Mathematical model of a fountain with a water picture in the shape of an hourglass

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Abstract. The work presents a mathematical model of a fountain providing an hourglass-shaped water image based on four of its basic parameters: height $H$, coefficient of contraction $\chi$, diameter of $n$ nozzles arrangement $D_n$, and coefficient of contraction location $\sigma$. The developed model makes possible to determine the angle $\alpha$ specifying the slope of the nozzles to the horizontal plane and determining the remaining parameters of the fountain, especially its outer diameter $D_f$ as well as the diameter of the position of the stream tops $D_{max}$. The developed model has been implemented in an Excel spreadsheet, which enables quick conversion of desired water images. The developed model works for $n \leq 24$ nozzles, however, derived dependencies allow for a quick extension of the model for $n > 24$.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>area</td>
</tr>
<tr>
<td>$d, D$</td>
<td>diameter</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$H$</td>
<td>height</td>
</tr>
<tr>
<td>$h$</td>
<td>vertical coordinate of jet path</td>
</tr>
<tr>
<td>$l$</td>
<td>horizontal coordinate of jet path</td>
</tr>
<tr>
<td>$L$</td>
<td>jet horizontal range</td>
</tr>
<tr>
<td>$n$</td>
<td>the number of nozzles in the assembly</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
</tr>
<tr>
<td>$\dot{v}$</td>
<td>volume flow rate</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>the angle of the nozzle inclination to</td>
</tr>
<tr>
<td>$\chi$</td>
<td>the level</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>coefficient of contraction</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>coefficient of contraction location</td>
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Indexes

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<tr>
<td>$c$</td>
<td>contraction</td>
</tr>
<tr>
<td>$max$</td>
<td>maximum value, e.g. height, range</td>
</tr>
<tr>
<td>$n$</td>
<td>nozzle</td>
</tr>
<tr>
<td>$na$</td>
<td>nozzle assembly</td>
</tr>
<tr>
<td>$0$</td>
<td>initial value</td>
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1 Introduction

Fountains for many centuries have been enriched by landscapes of castle parks, towns and rural areas. Their producers, as part of the competition, constantly introduce innovative improvements in the form of the variability of the water image, light effects and musical effects. Control of such complex facilities requires the use of modern computational techniques, usually PLC controllers. In turn, programming the controllers requires a mathematical model. The development of a mathematical model providing an hourglass-shaped water image is the subject of this paper.

Fig. 1. Basic geometric parameters of the designed fountain.
Fig. 1 shows the water image of the analysed fountain and its basic geometrical parameters: fountain height \( H_{\text{max}} \), nozzle assembly diameter \( D_{n} \), virtual contraction diameter of all jets \( D_{c} \), distance from the base of the fountain \( H_{f} \). Diameter of the fountain \( D_{f} \) is a parameter resulting from the setting of the above parameters.

2 Mathematical model

2.1 Path of free water jet

Outlet jet of water flows from nozzle of diameter \( d_{n} \) at an angle \( \alpha \) to the horizontal level with the initial speed \( v_{0} \). The initial velocity is treated as the average velocity over the entire nozzle cross-section, so that the product of this velocity and the transverse cross-section of the nozzle determines the volume flow through one nozzle

\[
\nu_{0} = \frac{d_{n}^{2}}{4} \pi \cdot v_{0} \quad \text{m}^{3} \cdot \text{s}^{-1},
\]

therefore the equation

\[
\nu_{0} \ldots d_{n} \cdot \ldots v_{0} \quad \text{m}^{3} \cdot \text{s}^{-1},
\]

determines the total flow rate through all the fountain nozzles.

The vector of the outgoing water velocity has two component vectors \( v_{0x} \) and \( v_{0y} \) (Fig. 1) of values \( v_{0x} = v_{0} \cos \alpha \) and \( v_{0y} = v_{0} \sin \alpha \).

![Fig. 2. Jet outgoing from nozzle a) throw with an angle to horizontal plane, b) schematic for rough analysis](image)

Water jet motion (its volume element) is the resultant motion of two straight-line movements [1]: uniform rectilinear with initial velocity \( v_{1} = v_{0x} = v_{0} \cos \alpha \) in the horizontal direction and non uniform rectilinear with initial velocity \( v_{0y} = v_{0} \sin \alpha \) in vertical direction. Therefore after the time \( t \) from the moment the traffic starts the observed water particle had travelled in a horizontal direction

\[
l = (v_{0} \cos \alpha) t
\]

and in the vertical direction

\[
h = (v_{0} \sin \alpha) t - \frac{gt^{2}}{2}
\]

Determining \( t \) from the equation (3) and substituting into (4) we obtain the equation of trajectory of the jet

\[
h = \frac{l \sin \alpha}{\cos \alpha} - \frac{gt^{2}}{2v_{0}^{2} \cos^{2} \alpha}
\]

This is a quadratic equation. Its graph, and hence the path of movement of the jet occurring from the nozzle inclined at an angle \( \alpha \) to the horizontal level is a parabola.

Horizontal range of the jet we obtain after substituting to path trajectory value of (for simplicity, we assume that the bottom of the fountain is horizontal)

\[
0 = \frac{L \sin \alpha}{\cos \alpha} - \frac{gL^{2}}{2v_{0}^{2} \cos^{2} \alpha} \quad \text{or} \quad \frac{L(v_{0}^{2} \sin 2 \alpha - gL)}{2v_{0}^{2} \cos^{2} \alpha} = 0
\]

Comparing nominator of equation (6) to zero, we obtain \( L(gL - v_{0}^{2} \sin 2 \alpha) = 0 \). This means that either \( L = 0 \), or the expression in brackets is zero. The first condition means the point from which the jet flows out, whereas means the point, where the jet drops. Therefore is a correct relationship \( gL - v_{0}^{2} \sin 2 \alpha = 0 \), so the range of the fountain \( l = L \) is

\[
l = \frac{v_{0}^{2} \sin 2 \alpha}{g}
\]

When designing fountains one of the basic parameters of the output is the maximum height to which the water jet reaches. This maximum height of the jet, we obtain by comparing to zero first derivative of the function \( h = f(l) \) defined by the formula (5). Therefore

\[
\frac{dh}{dl} = \frac{-2gL}{2v_{0}^{2} \cos^{2} \alpha} + \frac{\sin \alpha}{\cos \alpha} = 0
\]

whence

\[
l_{\text{sd}} = \frac{2v_{0}^{2} \sin \alpha \cos \alpha}{2g} = \frac{v_{0}^{2} \sin 2 \alpha}{2g} = 0,5L
\]

As is clear from comparison with formula (7) the maximum height is in the middle of the jet range. After substitution of the obtained value \( l_{\text{sd}} = 0,5L \) to the path equation (5), we obtain formula which specifies the maximum height of the water stream

\[
h_{\text{max}} = H = \frac{v_{0}^{2} \sin^{2} \alpha}{2g}
\]

2.2 Outflow geometry for the fountain image of hourglass type.

Water image of fountain in a hourglass shape can be determined on the basis of theoretical analysis of the spatial geometric relationships. It is assumed that it is a boundary of streams flowing from \( n \) nozzles of inner diameter \( d_{n} \) distributed evenly on a circle of diameter \( D_{n} \). The primary task was to find a mathematical model of analysed fountain, based on which is set the inclination angle \( \alpha \) of the nozzle to the horizontal plane. For analysis generalization were introduced dimensionless parameters:

- coefficient of fountain contraction

\[
\chi = \frac{D_{c}}{D_{n}}
\]

- coefficient of fountain contraction location

\[
\chi = \frac{D_{c}}{D_{n}}
\]
\[ \sigma = \frac{h}{h_{\text{max}}} \]  

(12)

In the light of formulas (11) and (12) we can write that \( D_c = \chi D_m \) and \( h_c = \sigma h_{\text{max}} \).

Fig. 3. Geometry of one jet: a) horizontal projection, b) projection on jet plane

Design parameters defined in this way define the basic parameters of the fountain water image. The jet plane jet is deflected from the plane passing through the axis of the fountain lying at point C and the point N by an angle of \( \beta \). Angle \( \beta \) can be determined from the rectangular triangle \( \Delta \) (Fig. 3a):

\[ \sin \beta = \frac{D_c}{D_m} = \frac{\chi D_m}{D_m} = \chi \quad \text{or} \quad \beta = \arcsin \chi \]  

(13)

In subsection 2.1 path of jet outgoing from nozzle (5) were determined. This jet should pass through the point \( A' \). From a mathematical point of view, this means that it must satisfy the equation of the jet path for the coordinates of this point (Fig. 3b).

After substituting \( l_c \) and \( h_c \) into equation of trajectory we get

\[ h_c = l_c \sin \alpha - \frac{g l_c^2}{2v_0^2 \cos^2 \alpha} \]  

(14)

From equation (14) we can determine the slope angle \( \alpha \) between the nozzle axis and horizontal level, however, it is necessary to know the initial velocity \( v_0 \) of outgoing jet. This speed can be counted from the equation (10)

\[ h_{\text{max}} = \frac{v_0^2 \sin^2 \alpha}{2g} \rightarrow v_0 = \sqrt{\frac{2gh_{\text{max}}}{\sin^2 \alpha}} \]  

(15)

After substituting for \( v_0^2 \) to (14) and using the geometric formula \( l_c = 0.5D_a \cos \beta = 0.5D_a \sqrt{1 - \chi^2} \) (Fig. 3a) and \( h_c = \sigma h_{\text{max}} \) we get

\[ 0.25D_m^2 (1 - \chi^2) \tan^2 \alpha + 2D_m h_{\text{max}} \sqrt{1 - \chi^2} \tan \alpha + 4\sigma h_{\text{max}}^2 = 0 \]  

(16)

This is a quadratic equation, so you can expect two angles: \( \alpha_1 \) for an ascending jet and \( \alpha_2 \) for falling jet.

This may be solved by using the formula:

\[ \tan \alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

where from equation (16): \( a = 0.25D_m^2 (1 - \chi^2) \), \( b = -2D_m h_{\text{max}} \sqrt{1 - \chi^2} \), and \( c = 4\sigma h_{\text{max}}^2 \).

If \( \Delta = b^2 - 4ac \)

\[ = \left( 2D_m h_{\text{max}} \sqrt{1 - \chi^2} \right)^2 - 4 \cdot 0.25D_m^2 (1 - \chi^2) \cdot 4\sigma h_{\text{max}}^2 \]  

(17)

hence

\[ \tan \alpha_{1,2} = \frac{2D_m h_{\text{max}} \sqrt{1 - \chi^2} \pm \sqrt{\Delta}}{2 \cdot 0.25D_m^2 \left( 1 - \chi^2 \right)} \]  

(18)

\[ = \frac{4 \left( 1 \pm \sqrt{1 - \sigma} \right) h_{\text{max}}}{D_m \sqrt{1 - \chi^2}} \]

therefore

\[ \alpha_{1,2} = \arctan \left( \frac{4 \left( 1 \pm \sqrt{1 - \sigma} \right) h_{\text{max}}}{D_m \sqrt{1 - \chi^2}} \right) \]  

(19)

In degrees slop of nozzle is determined by equation:

\[ \alpha_{1,2} = \frac{180^\circ}{\pi} \arctan \left( \frac{4 \left( 1 \pm \sqrt{1 - \sigma} \right) h_{\text{max}}}{D_m \sqrt{1 - \chi^2}} \right) \quad [^\circ] \]  

(20)

Fig. 4. Jet path on \((l, h)\) plane for selected design parameters

Fig. 4 shows the path of the water jet flowing from the nozzle at an angle of \( \alpha \) determined on the basis of selected design parameters of the fountain: \( D_m = 0.29 \text{ m} \), \( \chi = 0.5 \) (\( D_1 = 0.145 \text{ m} \)), \( H = h_{\text{max}} = 4 \text{ m} \), and \( \sigma = 0.5 \) (\( H_c = 2 \text{ m} \)). For this case, equation (16) becomes:

\[ 0.0158 \tan^2 \alpha - 2.0092 \tan \alpha + 32 = 0 \]

and the angle \( \alpha \) determined from equation (19) or (20) have values:

\[ \alpha_1 = 1.517256 \text{ rad} \quad \alpha_1 = 86.93234^\circ \]

\[ \alpha_2 = 1.561602 \text{ rad} \quad \alpha_2 = 89.47318^\circ \]
2.3 The diameters of the fountain image of hourglass type.

The basic parameters of the fountain include: its radius \( R_f \) which is measured at ground level and radius \( R_{\max} \) which specifies the distances of the \( n \) jets tops the axis passing through the point C (the centre of the fountain). Both of these parameters are dependent on the basic construction parameters, i.e., \( R_f = f(D_m, \chi, H, \sigma) \) and \( R_{\max} = f(D_m, \chi, H, \sigma) \).

![Diagram of fountain with dimensions](image)

**Fig. 5.** Top view of fountain – basic dimensions

Fig. 5 presents a fountain diagram in a horizontal view. In this drawing both parameters discussed above are visible. To determine the \( R_{\max} \) and \( R_f \) ancillary diagrams were prepared, respectively b) and c). All \( n \) traces have a length \( L \) defined by formula (7). To this formula was substituted square of initial velocity \( v_0 = \sqrt{2gh} \) designated by the formula (15). Formula defining the range of the stream, therefore takes the form

\[
L = \frac{4h_{\max} \sin \alpha \cos \alpha}{\sin^2 \alpha} = 4h_{\max} \tan \alpha
\]

After substituting for the equation (21) value of previously determined angle \( \alpha \), (the range of each stream, which is the part of the fountain), is specified by a formula

\[
L = 4h_{\max} \tan^{-1} \left( \frac{4 \left(1 \pm \sqrt{1 - \chi^2} \right) h_{\max}}{D_n \sqrt{1 - \chi^2}} \right)
\]

\[
R_{\max} = \sqrt{\left(0.5D_n \right)^2 + (0.5L)^2 - 0.5D_n L \cos \beta}
\]

3 Simulation of a fountain picture

Fig. 6 shows a fountain diagram (horizontal projection) illustrating geometrical relationships allowing the simulation of a fountain image.

![Diagram used to determine the basic parameters of the simulation](image)

The simulation was carried out in a few steps

- determine the angle \( \phi_i \)

Angle \( \phi_i \) is measured from the straight line connecting the centre of the fountain C with the axis of the first nozzle \( n = 1 \) in counter clockwise direction, it is determined by the relation:

\[
\phi_i = \frac{2\pi}{n} (i - 1) \text{ rad}
\]
determination of the $y_i$ coordinate of the position of 
the vector origin $L_y$ in the axis $y$ – e.g. point 6, coordinate in Fig. 6

$$y_i = -0.5D_{oi} \cos \phi_i \quad m \quad (28)$$

Minus in formula (27) follows from the fact that the 
nozzle 1 is located in the negative portion of the axis $y$.

- determination of the angle $\gamma_i$, which determines the 
  deviation of the trace of the stream flow surface from 
  the straight line $C-1$.

$$\gamma_i = \phi_i - \beta \quad \text{where} \quad \beta = \arcsin \chi \quad (29)$$

- determining of jest horizontal projection length $L_i$ 
  and its division into $j$ calculation points. 
The simulation presented in the paper was based on $m = 41$ calculation points. Therefore, the coordinate of the calculation points is

$$l_j = \frac{L}{m} \quad (30)$$

- determination of calculation points coordinates on 
  the axis $y$.

$$l_{yi} = y_i + l_j \cos \gamma_i \quad (31)$$

where coordinate $y_i$ is constant for a given jet (27), 
e.g. for $i = 6$, $y_6 = -0.0375 m$.

- creation of a matrix consisting of $m$ rows (the 
  number of calculation points for each stream), and $n$ 
  columns (number of nozzles) on the basis of formula 
  (31).

- to each column should be assigned values

$$h_i = f(l_i)$$

which will be visible on the chart. It was 
the most time-consuming step of simulation 
preparing. After the introduction of the first pair 
of series of data, there was created a graph illustrating 
the course of the value projection e.g. stream from 
the nozzle 1 on the plane of the desired view. Then 
you had to choose a chart, go to the Chart Tools on 
the command bar to the Design tab and choose the 
Select Date (Fig. 7).

This causes calling the Select Data Source dialog. If 
you want to add another pair of data series should in the 
window Legend Entries (Series), click the Add button.

![Fig. 7. Chart Tools bar with a button Select Data](image)

**Fig. 7. Chart Tools bar with a button Select Data**

**Fig. 8. Dialogue window Select Data Source**

4 Conclusions

- From the practical point of view, the most significant 
is the formula (19) allowing to determine the angle of 
inclination of the nozzle to the ground horizontal 
level $\alpha$, i.e. the basic parameter which defines the 
angle of outflow of the nozzle, ensuring achievement 
of the assumed construction parameters of the 
fountain.

The formulas (24) and (26) allow calculation the 
diameter of the fountain $D_i$ and diameter of tops jets 
distribution, in Fig. 4 named as $M_1$. It may seem 
surprising that the design parameter $h_{max}$ does not 
exist in these formulas. However, it is hidden in 
the factor $\sigma$.

- Described simulation enables rapid image analysis of 
water fountains, which is important from the point of 
view of the manufacturer fountains negotiations with 
the client. Fountains sample image shown in Fig. 10, 
Fig. 11, Fig.12, and Fig 13.

- Fig. 10 shows the fountain picture in a scale of length 
design parameters sizes similar to the actual 
dimensional ratio. $h/l \approx 1$. Time to the new image 
fountains, including to enter of new parameters is 
measured in seconds.

- Figs 11, 12, 13 are presented in a reduced height 
scale $h/l \vert \|$.

- Shortly before this work was written, preliminary 
tests were carried out. The result seems to confirm 
the theory of the model, see Fig. 14. However, with 
the final conclusion we should wait for the device to 
be made, which allows the exact $\alpha$ angle setting.

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Fig. 10. Simulation effect for $H = 4\text{m}$, $D_m = 0.29\text{m}$, $\chi = 0.5$, and $\sigma = 0.5$

Fig. 11. Simulation effect for $H = 4\text{m}$, $D_m = 0.29\text{m}$, $\chi = 0.3$, and $\sigma = 0.5$

Fig. 12. Simulation effect for $H = 4\text{m}$, $D_m = 0.29\text{m}$, $\chi = 0.5$, and $\sigma = 0.3$

Fig. 13. Simulation effect for $H = 4\text{m}$, $D_m = 0.29\text{m}$, $\chi = 0.5$, and $\sigma = 0.7$

Fig. 14. (Left) Very initial test results. (Right) Assemble of nozzles – the paper treats the outer set.

References