

# Dynamics of a System of Two Connected Bodies Moving along a Circular Orbit around the Earth

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**Abstract.** Symbolic–numeric methods are used to investigate the dynamics of a system of two bodies connected by a spherical hinge. The system is assumed to move along a circular orbit under the action of gravitational torque. The equilibrium orientations of the two-body system are determined by the real roots of a system of 12 algebraic equations of the stationary motions. Attention is paid to the study of the conditions of existence of the equilibrium orientations of the system of two bodies refers to special cases when one of the principal axes of inertia of each of the two bodies coincides with either the normal of the orbital plane, the radius vector or the tangent to the orbit. Nine distinct solutions are found within an approach which uses the computer algebra method based on the algorithm for the construction of a Gröbner basis.

## 1 Introduction

In this paper, we apply symbolic–numeric methods to investigate the dynamics of a system of two bodies (satellite and stabilizer) connected by a spherical hinge. The system moves in a central Newtonian force field along a circular orbit. The problem is of practical interest for designing composite gravitational orientation systems of satellites that can stay on the orbit for a long time without energy consumption. The dynamics of various composite schemes for satellite–stabilizer gravitational orientation systems was discussed in detail in [1].

We analyze the spatial equilibria (equilibrium orientations) of the satellite–stabilizer system in the orbital coordinate system for certain values of the principal central moments of inertia of the two bodies refers to special cases when one of the principal axes of inertia of each of the two bodies coincides with either the normal of the orbital plane, the radius vector or the tangent to the orbit. Equilibrium orientations of the satellite–stabilizer system are determined by real roots of a system of algebraic equations. To find equilibrium solutions, algorithms for Gröbner basis construction were used [2]. Some classes of equilibrium solutions are obtained explicitly from algebraic equations included in the Gröbner basis. The parameters values that cause the change in the number of equilibrium orientations for the satellite–stabilizer system are found numerically.

The combination of the computer algebra and linear algebra methods for the investigation of equilibrium orientations of the system of two bodies connected by a spherical hinge on a circular

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orbit under certain constraints imposed on the parameters of the bodies were successfully used in [3]. It has been shown that, on a circular orbit, the two-body system can have both planar and spatial configurations of the equilibrium orientation.

## 2 Equations of motion

Consider a system of two bodies connected by a spherical hinge that moves along a circular orbit [3]. To write the equations of motion of the two bodies, we introduce the following right-handed Cartesian coordinate systems: the absolute coordinate system  $CX_aY_aZ_a$  with the origin at the Earth's center of mass  $C$ ; the plane  $CX_aY_a$  coincides with the equatorial plane and the  $CZ_a$  axis coincides with the Earth axis of rotation, and the orbital coordinate system  $OXYZ$ . The  $OZ$  axis is directed along the radius vector that connects the Earth center  $C$  with the center of mass of the two-body system  $O$ , the  $OX$  axis is directed along the linear velocity vector of the center of mass  $O$ . Then, the  $OY$  axis is directed along the normal to the orbital plane. The coordinate system for the  $i$ th body ( $i = 1, 2$ ) is  $Ox_iy_iz_i$ , where  $Ox_i$ ,  $Oy_i$ , and  $Oz_i$  are the principal central axes of inertia for the  $i$ th body. The orientation of the coordinate system  $Ox_iy_iz_i$  with respect to the orbital coordinate system is determined using the pitch ( $\alpha_i$ ), yaw ( $\beta_i$ ), and roll ( $\gamma_i$ ) angles. The direction cosines  $a_{ij}$  of the first body and  $b_{ij}$  of the second body in the transformation matrix between the orbital coordinate system  $OXYZ$  and  $Ox_iy_iz_i$  are expressed in terms of the aircraft angles [1]. Suppose that  $(a_i, b_i, c_i)$  are the coordinates of the spherical hinge  $P$  in the body coordinate system  $Ox_iy_iz_i$ ,  $A_i, B_i, C_i$  are principal central moments of inertia;  $M = M_1M_2/(M_1 + M_2)$ ;  $M_i$  is the mass of the  $i$ th body;  $p_i, q_i$ , and  $r_i$  are the projections of the absolute angular velocity of the  $i$ th body onto the axes  $Ox_i, Oy_i$  and  $Oz_i$ ; and  $\omega_0$  is the angular velocity for the center of mass of the two-body system moving along a circular orbit. The kinetic energy of the system writes

$$T = 1/2(A_1p_1^2 + (B_1 + Ma_1^2)q_1^2 + (C_1 + Ma_1^2)r_1^2) + 1/2(A_2p_2^2 + (B_2 + Ma_2^2)q_2^2 + (C_2 + Ma_2^2)r_2^2) - Ma_1a_2((r_1a_{12} - q_1a_{13})(r_2b_{12} - q_2b_{13}) + (r_1a_{22} - q_1a_{23})(r_2b_{22} - q_2b_{23}) + (r_1a_{32} - q_1a_{33})(r_2b_{32} - q_2b_{33})). \quad (1)$$

The force function which determines the effect of the Earth gravitational field on the system of two bodies connected by a hinge [1] is given by

$$U = 3/2\omega_0^2((C_1 - A_1 + Ma_1^2)a_{31}^2 + (C_1 - B_1)a_{32}^2) + 3/2\omega_0^2((C_2 - A_2 + Ma_2^2)b_{31}^2 + (C_2 - B_2)b_{32}^2) - Ma_1a_2\omega_0^2(a_{11}b_{11} + a_{21}b_{21} + a_{31}b_{31}). \quad (2)$$

The equations of motion for this system can be written as Lagrange equations of the second kind by symbolic differentiation in the Maple system [4] in the case when  $b_1 = b_2 = c_1 = c_2 = 0$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial T}{\partial x_i} - \frac{\partial U}{\partial x_i} = 0, \quad i = \overline{1, 6}, \quad (3)$$

and kinematic Euler equations

$$\begin{aligned} p_1 &= (\dot{\alpha}_1 + 1)a_{21} + \dot{\gamma}_1, & p_2 &= (\dot{\alpha}_2 + 1)b_{21} + \dot{\gamma}_2, \\ q_1 &= (\dot{\alpha}_1 + 1)a_{22} + \dot{\beta}_1 \sin \gamma_1, & q_2 &= (\dot{\alpha}_2 + 1)b_{22} + \dot{\beta}_2 \sin \gamma_2, \\ r_1 &= (\dot{\alpha}_1 + 1)a_{23} + \dot{\beta}_1 \cos \gamma_1, & r_2 &= (\dot{\alpha}_2 + 1)b_{23} + \dot{\beta}_2 \cos \gamma_2. \end{aligned} \quad (4)$$

In (3)  $x_1 = \alpha_1, x_2 = \alpha_2, x_3 = \beta_1, x_4 = \beta_2, x_5 = \gamma_1, x_6 = \gamma_2$ .

### 3 Investigation of equilibria

Assuming the initial condition  $(\alpha_i, \beta_i, \gamma_i) = (\alpha_{i0} = \text{const}, \beta_{i0} = \text{const}, \gamma_{i0} = \text{const})$ , also  $A_i \neq B_i \neq C_i$ , we obtain from (3) and (4) the stationary equations

$$\begin{aligned} a_{22}a_{23} - 3a_{32}a_{33} &= 0, \\ (a_{23}a_{21} - 3a_{33}a_{31}) + m_1(a_{23}b_{21} - 3a_{33}b_{31}) &= 0, \\ (a_{22}a_{21} - 3a_{32}a_{31}) - n_1(a_{22}b_{21} - 3a_{32}b_{31}) &= 0, \\ b_{22}b_{23} - 3b_{32}b_{33} &= 0, \\ (b_{23}b_{21} - 3b_{33}b_{31}) + m_2(b_{23}a_{21} - 3b_{33}a_{31}) &= 0, \\ (b_{22}b_{21} - 3b_{32}b_{31}) - n_2(b_{22}a_{21} - 3b_{32}a_{31}) &= 0, \end{aligned} \tag{5}$$

which allow us to determine the equilibrium orientations of the system of two bodies connected by a spherical hinge in the orbital coordinate system. In (5):  $m_1 = Ma_1a_2/((A_1 - C_1) - Ma_1^2)$ ;  $m_2 = Ma_1a_2/((A_2 - C_2) - Ma_2^2)$ ;  $n_1 = Ma_1a_2/((B_1 - A_1) + Ma_1^2)$ ;  $n_2 = Ma_1a_2/((B_2 - A_2) + Ma_2^2)$ .

Taking into account the orthogonality conditions for the direction cosines,

$$\begin{aligned} a_{21}^2 + a_{22}^2 + a_{23}^2 - 1 &= 0, & b_{21}^2 + b_{22}^2 + b_{23}^2 - 1 &= 0, \\ a_{31}^2 + a_{32}^2 + a_{33}^2 - 1 &= 0, & b_{31}^2 + b_{32}^2 + b_{33}^2 - 1 &= 0, \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0, & b_{21}a_{31} + b_{22}b_{32} + b_{23}b_{33} &= 0, \end{aligned} \tag{6}$$

the equations (5) and (6) form a closed algebraic system of equations in the 12 unknown direction cosines that determine the equilibrium orientations of the two body system. For this system the following problem is formulated: for given  $m_1, m_2, n_1$ , and  $n_2$ , determine all twelve direction cosines. The other six direction cosines ( $a_{1i}$  and  $b_{1i}$ ) can be obtained from the orthogonality conditions. In [5] planar oscillations of the two-body system were analyzed, all equilibrium orientations were determined, and sufficient conditions for the stability of the equilibrium orientations were obtained using the energy integral as a Lyapunov function. In [3] for this case the system of 12 algebraic equations (5) and (6) was decomposed using linear algebra methods and algorithms for the Gröbner basis construction. Some classes of spatial equilibrium solutions were obtained from the algebraic equations included in the Gröbner basis.

Construction of the Gröbner basis for the system (5) and (6) of 12 second-order algebraic equations, the coefficients of which depend on 4 parameters, is a very complicated algorithmic problem. In the general case, the system of algebraic equations (5) and (6) cannot be solved by direct application of the Gröbner basis construction methods.

The system (5) and (6) was solved in the special cases when one of the principal axes of inertia of each of the two bodies coincides with either the normal of the orbital plane, the radius vector or the tangent to the orbit.

Case 1:  $a_{22}^2 = 1, a_{12} = a_{32} = a_{21} = a_{23} = 0, b_{22}^2 = 1, b_{12} = b_{32} = b_{21} = b_{23} = 0$ . In this case the system (5) has the simple form

$$a_{33}(a_{31} + m_1b_{31}) = 0, \quad b_{33}(b_{31} + m_2a_{31}) = 0, \quad a_{31}^2 + a_{33}^2 = 1, \quad b_{31}^2 + b_{33}^2 = 1. \tag{7}$$

Eqs. (7) define equilibrium solutions for the system of two bodies in the orbital plane. In [5] planar oscillations of the two-body system were analyzed and all equilibrium orientations were determined.

Case 2:  $a_{23}^2 = 1, b_{23}^2 = 1$  (axis  $Oz_1$  of the satellite and axis  $Oz_2$  of the stabilizer coincides with the normal  $OY$  to the orbital plane).

Case 3:  $a_{32}^2 = 1, b_{32}^2 = 1$ , and Case 4:  $a_{33}^2 = 1, b_{33}^2 = 1$  are similar to the Case 1.

Case 5:  $a_{12}^2 = 1, a_{11} = a_{13} = a_{22} = a_{32} = 0, b_{12}^2 = 1, b_{11} = b_{13} = b_{22} = b_{32} = 0$ . In this case, when  $a_{12} = b_{12} = 1$  or  $a_{12} = b_{12} = -1$  the system (5) takes the following form:

$$\begin{aligned} 4a_{21}a_{23} + m_1(a_{23}b_{21} + 3a_{21}b_{23}) &= 0, & a_{21}^2 + a_{23}^2 - 1 &= 0, \\ 4b_{21}b_{23} + m_2(a_{21}b_{23} + 3a_{23}b_{21}) &= 0, & b_{21}^2 + b_{23}^2 - 1 &= 0. \end{aligned} \tag{8}$$

To solve the algebraic system (8) we have applied the algorithm of constructing the Gröbner bases [2]. Using the Groebner[Basis] Maple 17 package [4] for constructing Gröbner bases the lexicographic monomial order was chosen. We constructed the Gröbner basis for the system of four polynomials (8) with the four direction cosines  $a_{21}, a_{23}, b_{21}, b_{23}$  taken for variables. In the list of polynomials we included the polynomials from the left-hand sides  $f_i$  ( $i = 1, 2, 3, 4$ ) of the algebraic equations (8). Here we write down the polynomial in the Gröbner basis that depends only on the variable  $x = a_{23}$ . This polynomial has the form

$$P(x) = x(x^2 - 1)P_1(x) = 0, \quad P_1(x) = p_0x^4 + p_1x^2 + p_2, \tag{9}$$

where  $p_0 = 64(m_1m_2 - 1)(m_1m_2 - 4), p_1 = -32(m_1^2 + 2)(m_1m_2 - 1)(m_1m_2 - 4), p_2 = 9m_1^2(4(m_1m_2 - 2)^2 - (m_1 + m_2)^2)$ . It is necessary to consider three cases:  $a_{23} = 0, a_{23}^2 = 1$ , and  $P_1(a_{23}) = 0$  to investigate the Case 5. Under  $a_{23} = 0$ , we have the following solutions:  $a_{12}^2 = 1, a_{21}^2 = 1, a_{11} = a_{13} = a_{22} = a_{23} = a_{31} = a_{32} = 0, b_{12}^2 = 1, b_{21}^2 = 1, b_{11} = b_{13} = b_{22} = b_{23} = b_{31} = a_{32} = 0$ . Under  $a_{23} = 1$ , we have the following solutions:  $a_{12}^2 = 1, a_{31}^2 = 1, a_{11} = a_{13} = a_{21} = a_{22} = a_{32} = a_{33} = 0, b_{12}^2 = 1, b_{31}^2 = 1, b_{23}^2 = 1, b_{11} = b_{13} = b_{21} = b_{22} = b_{32} = b_{33} = 0$ .

Let us consider the case when the satellite equilibria are determined by the real roots of the biquadratic equation  $P_1(x) = 0$ . The number of real roots of the biquadratic equation is even and not larger than 4. For each solution one can find two values of  $a_{23}$  and then their respective values  $a_{21}$ . For each set of values  $a_{21}$  and  $a_{23}$  one can unambiguously determine from the original system (5), (6) the values of the other direction cosines. Thus, each real root of the biquadratic equation (9) is matched with two sets of values  $a_{21}$  (two equilibrium orientations), and since the number of its real roots does not exceed 4, the satellite–stabilizer in the case 5 can have no more than 8 equilibrium orientations. Solutions of the biquadratic equation (9) exist under the conditions  $m_1m_2 < 1, m_1m_2 > 4$ .

Case 6:  $a_{13}^2 = 1, b_{13}^2 = 1$ , is similar to the case 5. In the case 6 all equilibrium orientations are determined by the equation

$$P(a_{22}) = a_{22}(a_{22}^2 - 1)P_2(a_{22}) = 0, \quad P_2(a_{22}) = p_3a_{22}^4 + p_4a_{22}^2 + p_5, \tag{10}$$

where  $p_3 = 64(n_1n_2 - 1)(n_1n_2 - 4), p_4 = 32(n_1^2 - 2)(n_1n_2 - 1)(n_1n_2 - 4), p_5 = n_1^2(4(n_1n_2 + 2)^2 - 9(n_1 + n_2)^2)$ .

In Case 7:  $a_{11}^2 = 1, b_{11}^2 = 1$  we have very simple equations

$$a_{22}a_{23} = 0, \quad a_{22}^2 + a_{23}^2 = 1, \quad b_{22}b_{23} = 0, \quad b_{22}^2 + b_{23}^2 = 1, \tag{11}$$

which not depend on the parameters of the two body system.

Case 8:  $a_{21}^2 = 1, b_{21}^2 = 1$  and Case 9:  $a_{31}^2 = 1, b_{31}^2 = 1$  are similar to the Case 7.

## References

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