Reflection Method for Mathematical Modeling of Potential Fields in Multi-Layer Media

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Abstract. In this paper, potential fields in areas with plane and circular symmetry have been studied. In this case, the field potential is defined as the sum of solutions of Dirichlet model boundary value problems. The reflection method is used for the modeling of stationary thermal fields in multilayer media. By applying the reflection method, we found analytical solutions of boundary value problems with boundary conditions of the fourth kind for the Laplace equations and developed new computational algorithms. The developed algorithms can be easily implemented and transformed into a computer code. First of all, these algorithms implement consistent solutions of the Dirichlet problems for the model domains that allows using libraries of subroutines. Secondly, they have high algorithmic efficiency. It has been shown that reflection method is identical to the method of transformation operators and proved that transformation operator can be decomposed into a series of successive reflections from the external and internal boundaries. Finally, a physical interpretation of the reflection method has been discussed in detail.

1 Introduction

We note the methods of the theory of functions of a complex variable as an analytical methods for solving boundary value problems in multilayer media. Such methods lead to Riemann boundary value problems, the solution of which is accompanied by significant difficulties, for example, the calculation of the index of the boundary value problem and integrals of Cauchy type [1]. Numerical methods: finite difference method (FDM see [4]), finite element method (FEM see [2]), radial basis function method (RBFM see [3]), - also have disadvantages. For example, in the FDM- method the solution is obtained only in the grid nodes. Any area can be described with FDM, since triangles and tetrahedra easily cover even complex domains. In all the subdomains, you can easily increase the density of the computational grid to improve the accuracy of calculations. However, the general method of error estimation in FDM and RBFM is absent today.

The reflection method is one of the well-known analytical methods for solving boundary value problems. Due to the wide spread of multilayer materials, the spread of this method for boundary value problems in multilayer media becomes relevant. In our article it will be shown that the method of transformation operators previously developed by the authors (see [15,16]) allows us to present the solution of the problem as a sum of successive reflections. Also in the article it is established that the method of transformation operators and the method of reflections are identical. As a consequence of this result, we have found new analytical formulas for solving boundary value problems in multilayer media.

The article offers an interpretation of the mathematical model of multilayer media as a deformation of the model of homogeneous media. New formulas for solutions obtained in the article allowed to develop effective computational algorithms on their basis.

The reflection method was firstly proposed by Kelvin, (see [5]). Let us illustrate on a simple example the idea of this method, i.e., we show how the solution of the boundary value problem in domain with complex N - connected boundary is constructed based on solutions of model boundary-value problem for domain with one - connected border. The Kelvin method consists in performing an iterative procedure of consistently reflecting the model solution from the external and internal boundaries of the domain. Consider the Dirichlet problem for the Laplace equation in the strip. We find the solution of the Laplace equation in the strip

$$u''_{yy} + a^2 u''_{xx} = 0, \quad 0 < x < l, y \in R \quad (1)$$

with boundary conditions

$$u(0, y) = f(y), u(l, y) = 0. \quad (2)$$

Consider the Dirichlet model problem for the half-plane

$$\left\{ \begin{array}{l} \tilde{u}''_{yy} + a^2 \tilde{u}''_{xx} = 0, 0 < x, y \in R \\ \tilde{u}(0, y) = f(y), \\ |f(y)| \leq \frac{M}{1+y^2}. \end{array} \right. \quad (3)$$

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We choose the solution of the model problem (3) i.e. the function $\tilde{u}(x, y)$ as zero- approximation for the solution of the boundary value problem (1)-(2). The zero- approximation satisfies the Laplace equation and the first boundary condition, but does not satisfy the second boundary condition. The solution of the model problem (3) reflected from the right boundary has the form $\tilde{u}(2l - x, y)$. Then the first-order approximation for the solution of the boundary value problem (1)-(2) has the form

$$u^1(x, y) = \tilde{u}(x, y) - \tilde{u}(2l - x, y).$$

For this approximation, the second boundary condition is satisfied, but the first boundary condition is not satisfied, because

$$u^1(0, y) = \tilde{u}(0, y) - \tilde{u}(2l, y) = f(y) - \tilde{u}(2l, y).$$

Iterations continue until the required accuracy is obtained. As a result, for the approximation obtained by $(m+1)$ - reflections from the boundary $x = l$, we get

$$u^m(x, y) \cong \sum_{j=0}^m \tilde{u}(x + 2lj, y) - \tilde{u}(2l - x + 2lj, y). \quad (4)$$

You can set assertions:

- 1) the function $u^m(x, y)$ satisfies the Laplace equation and the second boundary condition;
- 2) if the boundary value $f(y)$ satisfies the condition at infinity

$$(1 + y^2)|f(y)| \leq M,$$

then the algorithm of successive reflection from the boundaries converges to the exact solution of the Dirichlet problem (1)-(2);

- 3) series (4) converges uniformly on x ;
- 4) the error of formula (4) has the form

$$|u^m(x, y) - u(x, y)| \leq \frac{4Mlm}{4l^2m^2 + y^2}, M = \sup_{y \in R} |f(y)|.$$

Passing to the limit as $n \rightarrow \infty$ in formula (4), we obtain the analytical formula

$$u(x, y) = \sum_{j=0}^{\infty} \tilde{u}(x + 2lj, y) - \tilde{u}(2l - x + 2lj, y).$$

Let $l = \pi$, then, from Poisson's formula for the half-plane [5]

$$\tilde{u}(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - \eta)^2} f(\eta) d\eta.$$

and identities [5]

$$\sum_{j=0}^{\infty} \left(\frac{x + 2\pi j}{(x + 2\pi j)^2 + (y - \eta)^2} - \frac{2\pi - x + 2\pi j}{(2\pi - x + 2\pi j)^2 + (y - \eta)^2} \right) = \frac{1}{2} \frac{\sin x}{ch(y - \eta) - \cos x}$$

we obtain well-known result for solution the Dirichlet problem in the strip [5]

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin x}{ch(y - \eta) - \cos x} f(\eta) d\eta.$$

2 Main results

2.1 Reflection Method in Dirichlet problem for Laplace equation in two-layer strip

Modeling of a stationary thermal field in a two-layer plate bounded in the direction of the Ox - axis leads to the problem of solving a system of separate Laplace equations

$$\begin{cases} u''_{1yy} + a_1^2 u''_{1xx} = 0, & 0 < x < l, y \in R, \\ u''_{2yy} + a_2^2 u''_{2xx} = 0, & l < x < L, y \in R, \end{cases} \quad (5)$$

with Dirichlet boundary conditions

$$u_1(0, y) = f(y), u_2(L, y) = 0, \quad (6)$$

under the boundary condition of fourth kind on the line $x = l$

$$u_1(l, y) = u_2(l, y), \quad \lambda_1 u'_{1x}(l, y) = \lambda_2 u'_{2x}(l, y). \quad (7)$$

The Dirichlet model problem for the strip has the form:

$$\begin{cases} \tilde{u}''_{0yy} + \tilde{u}''_{0xx} = 0, & 0 < x < \tilde{L}, \tilde{L} = \frac{L-l}{a_2} + \frac{l}{a_1}, y \in R, \\ \tilde{u}_0(0, y) = f(y), \\ \tilde{u}_0(\tilde{L}, y) = 0. \end{cases} \quad (8)$$

Let the solution of the model problem (8) for the strip be known. We define the first approximation of the solution of the boundary value problem (5)-(7). The second component of the solution is given by the formula

$$u_2^1(x, y) = \frac{2\chi}{1+\chi} \tilde{u}_0\left(\frac{x-l}{a_2} + \frac{l}{a_1}, y\right), l < x < L, y \in R, \quad \chi = \frac{\lambda_1 a_2}{\lambda_2 a_1},$$

To obtain the first component of the approximation $u_1^1(x, y)$, we perform the reflection of the model solution with respect to the line $x = l$. Then we obtain

$$u_1^1(x, y) = \tilde{u}_0\left(\frac{x}{a_1}, y\right) - \frac{1-\chi}{1+\chi} \tilde{u}_0\left(\frac{2l-x}{a_1}, y\right), 0 < x < l.$$

The first approximation satisfies the Laplace equation, the boundary condition (6) on the line $x = \tilde{L}$, and on the line $x = 0$ satisfies the boundary condition

$$u_1^1(0, y) = f(y) - \frac{1-\chi}{1+\chi} \tilde{u}_0\left(\frac{2l}{a_1}, y\right).$$

The first approximation has the form

$$u_1^1(x, y) = \tilde{u}_0\left(\frac{x}{a_1}, y\right) - \frac{1-\chi}{1+\chi} \tilde{u}_0\left(\frac{2l-x}{a_1}, y\right), 0 < x < l;$$

$$u_1^1(x, y) = \frac{2\chi}{1+\chi} \tilde{u}_0\left(\frac{x-l}{a_2} + \frac{l}{a_1}, y\right), l < x < L.$$

Consider the following Dirichlet model problem on the strip

$$\begin{cases} \tilde{u}_{1yy}'' + \tilde{u}_{1xx}'' = 0, & 0 < x < \tilde{L}, \tilde{L} = \frac{L-l}{a_2} + \frac{l}{a_1}, y \in R; \\ \tilde{u}_1(0, y) = \frac{1-\chi}{1+\chi} \tilde{u}_0\left(\frac{2l}{a_1}, y\right); \\ \tilde{u}_1(L, y) = 0. \end{cases} \quad (9)$$

We use the solution $\tilde{u}_1(x, y)$ of the model problem (9) to obtain the second approximation. For the first and second components of the second approximation we have expressions, respectively

$$u_1^2(x, y) = u_1^1(x, y) + \tilde{u}_1\left(\frac{x}{a_1}, y\right) - \frac{1-\chi}{1+\chi} \tilde{u}_1\left(\frac{2l-x}{a_1}, y\right), 0 < x < l;$$

$$u_1^2(x, y) = u_1^1(x, y) + \frac{2\chi}{1+\chi} \tilde{u}_1\left(\frac{x-l}{a_2} + \frac{l}{a_1}, y\right), l < x < L.$$

As a result, for the approximation obtained by (m+1) reflections from the boundary $x=L$, we obtain the formulas

$$\begin{cases} u_1^m(x, y) \approx \sum_{j=0}^m \tilde{u}_j\left(\frac{x}{a_1}, y\right) - \frac{1-\chi}{1+\chi} \tilde{u}_j\left(\frac{2l-x}{a_1}, y\right), 0 < x < l, \\ u_2^m(x, y) \approx \frac{2\chi}{1+\chi} \sum_{j=0}^m \tilde{u}_j\left(\frac{x-l}{a_2} + \frac{l}{a_1}, y\right), l < x < L. \end{cases} \quad (10)$$

Considering known the solution $\tilde{u}_j$ of the Dirichlet model problem for the strip (9) we define $\tilde{u}_{j+1}$ as the solution of the following model problem

$$\begin{cases} \tilde{u}_{j+1yy}'' + \tilde{u}_{j+1xx}'' = 0, & 0 < x < \tilde{L}, \tilde{L} = \frac{L-l}{a_2} + \frac{l}{a_1}, y \in R, \\ \tilde{u}_{j+1}(0, y) = \frac{1-\chi}{1+\chi} \tilde{u}_j\left(\frac{2l}{a_1}, y\right), \\ \tilde{u}_{j+1}(L, y) = 0. \end{cases}$$

Thus, starting from the solution $\tilde{u}_0$, we obtain a sequence of model solutions $\{\tilde{u}_j\}$, with the help of which, according to formulas (10), we obtain a sequence of approximations for the solution of the problem (5)-(7). Iterations continue until the required accuracy is obtained. The process ends if the function $\tilde{u}_{m+1}(0, y)$ satisfies the specified accuracy in a uniform norm.

Remark 1. The method is transferred to the case of the number of layers greater than two. In this case, the reflections should be carried out sequentially with respect to each of the internal boundaries, moving from right to left to the boundary $x=0$.

2.2 Reflection Method for Dirichlet problem in two-layer half-plane

Modeling of a stationary thermal field in a two-layer plate semi-bounded in the Ox -axis direction leads to the problem of solving a system of separate Laplace equations in two-layer half-plane

$$E_{1,+} = \{(x, y) : y \in R, x \in (0, l) \cup (l, \infty)\}$$

$$u''_{jyy} + a_j^2 u''_{jxx} = 0, (x, y) \in E_{1,+}, j = 1, 2 \quad (11)$$

with boundary condition on the line $x=0$

$$u_1(0, y) = f(y), \quad (12)$$

under boundary condition of fourth kind on straight line $x=l$

$$u_1(l, y) = u_2(l, y), \quad \lambda_1 u'_{1x}(l, y) = \lambda_2 u'_{2x}(l, y). \quad (13)$$

Let the function $\tilde{u} = \tilde{u}(x, y)$ be the solution of the Dirichlet problem (3) for the Laplace equation on the half-plane $x > 0$. In [15] it is proved that the solution of the Dirichlet problem (11)-(13) has the form

$$u_1(x, y) = \sum_{j=0}^{\infty} \left(\frac{1-\chi}{1+\chi} \right)^j \left(\tilde{u}\left(\frac{x+2lj}{a_1}, y\right) - \frac{1-\chi}{1+\chi} \tilde{u}\left(\frac{2l-x+2lj}{a_1}, y\right) \right), 0 < x < l; \quad (14)$$

$$u_2(x, y) = \frac{2\chi}{1+\chi} \sum_{j=0}^{\infty} \left(\frac{1-\chi}{1+\chi} \right)^j \tilde{u}\left(\frac{x-l}{a_2} + \frac{l+2lj}{a_1}, y\right), l < x; \quad \chi = \frac{\lambda_1 a_2}{\lambda_2 a_1}. \quad (15)$$

The computational algorithm based on the exact formulas (14)–(15) consists in replacing the sum of a series by the sum of a finite number of terms. This algorithm can be interpreted as a sequence of reflections with respect to the inner and outer boundaries. This approach allows us to find out the iterative nature of the algorithm and speed up the calculation process.

Step 1. Find the first-order approximation

$$u^1(x, y) = \begin{cases} u_1^1(x, y) \equiv \tilde{u}_1\left(\frac{x}{a_1}, y\right) - \frac{1-\chi}{1+\chi} \tilde{u}_1\left(\frac{2l-x}{a_1}, y\right), 0 < x < l, \\ u_2^1(x, y) \equiv \frac{2\chi}{1+\chi} \tilde{u}_1\left(\frac{x-l}{a_2} + \frac{l}{a_1}, y\right), l < x; \quad \chi = \frac{\lambda_1 a_2}{\lambda_2 a_1}, \end{cases}$$

where

$$\tilde{u}_1(x, y) = \tilde{u}(x, y).$$

In this case, the first-order approximation the function $u^1(x, y)$ satisfies

- 1) a separate system of Laplace equations (11),
- 2) boundary conditions of fourth kind (13),
- 3) the error in the boundary condition (12) has the order $(1 - \chi) / (1 + \chi) \|\tilde{u}\|$ in the uniform metric.

Step 2. Improving the approximation. If the function $u^k(x, y)$ is a K-order approximation, then the (k+1)-order approximation is defined by the formula,

$$u^{k+1}(x, y) = \begin{cases} u_1^{k+1}(x, y), & 0 < x < l; \\ u_2^{k+1}(x, y), & l < x, \end{cases}$$

where

$$u_1^{k+1}(x, y) = \tilde{u}_{k+1}\left(\frac{x}{a_1}, y\right) - \frac{1 - \chi}{1 + \chi} \tilde{u}_{k+1}\left(\frac{2l - x}{a_1}, y\right), \quad 0 < x < l;$$

$$u_2^{k+1}(x, y) = \frac{2\chi}{1 + \chi} \tilde{u}_{k+1}\left(\frac{x - l}{a_2} + \frac{l}{a_1}, y\right), \quad l < x; \quad \chi = \frac{\lambda_1 a_2}{\lambda_2 a_1},$$

here $\tilde{u}_{k+1}(x, y)$ is the solution of the Dirichlet model problem (3) for a half-plane with Dirichlet condition on the boundary $x = 0$

$$\tilde{u}_{k+1}(0, y) = \frac{1 - \chi}{1 + \chi} \tilde{u}_k\left(\frac{2l}{a_1}, y\right).$$

Remark 1. The following formula is valid

$$\tilde{u}_k(x, y) = \left(\frac{1 - \chi}{1 + \chi}\right)^k \tilde{u}\left(\frac{x + 2lk}{a_1}, y\right).$$

Remark 2. The described algorithm can be interpreted in terms of the reflection method. The algorithm extends to the case of the number of layers greater than two.

2.3 Reflection Method for increasing the number of layers in Dirichlet problem for (n+1) - layer half-plane

As a model problem, we consider the Dirichlet problem for the Laplace equation in a half-plane $x > 0$ with under boundary condition of fourth kind on the lines

$$x = l_i, i = 2, \dots, n.$$

Let in (n+1) - layer half-plane

$$E_{n+1} = \left\{ (x, y) : y \in R, x \in (\tilde{l}_0, l_2) \cup \bigcup_{i=3}^{n+1} (l_{i-1}, l_i) \right\},$$

the solution of a separate system of Laplace equations is known

$$\tilde{u}_{jyy}'' + a_j^2 \tilde{u}_{jxx}'' = 0, \quad (x, y) \in E_{n+1}; j = 2, 3, \dots, n \quad (16)$$

with boundary condition

$$\tilde{u}_2(\tilde{l}_0, y) = f(y), \quad (17)$$

under boundary conditions of fourth kind on straight lines $x = l_i, i = 2, \dots, n$

$$\tilde{u}_i(l_i, y) = \tilde{u}_{i+1}(l_i, y), \quad \lambda_{1i} \tilde{u}'_{ix}(l_i, y) = \lambda_{2i} \tilde{u}'_{2ix}(l_i, y). \quad (18)$$

We will show how you can add a new layer for the class of problems under consideration. To this aim, we consider a new (n+2)- layer boundary value problem for determining the solution of a separate system of the Laplace equations

$$u_{jyy}'' + a_j^2 u_{jxx}'' = 0, \quad (x, y) \in E_{n+1}, j = \overline{1, n}, \quad (19)$$

with boundary condition

$$u_1(l_0, y) = f(y), \quad (20)$$

under boundary conditions of fourth kind on straight lines $x = l_i, i = 2, \dots, n$

$$u_i(l_i, y) = u_{i+1}(l_i, y), \quad \lambda_{1i} u'_{ix}(l_i, y) = \lambda_{i+1} u'_{i+1,x}(l_i, y). \quad (21)$$

We will look for the solution of the problem (16)-(18) through the solution of the model problem (19)-(21). Let's choose the left boundary in the model problem as $x = \tilde{l}_0$ in the form $\tilde{l}_0 = (l_0 - l_1)a_2 / a_1 + l_1$. The iterative computational algorithm consists of the following steps.

Step 1. The first- order approximation is calculated by the formulas

$$u_1^1(x, y) = \tilde{u}_2\left(\frac{a_2(x - l_1)}{a_1} + l_1, y\right) - \frac{1 - \chi_1}{1 + \chi_1} \tilde{u}_2\left(\frac{a_2(l_1 - x)}{a_1} + l_1, y\right), \quad l_0 < x < l_1;$$

$$u_j^1(x, y) = \frac{2\chi_1}{1 + \chi_1} \tilde{u}_j(x, y), \quad l_{j-1} < x < l_j, j = 2, \dots, n+1, \chi_1 = \frac{\lambda_1 a_2}{\lambda_2 a_1}.$$

It is directly verified that the first-order approximation $u^1(x, y)$

- 1) satisfies a separate system of Laplace equations (19),
- 2) satisfies the boundary conditions of fourth kind (21),
- 3) has an error of order $(1 - \chi_1) / (1 + \chi_1) \|\tilde{u}\|$ in the boundary condition (12) in the uniform metric, since the equality is satisfied on the boundary $x = \tilde{l}_0$

$$u_1^1(l_0, y) = f(y) - \frac{1 - \chi_1}{1 + \chi_1} \tilde{u}_2(2l_1 - \tilde{l}_0, y), \quad l_0 < x < l_1.$$

Step 2. Assuming a given k-order approximation $u_j^k(x, y)$, the (k+1)- order approximation is defined so that the error in the boundary condition has order

$(1 - \chi_1)^2 / (1 + \chi_1)^2 \|\tilde{u}\|$. In this case, we obtain the formula

$$u_1^{k+1}(x, y) = u_1^k(x, y) + \tilde{u}_1^{k+1} \left(\frac{a_2(x-l_1)}{a_1} + l_1, y \right) - \frac{1 - \chi_1}{1 + \chi_1} \tilde{u}_2^{k+1} \left(\frac{a_2(l_1 - x)}{a_1} + l_1, y \right), l_0 < x < l_1;$$

$$u_j^{k+1}(x, y) = u_j^k(x, y) + \frac{2\chi_1}{1 + \chi_1} \tilde{u}_j^{k+1}(x, y), l_{j-1} < x < l_j,$$

here there $\tilde{u}_j^{k+1}(x, y)$ is a j -component solution of the Dirichlet problem (16)-(18) for a $(n+1)$ -layer half-plane with a boundary condition (17) of the form

$$\tilde{u}_1^{k+1}(l_0, y) = \frac{1 - \chi_1}{1 + \chi_1} \tilde{u}_2^k(2l_1 - \tilde{l}_0, y).$$

Thus, the k -order approximation is defined by the formulas:

$$u_1^k(x, y) = u_1^1(x, y) + \sum_{j=1}^k \tilde{u}_1^j \left(\frac{a_2(x-l_1)}{a_1} + l_1, y \right) - \frac{1 - \chi_1}{1 + \chi_1} \tilde{u}_2^j \left(\frac{a_2(l_1 - x)}{a_1} + l_1, y \right), l_0 < x < l_1;$$

$$u_p^k(x, y) = u_p^1(x, y) + \frac{2\chi_1}{1 + \chi_1} \sum_{j=1}^k \tilde{u}_p^j(x, y), l_{p-1} < x < l_p, p = \overline{2, n+1}.$$

K-order approximation function

- 1) satisfies the system of Laplace equations (19),
- 2) satisfies the boundary condition of fourth kind (21),
- 3) has an error of order $(1 - \chi_1)^k / (1 + \chi_1)^k \|\tilde{u}\|$ in the boundary condition (20) in the uniform metric, since the equality is satisfied on the boundary $x = \tilde{l}_0$

$$u_1^k(l_0, y) = f(y) - \frac{1 - \chi_1}{1 + \chi_1} \tilde{u}_2^k(2l_1 - \tilde{l}_0, y).$$

Calculations are completed when the values of the function $u_1^k(l_0, y)$ in the uniform metric do not exceed the specified accuracy.

3 Conclusion

The results given in the paper allow the extension to the case of arbitrary bounded regions D_0, D_1 with boundaries Γ_0, Γ_1 . We solve the problem of modeling a stationary thermal field in a bounded two-layer plate. We set the Dirichlet problem for a system of separate Laplace equations in a two-layer domain D_0

$$u_{1xx}'' + u_{1yy}'' = 0, (x, y) \in D_0 \setminus \bar{D}_1,$$

$$u_{2xx}'' + u_{2yy}'' = 0, (x, y) \in D_1 \quad (22)$$

with boundary condition on line Γ_0

$$u_1(x, y) = f(x, y), (x, y) \in \Gamma_0, \quad (23)$$

under boundary condition of fourth kind on line Γ_1

$$u_1(x, y) = u_2(x, y), \quad \lambda_1 u_{1n}'(x, y) = \lambda_2 u_{2n}'(x, y), (x, y) \in \Gamma_1 \quad (24)$$

in formula (24), the derivative is taken along the normal to the boundary Γ_1 .

Let us define by induction two sequences of harmonic functions $\{\tilde{u}_j\}, \{\tilde{v}_j\}$. The harmonic functions $\{\tilde{u}_1\}, \{\tilde{v}_1\}$ are given as solutions of two model problems: $\tilde{u}_1$ is a solution of the Dirichlet problem (1) in the domain D_0 ,

$$\begin{cases} \tilde{u}_{1xx}'' + \tilde{u}_{1yy}'' = 0, (x, y) \in D_0, \\ \tilde{u}_1(x, y) = f(x, y), (x, y) \in \Gamma_0 \end{cases} \quad (25)$$

$\tilde{v}_1$ is solution of the Dirichlet problem in the domain D_1 ,

$$\begin{cases} \tilde{v}_{1xx}'' + \tilde{v}_{1yy}'' = 0, (x, y) \in D_1, \\ \tilde{v}_1(x, y) = \tilde{u}_1(x, y), (x, y) \in \Gamma_1, \end{cases}$$

where the function $\tilde{v}_1(x, y)$ is regular at infinity, i.e. it has a finite limit as $x^2 + y^2 \rightarrow \infty$.

Considering a pair of functions $\tilde{u}_j, \tilde{v}_j$ as defined, we define a pair $\tilde{u}_{j+1}, \tilde{v}_{j+1}$ as solutions of internal and external boundary value problems:

$$\begin{cases} \tilde{u}_{j+1xx}'' + \tilde{u}_{j+1yy}'' = 0, (x, y) \in D_0, \\ \tilde{u}_{j+1}(x, y) = \tilde{v}_j(x, y), (x, y) \in \Gamma_0 \end{cases}$$

$$\begin{cases} \tilde{v}_{j+1xx}'' + \tilde{v}_{j+1yy}'' = 0, (x, y) \in \bar{D}_1, \\ \tilde{v}_{j+1}(x, y) = \tilde{u}_{j+1}(x, y), (x, y) \in \Gamma_1, \end{cases}$$

respectively. The check shows that the solution of the Dirichlet problem (22)-(24) has the form

$$u_1(x, y) = \sum_{j=0}^{\infty} \left(\frac{1 - \chi}{1 + \chi} \right)^j \left(\tilde{u}_j(x, y) - \frac{1 - \chi}{1 + \chi} \tilde{v}_j(x, y) \right), (x, y) \in D_1 \quad (26)$$

$$u_2(x, y) = \frac{2\chi}{1 + \chi} \sum_{j=0}^{\infty} \left(\frac{1 - \chi}{1 + \chi} \right)^j \tilde{v}_j(x, y), (x, y) \in D_0 \setminus \bar{D}_1; \quad \chi = \frac{\lambda_1 a_2}{\lambda_2 a_1} \quad (27)$$

Formulas are simplified for a circular area.

Example 1. Let the regions D_0, D_1 be two circles centered at the beginning, radii 1 and R respectively. Let also the function $\tilde{u}$ be the solution of the Dirichlet problem in a circle D_0 , then formulas (26)-(27) take the form:

$$u_1(x, y) = \sum_{j=0}^{\infty} \left(\frac{1 - \chi}{1 + \chi} \right)^j \left(\tilde{u}(R^{2j}x, R^{2j}y) - \frac{1 - \chi}{1 + \chi} \tilde{u} \left(\frac{R^{2j+2}x}{x^2 + y^2}, \frac{R^{2j+2}y}{x^2 + y^2} \right) \right), (x, y) \in D_1$$

$$u_2(x, y) = \frac{2\chi}{1 + \chi} \sum_{j=0}^{\infty} \left(\frac{1 - \chi}{1 + \chi} \right)^j \tilde{u}(R^{2j}x, R^{2j}y), (x, y) \in D_0 \setminus D_1; \quad \chi = \frac{\lambda_1 a_2}{\lambda_2 a_1}.$$

The computational algorithm, based on exact formulas (26)-(27), is to break the series. This approach allows us to identify the iterative nature of the algorithm and speed up the calculation process. An important advantage of the algorithm is the ability to access libraries for solving

internal and external Dirichlet boundary value problems for the domain D_0 .

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