EPJ Web of Conferences **221**, 01014 (2019) https://doi.org/10.1051/epjconf/201922101014
EPPS 2019

Structural-phenomenological stress-strain model for concrete

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Abstract. This work is devoted to the improvement mathematical apparatus for describing the stress-strain state of structures made of heterogeneous materials and modeling the processes of their deformation under load. The proposed mathematical model establishes the relationship between the components of the macrostrains tensor and the components of the macrostress tensor and explicitly takes into account the heterogeneity of the material. The model can be used in finite-elemental software packages.

1 Introduction

Used in construction concrete can be distinguished among the variety of heterogeneous materials. This material has heterogeneity of the internal structure, which is due to the presence of inclusions of coarse and fine aggregates in the matrix of cement stone, as well as cracks, voids, pores and other structural features. This explains difficulties of describing the stress-strain state of structures made of this material and of modeling the process of their deformation under load. It is very important to take into account the peculiar of material's internal structure when building a stress-strain model. The structure of material in the vicinity of the crack tip determines the state in which its fracture occurs and also affects the trajectory of its growth. Thus, it is impossible to create models allowing with an acceptable error to predict the stress-strain state of made of heterogeneous materials without taking into account the heterogeneity of internal structure of the material.

There are used concepts of micro-, meso- and macro levels while describing the stress-strain state of heterogeneous environment. The description of the stress-strain state of micro- and meso levels is made considering the geometry of the structure in each structural element. A macro level is a description of the state of an environment without including the internal geometry of the structure- stresses and strains are averaged over a certain region of space including many separate structural elements. A detailed description of the stress-strain state at the micro- and meso-levels requires storing a tremendous amount of information which is still impossible at this stage of computer technology's development. Therefor averaged values are usually used while calculating. It is with the use of these quantities that most of the existing mathematical models of materials deformation are constructed, describing the change in mechanical characteristics in the process of

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deformation and the relationship (non-linear relationship) between stresses and strains. Such models are built on the basis of phenomenological theories that reflect objective experimental data without deep penetration into the physical essence of the processes of changing the structure of material during deformation and often remain adequate in a narrow range of parameter values, which makes it difficult to use them in solving practical problems of structural design.

The structural-phenomenological approach [1-6] to the construction of models for the deformation of materials with a heterogeneous internal structure is the most modern and promising one. Accounting in mathematical models of structural changes in the material allows to obtain models that retain their adequacy in a wide range of parameter values and provide higher accuracy compared to other models.

Calculation and design of reinforced concrete structures is done using modern commercial finite-elemental [7] software systems such as SCAD, Lyra, ANSYS, Abacus, Fidesys and others. In advanced FEM software there are tools for calculating the effective characteristics of materials, for example Digimat, ANSYS, Meterial Designer etc. This software implement the following homogenization methods, such as the Mori-Tanaka method [8-9], the Hashin-Shtrikman method [10], and others. In addition, there are used homogenization methods [11], based on the finite-elemental modeling of the stress-strain state cell periodicity.

The purpose of this work is to develop the principles of constructing mathematical models and approaches to modeling the processes of deformation of inhomogeneous media, and to increase the accuracy of predicting the stress-strain state and structural state of structures from inhomogeneous materials.

The objective is developing a structural-phenomenological model of the deformation of a heterogeneous material such as concrete.

2 Structural model of representative volume of heterogeneous materials

Let's consider an infinite field of space $\Omega_\infty = \mathbb{R}^n$, $n = 2,3$ (Fig. 1), occupied by a heterogeneous material, for example-concrete. It is assumed that there exists a subdomain $\Omega \subset \mathbb{R}^n$, $n = 2,3$ the way that under the condition of its minimal size the mechanical properties of the material in it are similar to the mechanical properties of the material in the region of Ω_∞. The subdomain Ω may be called the representative volume of the material [12].

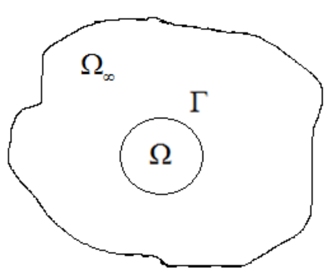

Fig. 1. Representative volume of material.

The boundary $\Gamma = \partial\Omega$ of Ω is strictly spherical in the 3D and round in 2D, so it should have circular symmetry:

$$\Omega^n = \{x \in R^n \mid x_1^2 + x_2^2 + ... + x_n^2 < R\}, \tag{1}$$

$$\Gamma^{n-1} = \left\{ x \in R^n \mid x_1^2 + x_2^2 + ... + x_n^2 = R \right\}, \tag{2}$$

where x is coordinate variables; R is radius; $n = 2$ (2D-case, circle) или $n = 3$ (3D-case, sphere). This form of boundary is a necessity for implementing the computational method which is the subject of this work and is described below.

In the representative volume of the material (the Ω area) can be distinguished subdomains Ω'_k, $k = 1, 2, \ldots$, representing inclusions from various quasi-homogeneous materials (Fig. 2). The phase boundary (Γ'_k) form the structure of the material and can have form of any complexity. The occupied by inclusions area is denoted as:

$$\Omega' = \bigcup_{k=1}^{n} \left(\Omega'_k \cup \Gamma'_k \right). \tag{3}$$

The Ω'_0 area is occupied by a bonding material which is also treated as a quasi-homogeneous material. Thus, the domain of representative material includes a subdomain of inclusions Ω' and a subdomain of the binder Ω'_0:

$$\Omega = \Omega' \cup \Omega'_0. \tag{4}$$

The boundaries of the domain Ω and the subdomains Ω'_k, can be mathematically described with the use of analytical geometry apparatus [14]. For geometric modeling will be used specialized computer-aided design (CAD) software.

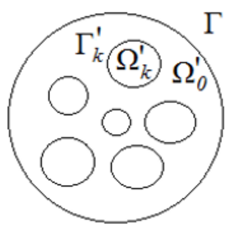

Fig. 2. Geometric model of representative volume of material.

This model is going to be called as a geometric (or structural) model of a representative volume of material.

First of all, this model describes the initial idealized structure of the material. In the process of deformation, the geometry of material structure changes and, for example, cracks are being formed. The change in the structure of the material can be described using the phenomenological approach – using the damage parameter (tensor or scalar):

$$d = d(x,y,z),\ x,y,z \in \Omega. \tag{5}$$

The tensor of irreversible plastic strains is also used often. Thus, the model can be surely called the structural-phenomenological model.

Structural-phenomenological modeling involves the selection in the material array of a certain minimum volume – the structural cell – which reflects the main features of the macroscopic behavior of the material. A cell is a considered as a construction, the functioning of which is ensured by its internal structure and the conditions of conjugation with the environment. The force interaction between elements is described using model potentials used in solid state physics.

3 Modeling of the stress-strain state of a representative volume of a heterogeneous material

Let's overview a semi-infinite region Ω_∞ of space occupied by an elastic area in which a sub-area Ω has circular/spherical shape (a region of representative material volume) with boundary Γ. When the elastic area is being deformed, the sub-area of representative volume of material (Ω) will also be deformed (Fig. 3).

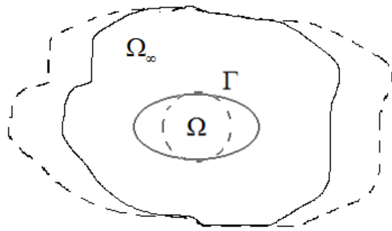

Fig. 3. Elastic medium deformation.

The stress-strain state of a representative volume of a heterogeneous material is described using the displacement vector, the strain tensor and the stress tensor, the relations between which are given by the following system of partial differential equations – the equations of elasticity, including:
- equations of static equilibrium

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = 0; \tag{6}$$

- geometric relations

$$\varepsilon_{ij} = \varepsilon_{ij}(u_k); \tag{7}$$

- physical relations

$$\sigma_{ij} = \sigma_{ij}(\varepsilon_{kl}), \tag{8}$$

where *u* is displacement; *σ* is stress tensor; *ε* is strain tensor; *f* is body forces; *x* is coordinates.

The physical relations here are given in general form and usually represented by a system of defining equations. For example, the following relation expressing the Hooke's law can be used for a linearly elastic area:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{9}$$

where *C* is tensor (rank 4) of the mechanical characteristics of the material.

The relation can be non-linear:

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl})\varepsilon_{kl}. \tag{10}$$

Geometric relations in the case of small displacements and deformations can be given using Cauchy formulas:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{11}$$

In the case of large displacements and deformations there ate used models of non-linear mechanics of a continuous medium based on the theory of finite deformations. The Lagragian approach is used to describing the motion of a continuous medium.

The unknowns are: displacements (*u(x, y, z)*), strains (*ε(x, y, z)*) and stresses (*σ(x, y, z)*); $x, y, z \in \Omega$.

The equations might be written in integral form (equations of continuum mechanics):
- mass conservation law

$$\int_{\Omega(t)} \frac{\partial \rho}{\partial t} d\Omega + \int_{\Gamma} \rho v d\Gamma = 0, \tag{12}$$

- equations of static equilibrium (moment conservation law)

$$\int_{\Omega} \rho f_i d\Omega + \int_{\Gamma} t_i^{(\hat{n})} d\Gamma = 0, \tag{13}$$

where $t_i^{(\hat{n})}$ is the stress vector $t_i^{(\hat{n})} = \sigma_{ji} n_j$, n_j is the normal vector to the plane.

- the energy equation, the equation of state and others.

For the system of differential equations (6)-(8) on the domain Ω with boundary Γ we formulate a boundary value problem. The boundary conditions in the absence of bulk forces completely determine the stress-strain state in the domain Ω. The boundary conditions can be two types:

- specified displacements

$$u(x, y, z),\ x, y, z \in \Gamma_1; \tag{14}$$

- specified stresses

$$t_i^{(\hat{n})}(x, y, z),\ x, y, z \in \Gamma_2. \tag{15}$$

At the boundaries of the subdomains of forming inclusions must be also specified boundary conditions in accordance with the accepted model of contact interaction. For example, in the simplest case-sticking conditions:

- normal gap

$$g = (\vec{x}_1 - \vec{x}_2) \cdot \vec{n} \geq 0, \tag{16}$$

where x_1, x_2 is radius-vectors of two material points of the contact surfaces in the current configuration; n is the unit normal vector of the contact surface.

- conditions of sameness of displacements

$$\vec{u}_1 = \vec{u}_2, \tag{17}$$

- normal stresses are equal in magnitude but opposite in direction

$$t_i^{(\hat{n}),1} = -t_i^{(\hat{n}),2}, \tag{18}$$

- shear stresses are equal in magnitude but opposite in direction

$$t_i^{(\hat{\tau}),1} = -t_i^{(\hat{\tau}),2}, \tag{19}$$

where τ is the unit vector tangent to the contact boundary.

The task can be solved using the finite-element method. After discretization of the computational domain on the finite-element mesh at the nodes at the boundaries of the regions, concentrated forces can be specified as boundary conditions:

$$F_i = \int_{\Gamma} t_i^{(\hat{n})} n_i d\Gamma. \tag{20}$$

The solution of contact problems is performed using the method of penalty functions or the method of Lagrange multipliers. The solution of the boundary value problem gives the stress-strain state of a representative element of the material's volume as well as information about the state of the material structure-damage, irreversible deformations. In addition, at the boundary of the computational domain can be determined reactive forces and displacements.

4 Mathematical description of the stress-strain state of the environment

Let's consider an elastic medium which has a certain stress-strain state, described by a tensor field $\sigma(x, y, z)$, $\varepsilon(x, y, z)$. We define some point P with coordinates (x_P, y_P, z_P). The components of the stress tensor at point P: $\sigma_{ij}(x_P, y_P, z_P)$. Select an elementary volume in the form of a sphere with radius $R \to 0$ centered at P. The sphere defines an area Ω with boundary Γ. Choose some point M on the surface of the sphere with coordinates x_M, y_M, z_M

$\in \Gamma$. Construct at this point an plane n_j, tangent to the sphere's surface. We denote $t_i^{(\hat{n})}$ – the stress vector (Fig. 4) at the plane dS with the unit normal n_i:

$$t_i^{(\hat{n})} = \sigma_{ji} n_j. \tag{21}$$

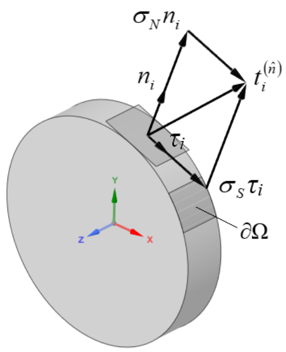

Fig. 4. Normal and tangential components of stress vector.

The stress vector can be represented as the sum of the normal and tangential components (Fig. 4). The normal component of the stress vector is

$$\sigma_N = t_i^{(\hat{n})} n_i = T^{(\hat{n})} \cdot N = \sigma_{ij} n_i n_j, \tag{22}$$

Tangent (shear) component of the stress vector:

$$\sigma_S^2 = t_i^{(\hat{n})} t_i^{(\hat{n})} - \sigma_N^2, \tag{23}$$

Let it be given on the surface of a sphere a finite-number (n) of points M_i, uniformly distributed over the surface Γ: $M_i \in \Gamma$, $i = 1, 2, \ldots, n$.

The tangent area to the surface of the sphere with the normal is given at each point. Thus, the site will approximate the sphere. For each site we draw a stress vector $t_i^{(\hat{n})}$ (Fig. 5).

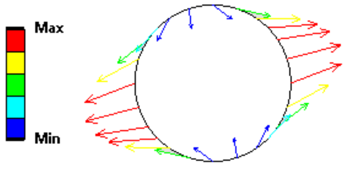

Fig. 5. Stress vectors on the surface of an elementary sphere.

The figure 5 shows two mutually perpendicular directions (main ones), where t_i^n and n_j are parallel. These vectors correspond to the principle stresses (σ_1, σ_2).

The Cartesian coordinate system is connected with the stress tensor. We introduce the polar (for the 2D case) coordinate system r, φ. Since the sphere is of unit radius, the coordinate $r = \text{const} = 1$. In the polar coordinate system, the stress vector can be represented by two components – normal ($t_N^{(\hat{n})}$) and tangent ($t_S^{(\hat{n})}$) as shown in Figure 4.

The components of the normal vector n_j in the Cartesian coordinate system are functions of the φ coordinate of the polar coordinate system:

$$n_x(\varphi) = \cos(\varphi), \tag{24}$$

$$n_y(\varphi) = \sin(\varphi). \tag{25}$$

Components of the stress vector $t_i^{(\hat{n})}$ in the Cartesian coordinate system:

$$t_x^{(\hat{n})}(\varphi) = \sigma_{xx} n_x(\varphi) + \sigma_{xy} n_y(\varphi), \tag{26}$$

$$t_y^{(\hat{n})}(\varphi) = \sigma_{yx} n_x(\varphi) + \sigma_{yy} n_y(\varphi). \tag{27}$$

Normal and tangential components of the stress vector $t_i^{(\hat{n})}$:

$$t_N^{(\hat{n})}(\varphi) = t_x^{(\hat{n})}(\varphi) n_x(\varphi) + t_y^{(\hat{n})}(\varphi) n_y(\varphi), \tag{28}$$

$$t_S^{(\hat{n})}(\varphi) = t_x^{(\hat{n})}(\varphi) n_x\left(\varphi + \frac{\pi}{2}\right) + t_y^{(\hat{n})}(\varphi) n_y\left(\varphi + \frac{\pi}{2}\right),$$

or

$$t_S^{(\hat{n})}(\varphi) = -t_x^{(\hat{n})}(\varphi) n_y(\varphi) + t_y^{(\hat{n})}(\varphi) n_x(\varphi). \tag{29}$$

Substituting (26)-(27) into (28)-(29) we get:

$$t_N^{(\hat{n})}(\varphi) = \sigma_{xx} n_x(\varphi) n_x(\varphi) + \sigma_{xy} n_x(\varphi) n_y(\varphi) + \sigma_{yx} n_y(\varphi) n_x(\varphi) + \sigma_{yy} n_y(\varphi) n_y(\varphi), \tag{30}$$

$$t_S^{(\hat{n})}(\varphi) = -\sigma_{xx} n_x(\varphi) n_y(\varphi) - \sigma_{xy} n_y(\varphi) n_y(\varphi) + \sigma_{yx} n_x(\varphi) n_x(\varphi) + \sigma_{yy} n_y(\varphi) n_x(\varphi). \tag{31}$$

Vector $t_N^{(\hat{n})}$ components in Cartesian coordinate system:

$$t_{Nx}^{(\hat{n})}(\varphi) = t_N^{(\hat{n})}(\varphi) n_x(\varphi), \tag{32}$$

$$t_{Ny}^{(\hat{n})}(\varphi) = t_N^{(\hat{n})}(\varphi) n_y(\varphi). \tag{33}$$

Vector $t_S^{(\hat{n})}$ components in Cartesian coordinate system:

$$t_{Sx}^{(\hat{n})}(\varphi) = t_x^{(\hat{n})}(\varphi) - t_{Nx}^{(\hat{n})}(\varphi), \tag{34}$$

$$t_{Sy}^{(\hat{n})}(\varphi) = t_y^{(\hat{n})}(\varphi) - t_{Ny}^{(\hat{n})}(\varphi). \tag{35}$$

Due to the analysis of the formulas (30)-(31) the function $t_N^{(\hat{n})}(\varphi)$ as all the completeness of information about the stress state on the surface of the sphere.

The function $t_N^{(\hat{n})}(\varphi)$ might be represented as a Fourier series expansion. In this case can be obtained formulas for calculating the components of the stress tensor similar to the known formulas for calculating the coefficients of the Fourier series:

- component σ_{11}:

$$\sigma_{11} = \frac{1}{2\pi} \int_0^{2\pi} t_N^{(\hat{n})}(\varphi) \cdot [3 - 4 \cdot n_y(\varphi) \cdot n_y(\varphi)] \cdot d\varphi, \tag{36}$$

- component σ_{22}:

$$\sigma_{22} = \frac{1}{2\pi} \int_0^{2\pi} t_N^{(\hat{n})}(\varphi) \cdot [3 - 4 \cdot n_x(\varphi) \cdot n_x(\varphi)] \cdot d\varphi, \tag{37}$$

- components σ_{12}, σ_{21}:

$$\sigma_{12} = \sigma_{21} = \frac{2}{\pi} \int_0^{2\pi} t_N^{(\hat{n})}(\varphi) \cdot n_x(\varphi) \cdot n_y(\varphi) \cdot d\varphi. \tag{38}$$

Thus, the components of the stress tensor can be calculated if the function $t_N^{(\hat{n})}(\varphi)$ is known. The integral formulas (36)-(37) can be calculated approximately using numerical integration methods, for example-quadrature formulas.

Formulas similar to formulas (36)-(37), can be obtained for the components of the strain tensor.

Thus, using formulas (36)-(37) the dependence between the components of the normal vector and the components of the stress tensor or the components of the strain tensor is performed by a linear function (transformation) (21), what means the components of the tensor are calculated for which this function describes the given dependence most accurately. Thus, these formulas can be considered as a method for solving the problem of approximating this dependence.

The first works, in which were used simpler mathematical entities (vectors and scalars) describing stresses and strains in several different directions instead of the strain tensor and the stress tensor, were works [15-17]. In these works, models of plastic deformation of materials with a quasi-homogeneous structure (metals) are described. In works [18-20] a similar approach was described for the construction of models for the deformation of concrete materials- one-dimensional deformation laws were used, introduced separately for each direction in the space specified by the normal vector to the plane.

Similar formulas were obtained in [21, 22]. In [23], a mathematical apparatus was described for constructing models for deforming materials based on the so-called N-directions approach in space.

The novelty of this work (its difference from the works [15-23]) lies in the fact that this approach has been applied to the construction of a model for the deformation of inhomogeneous materials and the determination of macrostrains of inhomogeneous media.

Formulas (36)-(37), in essence, are the averaging operator

$$\langle f \rangle = \frac{1}{\Omega_e} \int_{\Omega_e} f(x) d\Omega, \tag{39}$$

which applies to stresses and strains:

$$\langle \sigma \rangle = \frac{1}{\Omega_e} \int_{\Omega_e} \sigma(x) d\Omega, \ \langle \varepsilon \rangle = \frac{1}{\Omega_e} \int_{\Omega_e} \varepsilon(x) d\Omega. \tag{40}$$

In this paper, the derivation of formulas (36)-(37) is not given. In deriving formulas and their interpretation, the theory of equations of mathematical physics, the theory of harmonic functions, spherical functions, Erie functions, the Legendre polynomial, Green's theorem, Gauss-Ostrogradsky formula, and the theory of Fourier series are used.

5 Building a model of deformation

It is required to build a material deformation model – to describe mathematically the relationship between the components of the stress tensor (σ) and the components of the strain tensor (ε):

$$\sigma_{ij}^{mac} = \sigma_{ij}^{mac}\left(\varepsilon_{kl}^{mac}\right), \tag{41}$$

where σ_{ij}^{mac} and ε_{kl}^{mac} are the components of the macrostress tensor and the components of the macrostrain tensor.

Macrostresses and macrostrains are deformation and stress averaged over a representative volume of the material. Macro values describe the stress-strain state of a representative element of the volume as a whole (integral values). The averaged mechanical characteristics of the material are called effective characteristics [13]. There are many methods for calculating the effective characteristics of inhomogeneous materials. Methods [11], based on the finite element modeling of the stress-strain state of representative volume, are widely used and actively developed. Section 4 in this paper describes the mathematical apparatus for implementing such a method.

For a representative volume in the form of a circular region of radius R in the case $R \to 0$ (in regions, the sizes of which are incomparable or smaller) will be micro values, and in the case $R >> 0$ (in areas, the sizes are much larger than the dimensions of inclusions) – macro values.

The dependence (41) is constructed on the basis of the boundary problem, described in Section 3, and the mathematical apparatus for calculating the components of the stress and strain tensors, described in Section 4. The solution of the boundary problem and modeling the stress-strain state of a representative volume element are performed using the finite element method.

We introduce the polar (for the 2D case) coordinate system r, φ; $r = const = R$, where R is the radius of the region. The force boundary conditions (nodal forces) at the boundary Γ

of the domain Ω are described using the macrostress tensor and can be calculated using the formulas:

$$F_x(\varphi, r) = \int_0^{\Delta\theta} \left[\sigma_{xx} n_x(\varphi+\theta) + \sigma_{xy} n_y(\varphi+\theta)\right] \frac{2\pi r}{360} d\theta = \int_0^{\Delta\theta} t_x^{(\hat{n})}(\varphi+\theta) \frac{2\pi r}{360} d\theta, \tag{42}$$

$$F_y(\varphi, r) = \int_0^{\Delta\theta} \left[\sigma_{yx} n_x(\varphi+\theta) + \sigma_{yy} n_y(\varphi+\theta)\right] \frac{2\pi r}{360} d\theta = \int_0^{\Delta\theta} t_y^{(\hat{n})}(\varphi+\theta) \frac{2\pi r}{360} d\theta, \tag{43}$$

where $\Delta\theta$ is the angular size of the finite element face on the surface of the domain Ω. At the same time, additional kinematic constraints are required, which prohibit the motion of a rigid body as a rigid whole.

The kinematic boundary conditions (nodal displacements) on the boundary Γ of the domain Ω are described using the macrostrains tensor and can be calculated using the formulas:

$$u_x(\varphi, r) = r\varepsilon_{xx} n_x(\varphi) + \frac{1}{2} r\varepsilon_{xy} n_y(\varphi), \tag{44}$$

$$u_y(\varphi, r) = \frac{1}{2} r\varepsilon_{yx} n_x(\varphi) + r\varepsilon_{yy} n_y(\varphi). \tag{45}$$

The result of solving the boundary value problem is the force-reactions $F'(x, y, z)$ and displacements $u'(x, y, z)$ at the boundary of the region Ω. Having the distribution function of displacements $u'(x, y, z)$ along the boundary Γ of the domain Ω he components of the strain tensor (ε) can be obtained. First, the normal strains are calculated:

$$q_N^{(\hat{n})}(\varphi, r) = \left[u_x'(\varphi) n_x(\varphi) + u_y'(\varphi) n_y(\varphi)\right] \frac{1}{r}. \tag{46}$$

Then, using formulas similar to (36)-(37), the components of the strain tensor are calculated.

And if the distribution function of forces $F'(x, y, z)$ is known along the boundary Γ the subdomain Ω the components of the stress tensor (σ) can be obtained. At first normal stresses are calculated:

$$t_N^{(\hat{n})}(\varphi, r) = \left[F_x'(\varphi) n_x(\varphi) + F_y'(\varphi) n_y(\varphi)\right] \frac{360}{2\pi r}. \tag{47}$$

Then, using formulas similar to (36)-(37), the components of the strain tensor are calculated.

The resulting dependence (model) has a large computational complexity, so we construct a model close to it in terms of its properties, but less computationally complex (metamodel). We approximate the dependence between the components of the macrostress tensor and the components of the macrodeformation tensor by the following linear function (transformation):

$$\sigma_{ij}^{mac} = C_{ijkl}^{mac} \varepsilon_{kl}^{mac}, \tag{48}$$

where C is rank 4 tensor describing the mechanical characteristics (effective) of the material. The tensor can be represented by a 6x6 matrix:

$$C = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1123} & c_{1113} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2223} & c_{2213} \\ c_{3311} & c_{3322} & c_{3333} & c_{3312} & c_{3323} & c_{3313} \\ c_{1211} & c_{1222} & c_{1233} & c_{1212} & c_{1223} & c_{1213} \\ c_{2311} & c_{2322} & c_{2333} & c_{2312} & c_{2323} & c_{2313} \\ c_{1311} & c_{1322} & c_{1333} & c_{1312} & c_{1323} & c_{1313} \end{bmatrix}. \tag{49}$$

The coefficients of the matrix (components of the tensor C_{ijkl}) are found by the method of the least squares.

There are formulas for calculating the components of the tensor C, similar to (6) and (7) [18].

Thus, the spherical shape of a representative volume is natural. Usually, the shape of the region of a representative volume of material is assumed to be rectangular, since this provides computational efficiency. However, the spherical shape allows not only to perform finite element modeling, but also allows you to build mathematical models.

6 Example of the implementation of the proposed approach

Consider an inhomogeneous material with a regularly inhomogeneous structure. Calculations perform in 2D. We accept a round shape of inclusions with a diameter of D = 20 mm. A geometric model of a representative volume of material is shown in Figure 6. The diameter of the area Ω: D_Ω = 100 mm.

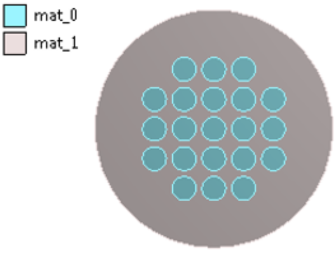

Fig. 6. A geometric model of a representative volume of material with a regularly heterogeneous structure.

The mechanical characteristics of the material of inclusions are accepted as follows: $E_0 = 6\cdot 10^{10}$ Pa, $\nu_0 = 0.28$. The mechanical characteristics of the matrix material are taken as follows: $E_1 = 3\cdot 10^{10}$ Pa, $\nu_1 = 0.28$.

Component deformation patterns are assumed to be linearly elastic. The calculations are performed in the formulation of a plane stress state. The stress-strain state is described by the system of equations (6), (9), (11). Snapping (16)-(20) is accepted at the phase boundaries.

Let the macrostrains tensor be given for the representative volume:

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix},$$

$$\varepsilon = \begin{bmatrix} 3.33\cdot 10^{-5} & 2.56\cdot 10^{-5} \\ 2.56\cdot 10^{-5} & -9.40\cdot 10^{-6} \end{bmatrix}.$$

The kinematic boundary conditions (given displacements) on the boundary of the domain Ω are defined using the formulas (44)-(45). Figure 7 shows the displacements defined on the boundary of the computational domain Ω.

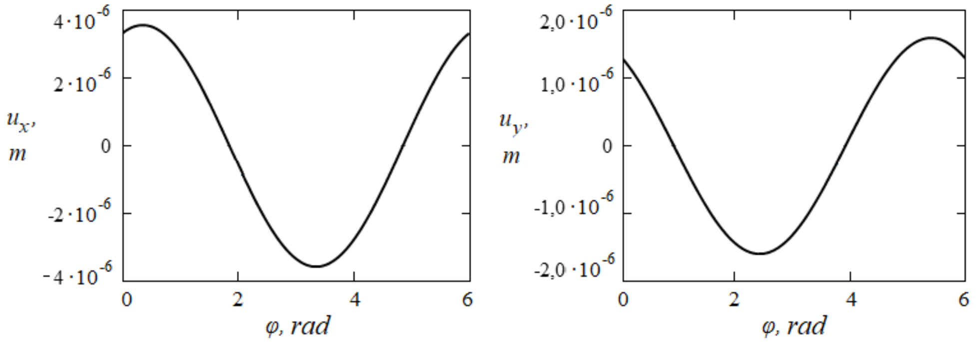

Fig. 7. Displacements defined on the boundary of the computational domain Ω.

Finite element discretization of the computational domain sown in Figure 8.

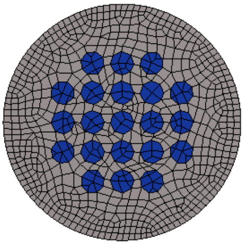

Fig. 8. Finite element discretization of the computational domain.

The calculations were performed using the ANSYS Mechanical finite element software package.

As a result of solving the boundary value problem described in Section 3, we obtain displacement, stress and strain fields in the domain Ω. Figure 9 shows the displacements.

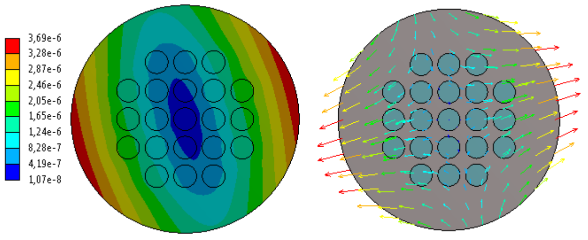

Fig. 9. Total displacements (*m*).

Figure 10 shows the microstrains fields and Figure 11 shows the microstresses fields.

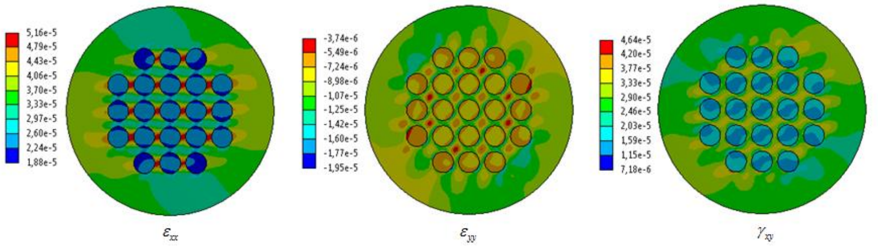

Fig. 10. Microstrains.

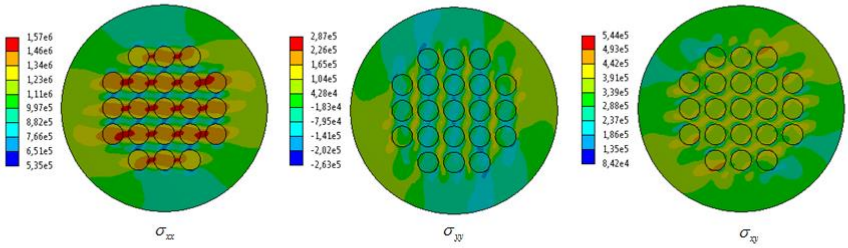

Fig. 11. Microstresses (*Pa*).

The stress and strain fields are not uniform. The reactive forces are determined at the boundary of the region Ω (Fig. 12).

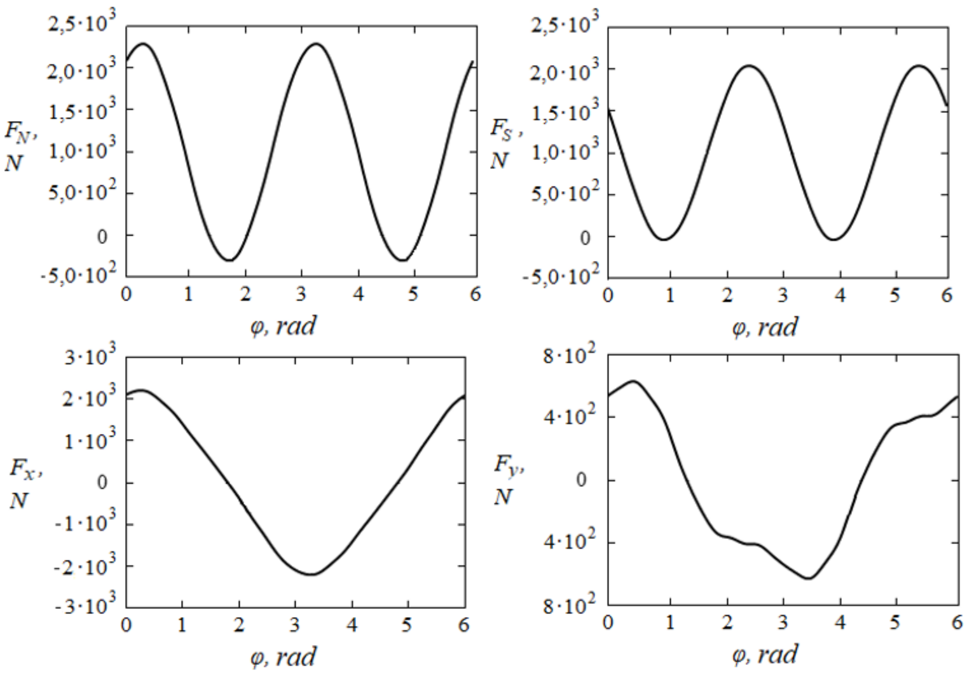

Fig. 12. Reaction forces.

Knowing the reactive forces at the boundary of the region, the tensor of macrostresses is calculated by the formulas (36)-(38):

$$\sigma = \begin{bmatrix} 1.21\cdot10^6 & 2.40\cdot10^5 \\ 2.40\cdot10^5 & -6.89\cdot10^4 \end{bmatrix}.$$

Figure 13 shows a comparison of the microstresses at the boundary of the Ω, obtained in the FEM calculation, and the macrostresses, given by the stress tensor, obtained by the formulas (36)-(38).

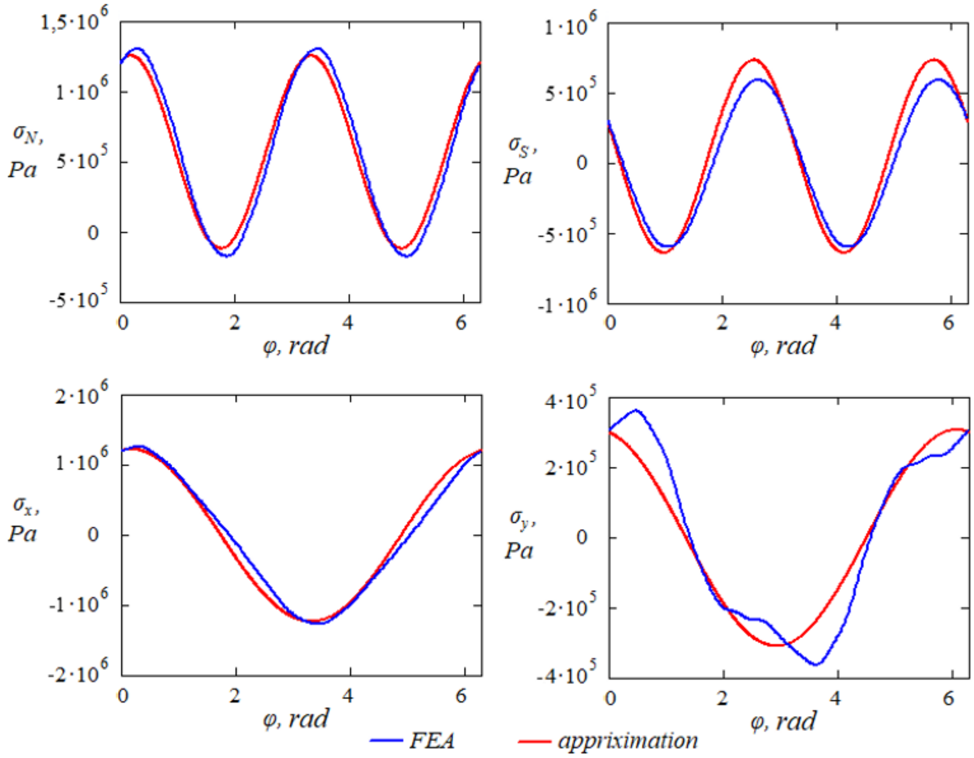

Fig. 13. Micro and macro stresses at the boundary of the region Ω.

Let's Calculate the tensor of the elastic characteristics of an inhomogeneous material using the least squares method. We get:

$$C = \begin{bmatrix} 5.121\cdot10^{10} & 1.278\cdot10^{10} & 2.112\cdot10^{-1} \\ 1.249\cdot10^{10} & 5.113\cdot10^{10} & 9.378\cdot10^{-2} \\ 2.139\cdot10^{-1} & 9.301\cdot10^{-2} & 1.961\cdot10^{10} \end{bmatrix}.$$

For comparison, we give the tensors of elastic characteristics of materials of inclusions

$$C_0 = \begin{bmatrix} 6.51\cdot10^{10} & 1.823\cdot10^{10} & 0 \\ 1.823\cdot10^{10} & 6.51\cdot10^{10} & 0 \\ 0 & 0 & 2.344\cdot10^{10} \end{bmatrix}$$

and matrix

$$C_1 = \begin{bmatrix} 3.255\cdot10^{10} & 9.115\cdot10^{10} & 0 \\ 9.115\cdot10^{10} & 3.255\cdot10^{10} & 0 \\ 0 & 0 & 1.172\cdot10^{10} \end{bmatrix},$$

which are calculated by the formulas of the theory of elasticity for a plane stressed state

$$C = \begin{bmatrix} \gamma & \gamma\cdot\nu & 0 \\ \gamma\cdot\nu & \gamma & 0 \\ 0 & 0 & \gamma\cdot\dfrac{1-\nu}{2} \end{bmatrix},$$

where $\gamma = \dfrac{E}{1-\nu^2}$, E is the modulus of elasticity; ν is the Poisson's ratio.

7 Conclusion

The proposed structural-phenomenological model describes the relationship between stresses and strains in heterogenous media such as concrete. The model explicitly takes into account the heterogeneity of the material. The structure of the material is described using a structural sub-model based on the mathematical apparatus of analytical geometry.

The model can be easily integrated into finite-element software systems, since it involves performing an analysis of the stress-strain state of a representative volume of a material using the finite element method. In software systems, the model can also be used in special subprograms to calculate the effective characteristics of inhomogeneous materials.

The method of calculating the components of the macrostrain tensor and the components of the macrostress tensor is considered on the basis of information on the stress-strain state at the boundary of the region of a representative volume element of a inhomogeneous medium.

It is shown that the spherical shape of the representative volume is natural.

The reported study was funded by RFBR according to the research project № 18-31-00103.

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