Hadronic light-by-light scattering in the muon $g - 2$

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Abstract. We briefly review the current status of the hadronic light-by-light scattering contribution to the anomalous magnetic moment of the muon. Based on various model calculations in the literature, we obtain the estimate $a_{\mu}^{\text{HLbL}} = (102 \pm 39) \times 10^{-11}$. Recent developments including more model-independent approaches using dispersion relations and lattice QCD, that could lead to a more reliable estimate, are also discussed.

1 Introduction

Since 1947, the anomalous magnetic moments of the electron and the muon have always triggered new approaches and developments in loop calculations in quantum field theories [1, 2]. The muon $g - 2$ thereby serves as an important precision test of the Standard Model (SM) [1, 3]. For some time already, there is a discrepancy $a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} \approx (300 \pm 80) \times 10^{-11}$ of $3 - 4\sigma$ between experiment and theory. This could be a sign of New Physics, but the theoretical uncertainties from hadronic vacuum polarization (HVP) and hadronic light-by-light scattering (HLbL) need to be better controlled to draw firm conclusions. This issue is even more pressing in view of more precise new experiments [4] at Fermilab and J-PARC, with a precision goal of $\delta a_{\mu}^{\text{exp}} = 16 \times 10^{-11}$ in a few years time.

For the HLbL contribution, the following estimates are frequently used:

$$a_{\mu}^{\text{HLbL}} = (105 \pm 26) \times 10^{-11}, \quad [5], \quad (\text{“Glasgow consensus”}), \quad (1)$$

$$a_{\mu}^{\text{HLbL}} = (116 \pm 39) \times 10^{-11}, \quad [1, 6]. \quad (2)$$

Note that they are both based on almost the same input from calculations by various groups using different hadronic models [7–10], which suffer from uncontrollable uncertainties. More model-independent approaches, using dispersion relations (data driven) [11–15] and lattice QCD [16–18], have been proposed and first promising results have been obtained recently. To come up with a more refined estimate for HLbL with a controlled uncertainty is also one of the goals of a recently formed “Muon $g - 2$ Theory Initiative” [19] that will accompany the upcoming new experiments.

2 Current status of HLbL: model calculations

The HLbL contribution to the muon $g - 2$ contains the QCD correlation function of four hadronic electromagnetic currents, connected by off-shell photons to the muon line, see Figure 1. Within a

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Figure 1. The different contributions to HLbL scattering in the muon $g-2$ and their chiral and large- N_c counting.

hadronic picture, the four-point function is decomposed into single-meson exchanges and loops of hadrons, e.g. pions. Often the QCD short-distance part is modelled by a dressed constituent quark loop, which raises issues of double counting. The couplings of the hadrons (and the constituent quarks) to the photons involve, in general, momentum dependent vertex functions (form factors). The different contributions have been classified in Ref. [20] according to their leading order in the chiral expansion p^2 and their large- N_c counting to bring some order and systematics into the calculations.

The relevant momentum scales in HLbL are around 0 – 2 GeV, i.e. in the non-perturbative resonance region of QCD. The QCD four-point function is, however, a very complicated object that involves many Lorentz structures [8, 11] that depend on several invariant photon momenta with mixed regions of small and large momenta. Therefore the distinction between low and high energies and the use of an effective field theory approach (chiral perturbation theory with hadronic resonances) at low momenta and of perturbative QCD at high momenta is not so straightforward. So far only hadronic models have been used to estimate the full HLbL contribution. A selection of these results and some compilations, including those quoted in Eqs. (1) and (2), is shown in Table 1. One important difference between these two compilations is the combination of the errors of the individual contributions. They are combined in quadrature in Ref. [5] and linearly in Refs. [1, 6], as was done in Ref. [8]. Since these are model errors, not experimental uncertainties, both ways of combining them can be questioned.

The contribution from the light pseudoscalars π^0, η, η' is numerically dominant according to most model calculations. Because of this observation, there are many evaluations of this contribution, see Refs. [1, 22, 23] and references therein. The central value is about $a_\mu^{\text{HLbL:P}} = 90 \times 10^{-11}$ with a spread for most calculations of about 15% (but 30% if the central values of all estimates are taken into account), which can be understood by looking at the relevant momentum regions in a 3-dimensional integral representation [1, 23], with model-independent weight functions that are peaked below 1 GeV for the pion and below about 1.5 – 2 GeV for η, η' . As long as the transition form factors fall off for large momenta, one obtains always very similar results. In Ref. [10] a QCD short-distance constraint from the operator product expansion (OPE) on the four-point function was derived by connecting it to the chiral triangle anomaly. The constraint is then saturated by the pion-pole contribution alone which is a model assumption. This leads to an increased value, since there is no pion transition form factor at the external vertex, but then no quark-loop contribution should be added.

The other contributions to HLbL are, however, not negligible at the level of the precision goal of $(15-20) \times 10^{-11}$ needed to match future experiments. For the dressed pion-loop there is a strong model-dependence, cf. the results obtained in Refs. [7, 8] as discussed in Refs. [10, 24]. There is also some

Table 1. Selection of model estimates for the various contributions to $a_\mu^{\text{HLbL}} \times 10^{11}$.

Contribution	[7]	[8]	[9]	[10]	[21]	[5]	[1, 6]
π^0, η, η'	82.7±6.4	85±13	83±12	114±10	–	114±13	99±16
axial vectors	1.7±1.7	2.5±1.0	–	22±5	–	15±10	22±5
scalars	–	–6.8±2.0	–	–	–	–7±7	–7±2
π, K loops	–4.5±8.1	–19±13	–	–	–	–19±19	–19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	9.7±11.1	21±3	–	–	–	2.3 (c-quark)	21±3
Total	89.6±15.4	83±32	80±40	136±25	110±40	105±26	116±39

numerical cancellation with the dressed quark-loop. On the other hand, recent reevaluations [25–27] of the axial-vector contribution lead to a much smaller estimate $a_\mu^{\text{HLbL};\text{axial}} = (8 \pm 3) \times 10^{-11}$ than in Ref. [10]. Using these new evaluations and the observation that the contribution from tensor mesons seems to be very small, $a_\mu^{\text{HLbL};\text{tensor}} = 1 \times 10^{-11}$ [15, 25], lead us to suggest the following update of our earlier estimate [1, 6] for the HLbL contribution (see also Ref. [27]):

$$a_\mu^{\text{HLbL}} = (102 \pm 39) \times 10^{-11}. \quad (3)$$

We have only updated the central value and kept the error estimate unchanged. Using the new estimate for the axial-vectors would also shift the “Glasgow consensus” downward to $a_\mu^{\text{HLbL}} = (98 \pm 26) \times 10^{-11}$.

3 Model-independent approaches to HLbL

3.1 HLbL from dispersion relations (data driven)

The approach with dispersion relations (DR) for HLbL proposed in Refs. [11–15] tries to relate parts of the contributions in the Feynman diagrams in Figure 1 from on-shell intermediate states, e.g. from the pseudoscalar-poles and from two pions (pion-loop), to in principle measurable form factors and cross-sections $\gamma^* \gamma^* \rightarrow \pi^0, \eta, \eta'$ and $\gamma^* \gamma^* \rightarrow \pi^+ \pi^-, \pi^0 \pi^0$. These contributions involving the lightest hadrons are expected to dominate numerically based on experiences of other uses of DR’s. It is also confirmed by the results from the model calculations in Table 1. Note that in a quantum field theoretical approach, the Feynman diagrams in Figure 1 contain off-shell hadrons and the individual contributions to HLbL are model-dependent [1]. This complicates the comparison of the results for the individual contributions in different models. The dispersive approach in Refs. [11, 12] considers first the fully off-shell four-point function and projects on the on-shell intermediate hadronic states with one or two pions. Only at the end the contribution to the muon $g - 2$ is evaluated. On the other hand, the approach in Ref. [13] starts from a DR for the Pauli form factor $F_2(k^2)$ and then evaluates $a_\mu^{\text{HLbL}} = F_2(k^2 = 0)$.

Assuming that experimental results for the two-photon processes above can be obtained directly, which is not yet the case at present for two off-shell photons, or indirectly, using other DR’s with purely hadronic processes or single-virtual photons, the hopefully numerically dominant contributions to HLbL from single light pseudoscalars and from the two-pion intermediate states, can be obtained with high precision, e.g. better than 10%, where the error largely relies on experimental input only, like

for the HVP. On the other hand, it should be possible to obtain the contributions from the presumably numerically subdominant 3π -intermediate states, e.g. axial-vectors, from further multi-pion states (heavier resonances) and from the dressed quark-loop from models and theoretical constraints, e.g. by matching with perturbative QCD, to 30% to obtain an overall error of about 20% ($\delta a_\mu^{\text{HLbL}} \approx 20 \times 10^{-11}$ if the central value stays the same as in Eq. (3)).

For the pseudoscalar-pole contribution to HLbL the double-virtual transition form factors are needed as input, see Figure 1. There exist already several measurements of the single-virtual transition form factors in certain momentum regions [28]. The double-virtual form factors have only been modelled so far. They can hopefully be measured at BESIII [23] or they can be obtained from a DR itself [29] or from lattice QCD [30]. It remains to be seen, which precision on the pseudoscalar-pole contribution can be obtained in this way [23, 30] and how much modelling will still be needed, e.g. for the high-energy region in the DR's or to parametrize experimental or lattice data.

Very recently, a first estimate for the 2π -contribution has been obtained in Ref. [12]. The contribution itself is split into two parts. The first is the pion-box contribution, see the first diagram on the right-hand side of Figure 1, which has a one-pion cut in the s - and the t -channel and is identical to scalar QED with all vertices dressed by the pion vector form factor obtained from a DR itself. The second part describes the remaining $\pi\pi$ -rescattering effects. In a first approximation only S -wave rescattering from the pion-pole in the left-hand cut (lhc) have been taken into account and the high-energy part in the DR is modelled. The results read:

$$a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}, \quad (4)$$

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole lhc}} = -8(1) \times 10^{-11}, \quad (5)$$

$$a_\mu^{\pi\text{-box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole lhc}} = -24(1) \times 10^{-11}. \quad (6)$$

The result in Eq. (6) is much more precise than the values for the pion-loop given in Table 1. Not surprisingly, the pion-box contribution in Eq. (4) is close to the estimate from Ref. [8] that uses full VMD. That evaluation was recently reanalyzed [22, 31] yielding $a_\mu^{\text{HLbL};\pi\text{-loop}} = (-20 \pm 5) \times 10^{-11}$ close to the original value and compatible with the estimate in Eq. (6). The question is, how much is not yet included in the truncated dispersive approach compared to the complete two-pion contribution.

Another approach was developed by the Mainz group by using HLbL forward scattering sum rules [14] to constrain, under the assumption of factorization, the transition form factors of various mesons with two off-shell photons from experimental data or from lattice QCD [18]. These transition form factors are then used to evaluate the corresponding contributions to HLbL in the $g-2$ [15].

3.2 HLbL from lattice QCD

The calculation of HLbL in lattice QCD was proposed about 10 years ago, but only recently some first, still incomplete, numerical results have been obtained [16]. After several changes in the strategy, the calculation is now performed in position space, one obtains directly $a_\mu^{\text{HLbL}} = F_2(k^2 = 0)$ and exact expressions for all photon propagators are used. The latest result with physical pion mass, a finite lattice spacing $a^{-1} = 1.73 \text{ GeV}$ and a box-size with $L = 5.5 \text{ fm}$ reads:

$$a_\mu^{\text{cHLbL}} = (116.0 \pm 9.6) \times 10^{-11}, \quad (\text{quark-connected diagrams}), \quad (7)$$

$$a_\mu^{\text{dHLbL}} = (-62.5 \pm 8.0) \times 10^{-11}, \quad (\text{leading quark-disconnected diagrams}). \quad (8)$$

The size of these estimates is in the ballpark of the model calculations, see Table 1. But note that the error is statistical only. Missing systematic uncertainties are potentially large power-law finite-volume effects from QED in a box $\sim 1/L^2$ (this has now been overcome by using infinite volume, continuum

QED in the last paper of Ref. [16], as proposed in Ref. [17]) and from the finite lattice spacing. Also subleading quark-disconnected diagrams could be around 10% of the numbers given. Finally, it was found empirically that the short-distance contribution < 0.6 fm dominates the HLbL integral.

Independently, the lattice group at Mainz [17] developed in the last few years an approach in position space. We obtain the master formula

$$a_{\mu}^{\text{HLbL}} = \frac{me^6}{3} \int d^4y \int d^4x \underbrace{\tilde{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\text{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\text{QCD}}, \quad (9)$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = - \int d^4z z_{\rho} \langle j_{\mu}(x) j_{\nu}(y) j_{\sigma}(z) j_{\lambda}(0) \rangle, \quad (10)$$

where the QED part is computed semi-analytically in the continuum and in infinite volume. Therefore there are no power-law $1/L^2$ finite-volume effects. We have kept Lorentz invariance manifest which allows us to parametrize the QED kernel by six weight functions (and derivatives thereof) that depend only on x^2, y^2 and $x \cdot y$ which we have pre-computed on a 3-dimensional grid. The QCD part will be computed on the lattice eventually.

As a numerical test on our approach we have calculated the presumably numerically dominant pion-pole contribution to HLbL with a simple VMD model in position space [17]. In contrast to the observations in Ref. [16] we find that one needs rather large lattices of $L \sim (5 - 10)$ fm to reproduce the known results for the physical pion mass. As a further check, we also reproduce the known results for a lepton-loop in QED with $m_{\text{loop}} = m_{\mu}, 2m_{\mu}$ at the percent level.

These numerical tests give us confidence in our approach and the lattice simulations to calculate HLbL in the muon $g - 2$ with full QCD will soon begin. Note that as another complementary way to tackle HLbL scattering, lattice QCD calculations have already been performed to constrain hadronic models for transition form factors using HLbL forward scattering sum rules [14, 18] and to evaluate the pion transition form factor with two off-shell photons on the lattice [30].

4 Conclusions

We have briefly reviewed the current approach to HLbL using hadronic models and given an updated value $a_{\mu}^{\text{HLbL}} = (102 \pm 39) \times 10^{-11}$ in Eq. (3) where the uncertainty is rather arbitrary. Hopefully, the data driven dispersive approaches and lattice QCD will soon be able to give a reliable estimate. The experimental results presented at this PHIPSI 2017 meeting (and to come in the near future) from various measurements below and above the ϕ -meson thereby serve as important input and constraints on the theoretical approaches. Note that in order to fill the gap between experiment and theory by HLbL alone, the HLbL contribution would have to be four times bigger than the above estimate, i.e. 400×10^{-11} . This would mean that the current model calculations need to be way off, which seems not very likely in my opinion. We have learnt a lot about the HLbL contribution from the theory side in the last 15 years and the model estimates did not change very much over time. In fact the very recent preliminary and still incomplete evaluations using DR's and lattice QCD seem to roughly confirm the estimates from model calculations, but have the potential to much better control the uncertainty.

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