational lensing in covariant emergent gravity.

1 Introduction

The problem of galactic rotation curves remains one of the outstanding problems in astrophysics and gravity. The fact that the observed velocity distribution curve in galaxies do not follow the predictions of standard gravity based on the observed luminous mass is usually interpreted as the presence of dark matter. However, we could also consider the possibility that the observed curve indicates a deviation from standard gravity. Since most experimental tests of General Relativity were performed within solar system scales, this might mean that our theories of gravity requires modification at long length-scales, such as the order of galactic lengths where the peculiar rotation curves are observed.

One candidate for an alternative theory of gravity is Weyl's conformal gravity. This theory is based on conformal symmetry, and solutions to this theory may possibly explain the galactic rotation curves [1, 3, 4]. Another feature of conformal gravity is that the quantisation of this theory is possibly renormalisable [7, 8].

Another possible explanation for the observed rotation curves might come from the idea of emergent gravity by Verlinde [9, 10], which argues that the competition between the volume and area contribution of a spacetime's entropy results in an additional force. This force presumably gives rise to the observed galactic rotation curves. Recently, Hossenfelder proposed [17] a covariant Lagrangian to cast Verlinde's theory in the form of an action principle.[1] Having an action, general field equations for arbitrary spacetime configurations can be derived [19].

As both these theories outline are metric theories of gravity, they also contain the phenomena of gravitational lensing. Both these theories give different bending angles compared to standard General Relativity. Therefore it is worthwile to explore, at least theoretically, the lensing properties of these theories so that subsequent checks can be made against observations.

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[1]See also, Ref. [18].

2 Conformal gravity and its spherically-symmetric solution

As was mentioned in the Introduction, conformal gravity is based on the idea of conformal symmetry, and thus has a conformally-invariant action given by

$$I = \alpha \int \mathrm{d}^4x\, \sqrt{-g}\, C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} + I_{\mathrm{m}}, \tag{1}$$

where $C_{\mu\nu\lambda\sigma}$ is the Weyl tensor and I_{m} is the matter action generating the stress tensor $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta I_{\mathrm{m}}}{\delta g^{\mu\nu}}$. The equations of motion extremising the action is

$$\left(2\nabla^\sigma\nabla^\lambda + R^{\sigma\lambda}\right)C_{\mu\sigma\lambda\nu} = \frac{1}{4\alpha}T_{\mu\nu}. \tag{2}$$

A spherically-symmetric vacuum solution to Eq. (2) is [1, 2]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{3}$$

with

$$f(r) = \sqrt{1-6m\gamma} - \frac{2m}{r} + \gamma r - kr^2. \tag{4}$$

This solution has three parameters m, k, and γ, where the former two are the familiar mass and a de Sitter-like parameter, respectively. This is easy to see by setting $\gamma = 0$, and the solution reduces to the Schwarzschild-de Sitter solution, and k is the inverse de Sitter length scale. Hence we regard γ as the parameter that characterises conformal gravity. It can be seen in Eq. (4) that a non-zero γ contributes a linear gravitational potential, and it is this linear term that was argued to explain the shape of the galactic rotation curves [3].

It is worth noting that this solution also occurs in other theories of gravity. In particular, the spherically-symmetric solution with a linear potential also ccurs in the dilaton-reduced action of gravity [5, 6], and also $f(R)$ gravity [11].

3 Gravitational lensing in conformal gravity

3.1 Equations of motion

Gravitational lensing in the spacetime (3) has already been considered before in earlier literature [12, 14, 15], but various works seem to disagree on the expression for the bending angle $\hat{\alpha}$. Here we attempt to resolve this agreement by deriving an exact expression for the bending angle, without any approximations as was done in the previous literature.

The motion of a time-like or null particle is described by a trajectory $x^\mu(\tau)$, where τ is an appropriate affine parametrisation. The geodesic equations are determined by the Lagrangian $L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, where over-dots denote derivatives with respect to τ. The equations of motion may be derived using the Euler-Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial\dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}$. Due to the spherical-symmetry of the spacetime, we may fix $\theta = \frac{\pi}{2}$ = constant without loss of generality.

Since $\partial/\partial t$ and $\partial/\partial\phi$ are Killing vectors of the spacetime, we have the first integrals of motion

$$\dot{t} = \frac{E}{f}, \quad \dot{\phi} = \frac{\Phi}{r^2}, \tag{5}$$

where E and Φ are constants of motion, which we may interpret as the energy and angular momentum of the particle, respectively.

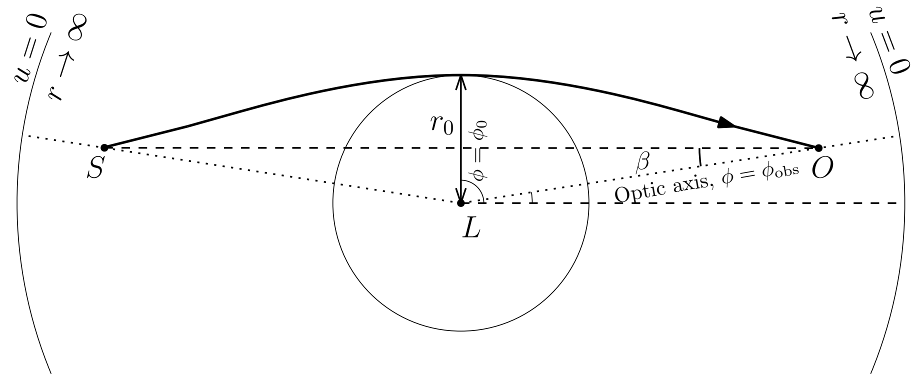

Figure 1. The trajectory of light from source S to observer O, passing at a distance of closest approach r_0 to the lens L. The asymptotic region $r \to \infty$ is represented by the outer circular arcs. We assume the trajectory does not cross either the cosmological or event horizons of the spacetime. Here we have drawn the angles ϕ_0 and $\phi_{\rm obs}$ to be relative to the horizontal dashed line, implying that this horizontal line is the $\phi = 0$ angle. However this is clearly an arbitrary choice and does not affect the analysis.

The equation of motion for r is

$$\dot{r}^2 + r^2 f \dot{\theta}^2 = E^2 - V^2. \tag{6}$$

where V^2 is the effective potential given by

$$V^2 = \frac{\Phi^2}{r^2} f. \tag{7}$$

For the case of null geodesics, the energy and angular momentum always appears in the combination $\frac{E^2}{\Phi^2}$. It follows from Eq. (6) that

$$\frac{E^2}{\Phi^2} = \frac{f(r_0)}{r_0^2}, \tag{8}$$

where $r = r_0$ is the 'distance of closest approach', which is the radial position when $\dot{r} = 0$. With these considerations, the equations of motion reduce to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = r^4 \left[\frac{f(r_0)}{r_0^2} - \frac{f(r)}{r^2}\right]. \tag{9}$$

One can prove that any solution to (9) is symmetric about the point $r = r_0$.

To calculate the bending angle, we consider photon trajectories where the particle reaches $r = r_0$ at the initial angle $\phi = \phi_0$, as shown in Fig. 1. (We assume throughout that r_0 lies outside the horizon of the spacetime.) It is convenient to introduce the substitution $u = 1/r$, so that Eq. (9) becomes

$$\begin{aligned}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 &= \frac{r_0\sqrt{1-6m\gamma} - 2m + \gamma r_0^2}{r_0^3} - \gamma u - \sqrt{1-6m\gamma}\, u^2 + 2mu^3, \\ &= 2m\,(u_+ - u)\,(u_0 - u)\,(u - u_-)\,. \end{aligned} \tag{10}$$

In the second line above we have factorised the third-order polynomial where the roots are given by

$$u_0 = \frac{1}{r_0}, \quad u_\pm = \frac{r_0\sqrt{1-6m\gamma} - 2m \pm \sqrt{r_0^2 + 2m\gamma r_0^2 + 4r_0 m\sqrt{1-6m\gamma} - 12m^2}}{4mr_0}. \tag{11}$$

Therefore, the equations can be solved by performing the integration

$$\int_{\phi_0}^{\phi} \mathrm{d}\phi' = \pm \int_{u_0}^{u} \frac{\mathrm{d}u'}{\sqrt{2m\,(u_+ - u')\,(u_0 - u')\,(u' - u_-)}}. \tag{12}$$

Because the spacetime is invariant under the reflection $\phi \to -\phi$, we can, without loss of generality, select the lower sign and evaluate the integral exactly, giving[2]

$$\phi(u) = \phi_0 + \frac{2}{\sqrt{2m\,(u_+ - u_-)}}\mathrm{F}\left(\sin^{-1}\sqrt{\frac{(u_+ - u_-)(u_0 - u)}{(u_0 - u_-)(u_+ - u)}}, \sqrt{\frac{u_0 - u_-}{u_+ - u_-}}\right), \tag{13}$$

where $\mathrm{F}(p, q)$ is the incomplete elliptic integral of the first kind. We can express u as a function of ϕ by inverting to obtain

$$u(\phi) = \frac{u_+(u_0 - u_-)\mathrm{sn}\left(\sqrt{\frac{m(u_+-u_-)}{2}}(\phi - \phi_0), \sqrt{\frac{u_0-u_-}{u_+-u_-}}\right)^2 - u_0(u_+ - u_-)}{(u_0 - u_-)\mathrm{sn}\left(\sqrt{\frac{m(u_+-u_-)}{2}}(\phi - \phi_0), \sqrt{\frac{u_0-u_-}{u_+-u_-}}\right)^2 - (u_+ - u_-)}, \tag{14}$$

where $\mathrm{sn}(p, q)$ is the Jacobi elliptic function of the first kind. Equation (14) can be verified independently against a numerical integration of the equations of motion.

Having a full expression for the photon trajectory, we are now able to calculate the bending angle exactly. We shall also apply the Rindler-Ishak approach to extract the bending angle from a coordinate-independent quantity [13].

To this end, we establish a Euclidean ortho-normal frame at the location of the observer,

$$\vec{e}_{(r)} = \sqrt{f(r)}\partial_r, \quad \vec{e}_{(\theta)} = \frac{1}{r}\partial_\theta, \quad \vec{e}_{(\phi)} = \frac{1}{r\sin\theta}\partial_\phi. \tag{15}$$

The photon's 4-velocity when it reaches the observer can easily be calculated from the solution (14). Taking the spatial components, we have the Euclidean 3-vector

$$\vec{v} = \vec{v}_{||} + \vec{v}_\perp; \quad \vec{v}_{||} = \dot{r}\partial_r, \quad \vec{v}_\perp = \dot{\theta}\partial_\theta + \dot{\phi}\partial_\phi. \tag{16}$$

Now, we define ψ as the angle between the photon's spatial 3-velocity and the optic axis. Since $\vec{e}_{(r)}$ is parallel to the optic axis, ψ may be calculated from

$$\cos\psi = \vec{e}_{(r)} \cdot \frac{\vec{v}}{|\vec{v}|}. \tag{17}$$

Using trigonometric identities and the equations of motion, we simplify the expression for ψ and obtain the invariant bending angle as

$$\hat{\alpha} \equiv 2\psi = 2\sin^{-1}\left.\sqrt{\frac{u(\phi)^2 f(1/u(\phi))}{u_0^2 f(1/u_0)}}\right|_{\phi=\phi_{\mathrm{obs}}}, \tag{18}$$

and the bending angle is simply $\hat{\alpha} = 2\psi$.

[2]The integral in the right-hand side of (12) can be found in 3.131-4, pg. 254 of [16].

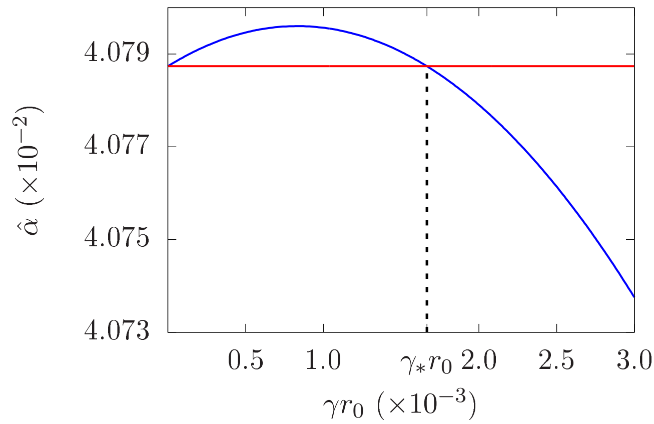

Figure 2. (Colour online.) Plot of $\hat{\alpha}$ vs. γr_0, for $m = 0.01 r_0$, $k = 0$, and $\beta = 0$. The horizontal red line corresponds to the Schwarzschild deflection value. In this case, we find that the deflection is greater than the Schwarzschild case for the range $0 < \gamma < \gamma_*$, where $\gamma_* \simeq 1.6657 \times 10^{-3} r_0^{-1}$.

3.2 Results

Having an exact formula for the bending angle, we are now ready to investigate how this value deviates from that of standard gravity as γ changes.

Fig. 2 shows how $\hat{\alpha}$ changes as γ is varied for $m/r_0 = 0.01$. In particular, we note that as γ is increased from zero, the bending angle is increased compared to standard gravity (which is deflection by the Schwarzschild metric), which is marked by the horizontal red line at $\hat{\alpha} \simeq 4.079 \times 10^{-2}$. However, if γ is increased beyond some γ_*, it becomes smaller compared to the Schwarzschild case. This behaviour seems to be generic for various choices of m/r_0. Therefore in each case, there seems to be a maximum γ for which conformal gravity predicts a positive correction to the standard bending angle.

Exploring the parameter space further, we can also see that when γ is too large, the solutions no longer exist. This is because for this range of parameters, the defocusing effect is too strong such that it is not possible for a beam of light to start from a source on the optic axis to be deflected back to the axis again where the observer is located.

In light of this, we explore the parameter space to see which of the three possibilities occur: (1) Positive correection to standard gravity, (2) negative correction to standard gravity, and (3) trajectory does not exist. These are shown visually in Fig. 3.

We can also find an approximate equation for the curve separating the lighter shade region and the white region. Namely, we are able to find the critical mass m_{crit} as a function of γ which tells us the maximum mass after which axis-to-axis lensing is no longer possible,

$$\frac{m_{\text{crit}}(\gamma)}{r_0} \sim \frac{1}{4}\gamma r_0 - \left(\frac{1}{8} + \frac{15}{256}\pi\right)\gamma^2 r_0^2 + \left(\frac{1}{16} + \frac{15}{1024}\pi + \frac{225}{8192}\pi^2\right)\gamma^3 r_0^3 + O(\gamma^4 r_0^4). \qquad (19)$$

From Eq. (19) and Fig. 3, we can see that in terms of lensing, we should not regard m and γ as independent parameters if we were to obtain a perturbative expression. In other words, for a given γ, the range of m must be chosen carefully so as the perturbation is not expanded into a non-existent solution, taking us into the white region in Fig. 3.

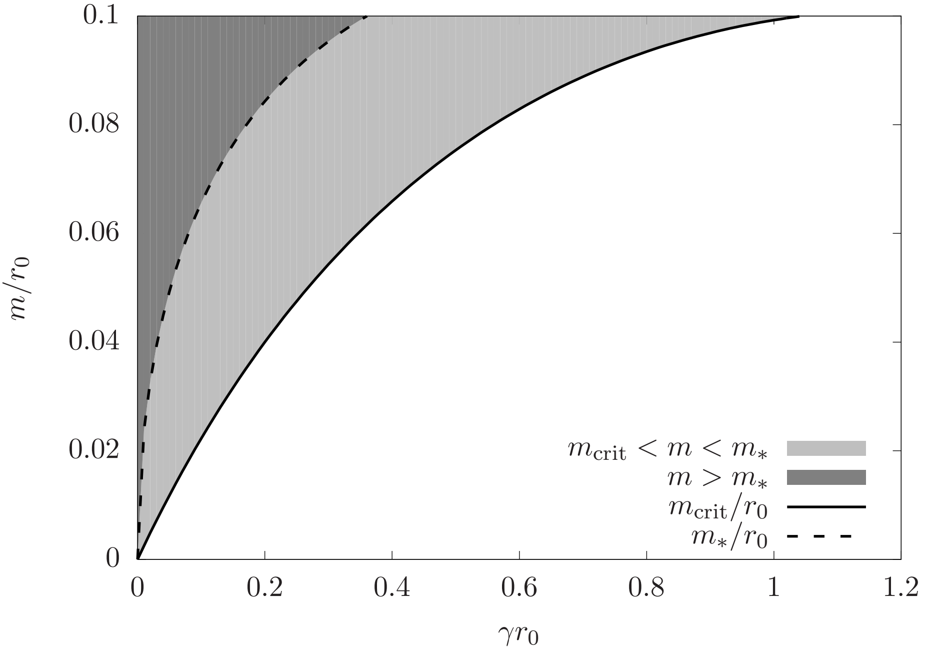

Figure 3. Plot of m/r_0 vs. γr_0, where $k = 0$. The shaded regions shows the range $m_{\rm crit} < m < m_*$ which gives reduced deflection, while the darker-shaded regions correspond to $m > m_*$ which gives enhanced deflection.

In order to take this into account, we express γ in terms of m by writing

$$\gamma r_0 = w\frac{m}{r_0}, \tag{20}$$

where now w is a free parameter. Written in this way, we now perform an expansion in small m/r_0, giving

$$\begin{aligned}\hat{\alpha} \sim\ & \sqrt{16-w^2}\frac{m}{r_0} + \frac{1}{\sqrt{16-w^2}}\left(-16+15\pi+8w-w^2+\frac{1}{2}w^3\right)\frac{m^2}{r_0^2} \\ & + \frac{1}{(16-w^2)^{3/2}}\left[\frac{6272}{3} - 480\pi + (-256+240\pi)\,w \right. \\ & \qquad \left. -\left(176-60\pi+\frac{225}{32}\pi^2\right)w^2 - (-32+30\pi)w^3 - \frac{25}{2}w^4 + w^5 + \frac{1}{3}w^6\right]\frac{m^3}{r_0^3} \\ & + \mathcal{O}\left(m^4/r_0^4\right). \end{aligned} \tag{21}$$

Here, we see that in order to have a real solution, w must not be greater than 4. This is consistent with Eq. (19) where the leading-order coefficient of $m_{\rm crit}$ is $1/4$.

4 Gravitational lensing in emergent gravity

Emergent gravity is described by the action

$$I = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{16\pi G}\left(R + \alpha\chi^{3/2}\right) - \mathcal{V}(u) + \frac{\beta}{2}\frac{u^\mu u^\nu}{u}T_{\mu\nu}\right), \tag{22}$$

where

$$\chi = 2a\nabla_\sigma u^\sigma \nabla_\lambda u^\lambda + (b+c)\nabla_\sigma u^\lambda \nabla^\sigma u^\lambda + (b-c)\nabla_\sigma u_\lambda \nabla^\lambda u^\sigma. \tag{23}$$

Normal (baryonic) matter feels the effective metric

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \beta\frac{u_\mu u_\nu}{u}. \tag{24}$$

The distinguishing feature of Hossenfelder's covariant action is that the emergent effects of the spacetime entropy is modelled by a vector field u^μ, and is called the *imposter field*. At the moment, we can see from the above that the theory carries two free parameters, α and β where the former parameterises the strength of the imposter field and the latter parametrises the coupling strength between baryonic matter and the imposter field.

Varying the action with respect to the metric gives the Einstein equations

$$\begin{aligned}\mathcal{G}_{\mu\nu} = 8\pi G&\left[T_{\mu\nu} + \frac{1}{2}\beta u^\lambda u^\sigma T_{\lambda\sigma}\left(g_{\mu\nu} + \frac{u_\mu u_\nu}{u^2}\right) + \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}u}\frac{u_\mu u_\nu}{u} - \mathcal{V}g_{\mu\nu}\right]\\ &- \alpha c\chi^{1/2}\left(\frac{3}{2}F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}F^{\sigma\lambda}F_{\sigma\lambda}g_{\mu\nu}\right) - \alpha\chi^{1/2}\Bigg(\frac{a}{2}\left(\epsilon^\sigma{}_\sigma\right)^2 g_{\mu\nu}\\ &\qquad - \frac{b}{4}\epsilon^{\sigma\lambda}\epsilon_{\sigma\lambda}g_{\mu\nu} + \frac{3b}{4}\epsilon^\lambda{}_\lambda\epsilon_{\mu\nu} - \frac{3b}{2}F_{\sigma(\mu}\epsilon^\sigma{}_{\nu)}\Bigg)\\ &- \frac{3\alpha}{2}u^\lambda\nabla_\lambda\left[\chi^{1/2}\left(a\epsilon^\sigma{}_\sigma g_{\mu\nu} + b\epsilon_{\mu\nu}\right)\right] - 3\alpha c\left[\nabla_\sigma\left(\chi^{1/2}F^\sigma{}_{(\mu}\right)\right]u_{\nu)},\end{aligned} \tag{25}$$

and varying with respect to u^μ gives the imposter equation

$$\begin{aligned}&\frac{3\alpha}{16\pi G}\nabla_\mu\left[\chi^{1/2}\left(a\epsilon^\sigma{}_\sigma g^{\mu\nu} + b\epsilon^{\mu\nu} + cF^{\mu\nu}\right)\right]\\ &\qquad = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}u}\frac{u^\nu}{u} + \frac{\beta}{2}\left(\frac{2u^\lambda T_\lambda{}^\nu}{u} + \frac{u^\sigma u^\lambda T_{\sigma\lambda}u^\nu}{u^3}\right).\end{aligned} \tag{26}$$

We now attempt to take a similar approach to the case of conformal gravity in Sec. 2, where we consider lensing by a spherically-symmetric lens mass. So far, we were able to obtain such a solution in the case $a = b = 0$, and $c = -1$, and is given by

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{(2)}, \quad u^\mu = \phi(r)\,\delta^\mu_t, \tag{27a}$$

$$f(r) = 1 - \frac{2M}{r} - \frac{\alpha q^3}{r}\ln\frac{r}{r_g}, \tag{27b}$$

$$\phi(r) = \frac{q}{f(r)}\ln\frac{r}{r_0}, \tag{27c}$$

This solution is parametrised by the mass parameter M and q which characterises the strength of the imposter field. Clearly, the case $q = 0$ reduces to that of the Schwarzschild solution.

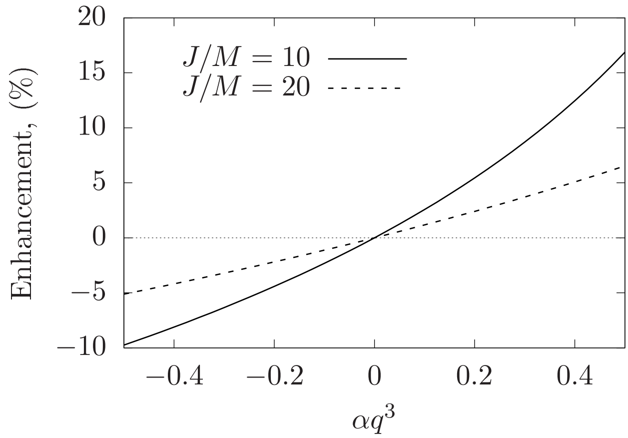

Figure 4. Percentage of bending-angle enhancement over standard GR, for impact parameters $J/M = 10$ and $J/M = 20$.

This solution is asymptotically flat, and we can calculate the bending angle by integrating $\frac{d\varphi}{dr}$ in the usual manner, namely by letting $r = 1/u$, giving

$$\Delta\varphi = 2\int_0^{u_{\min}} \frac{du}{\sqrt{u_{\min}^2 f(1/u_{\min}) - u^2 f(1/u)}}, \tag{28}$$

where $u_{\min} = 1/r_{\min}$ is the inverse of the coordinate r of closest approach.

Preliminary results are shown in Fig. 4, where the percentage enhancement against J/M is plotted. Here we define J as the impact parameter

$$J = \frac{r_{\min}}{f(r_{\min})}. \tag{29}$$

We see that if αq^3 is increased, emergent gravity predicts a positive contribution over standard gravity.

5 Conclusion

In this paper, we have considered gravitational lensing in two alternative theories of gravity, namely conformal gravity and covariant emergent gravity. The parameter space for which a positive correction to standard gravity is explored, and, in the case of conformal gravity, exact and perturbative expressions are obtained.

At this stage, it is important to emphasise that the results were obtained in an idealised case of a spherically-symmetric point mass, and that the source and observer are co-aligned with the lens. Of course, for usual astrophysical situations this may not occur. Though the theoretical results gives a rough idea of the expected observations compared to what we expect from standard General Relativity (possibly with dark matter).

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