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Multipartite Quantum Systems and Representations of Wreath Products

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Abstract. The multipartite quantum systems are of particular interest for the study of such phenomena as entanglement and non-local correlations. The symmetry group of the whole multipartite system is the wreath product of the group acting in the "local" Hilbert space and the group of permutations of the constituents. The dimension of the Hilbert space of a multipartite system depends exponentially on the number of constituents, which leads to computational difficulties. We describe an algorithm for decomposing representations of wreath products into irreducible components. The C implementation of the algorithm copes with representations of dimensions in quadrillions. The program, in particular, builds irreducible invariant projectors in the Hilbert space of a multipartite system. The expressions for these projectors are tensor product polynomials. This structure is convenient for efficient computation of quantum correlations in multipartite systems with a large number of constituents.

1 Introduction

The Hilbert space of a multipartite quantum system is the tensor product of the Hilbert spaces of the components: $\widetilde{\mathcal{H}} = \bigotimes_{x=1}^{N} \mathcal{H}_x$. The states of the multipartite system that can be represented as a weighted sum of tensor products of the states of the components are called *separable*. States that are not separable are called *entangled*. The vast majority of the states of a typical multipartite system are entangled. The concept of entanglement is the basis of quantum informatics. Entanglement leads to such experimentally observable phenomena as nonlocal quantum correlations (violation of Bell's inequalities), quantum teleportation, etc. Moreover, in recent years, the idea that the physical space itself is not a fundamental entity, but emerges as an approximate phenomenological structure within "the Hilbert space of the Universe" as a result of some process of statistical selection has become increasingly popular. The idea of an emergent space is attractive, in particular, because of the possibility to reformulate the problem of reconciling the quantum mechanics with the gravity, where the main difficulties arise from the fact that the quantum and space-time structures are considered to be equally fundamental. Typical approaches to identify geometric structures within a Hilbert space use the concept of entanglement (see, e.g., [1, 2]) and can be briefly described as follows. The quantum state in a large Hilbert space decomposes approximately into a tensor product of a large number of

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factors, which are interpreted as points (or bulks) of a geometric space, the distances between which (the metric) are determined by *entanglement measures*. A typical example of such a measure is the *quantum mutual information*.

Let $X \cong \{1, \ldots, N\}$ be a space with a finite set of points. To reproduce the usual property of the space, homogeneity, we will assume that the group of spatial symmetries $G = G(X)$ permutes transitively the components of a multipartite system. In this case, the Hilbert space of the system can be written as

$$\widetilde{\mathcal{H}} = \mathcal{H}^{\otimes N}, \tag{1}$$

where $\mathcal{H}$ is a representative of the G-orbit $\mathcal{H}_{xG}$. We will call $\mathcal{H}$ a *local Hilbert space*.

A quantum description can be made constructive if continuous groups of unitary evolution operators are replaced in the quantum formalism by unitary representations of finite groups. It is known that any linear (which is always unitary for a simple general reason) representation of a finite group is a subrepresentation of some permutation representation. In particular, the so-called *regular* representation, that is, the permutation representation of the action of a group on its own elements, contains all possible irreducible representations of a given group. Thus, we can embed any constructive quantum model into a suitable invariant subspace of some permutation representation [3, 4]. If we can decompose permutation representations of a group into irreducible components, then we can construct any representations of the group. Here we describe an algorithm for decomposing Hilbert space (1) into invariant subspaces with respect to the natural symmetry group of a multipartite quantum system. In particular, the algorithm outputs a complete set of projectors to irreducible invariant subspaces.

2 Irreducible invariant projectors of wreath product representation

Let $V \cong \{1, \ldots, M\}$ be a basis of the local Hilbert space $\mathcal{H}$ on which a group of *local symmetries* $F = F(V)$ acts. The sets X and V and the group F can be treated, respectively, as the *base*, the *typical fiber* and the *structure group* of a *fiber bundle*. A natural symmetry group that acts on the set of the *bundle sections* V^X and preserves the structure of the bundle is the *wreath product* of the groups F and G [5]: $\widetilde{W} = F \wr G \cong F^X \rtimes G$. The action of $\widetilde{W}$ on V^X is defined as $v(x)\left(f(x), g\right) = v\left(xg^{-1}\right) f\left(xg^{-1}\right)$, where $v \in V^X$, $f \in F^X$, $g \in G$. The *right-action* convention is used for all group actions.

In [6], we proposed an algorithm for decomposing the representations of finite groups based on the construction of a *complete set of mutually orthogonal irreducible invariant projectors*. These projectors are special elements of the *centralizer algebra*, which is defined as the algebra of matrices that commute with all matrices of the representation. The dimension of the centralizer algebra is called the *rank of the representation*. The computer implementation of the algorithm in [6] proved to be very effective in problems with low ranks. In particular, the program coped with many high dimensional representations of simple groups and their "small" extensions (which typically have low ranks), presented in the Atlas [7], in the computationally difficult case of characteristic zero [6].

The permutation representation $\widetilde{P}$ of the wreath product is a representation of $\widetilde{W}$ by $(0, 1)$-matrices of the size $M^N \times M^N$ that have the form $\widetilde{P}(\widetilde{w})_{u,v} = \delta_{u\widetilde{w},v}$, where $\widetilde{w} \in \widetilde{W}$; $u, v \in V^X$; δ is the *Kronecker delta*. To obtain all irreducible components of the representation $\widetilde{P}$ in the Hilbert space (1) it is sufficient to assume that the base field of this space is some abelian extension of the field of rational numbers $\mathbb{Q}$ that is a splitting field for the local group F.

Unfortunately, to split the representation $\widetilde{P}$ we cannot apply the algorithm [6] because wreath products are far from simple groups and their representations have too high ranks. However, it is possible to express the irreducible invariant projectors for the wreath product representation in terms of the projectors for the local group representation.

Let $B_1, \dots, B_L$ be the complete set of mutually orthogonal irreducible invariant projectors for the permutation representation of the local group F. Denote by $\overline{L}^X$ the set of all maps from X to $\overline{L}$, where $\overline{L} = \{1, \dots, L\}$. The action of $g \in G$ on the map $\ell = [\ell_1, \dots, \ell_N] \in \overline{L}^X$ is defined as $\ell g = \left[\ell_{1g}, \dots, \ell_{Ng}\right]$. Then we can prove the following

Proposition. *The irreducible invariant projector for the wreath product representation* $\widetilde{P}$ *is*

$$\widetilde{B}_k = \sum_{\ell \in kG} B_{\ell_1} \otimes \cdots \otimes B_{\ell_N}\,, \tag{2}$$

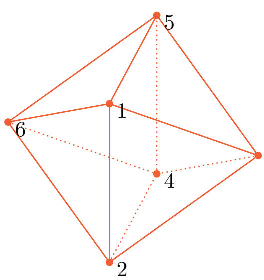
where kG *denotes the* G*-orbit of the map* k *on the set* $\overline{L}^X$.

The easily verifiable completeness condition $\sum_{i=1}^{\widetilde{K}} \widetilde{B}_{k^{(i)}} = \mathbb{1}_{M^N}$ holds. Here $\widetilde{K}$ is the number of irreducible components of the wreath product representation, $k^{(i)}$ denotes some numbering of the orbits of G on $\overline{L}^X$, $\mathbb{1}_{M^N}$ is the identity matrix in the Hilbert space (1).

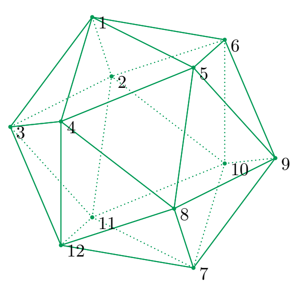
To compute the projectors (2), we wrote a program in C. The input data for the program are the generators of the spatial and local groups as well as the complete set of irreducible invariant projectors of the local group (obtained, e.g., by the program described in [6]).

3 Calculation examples

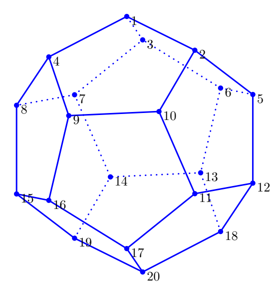
We give here two examples of calculations on a PC with 3.30GHz CPU and 16GB RAM. The space X in both cases is an icosahedron, and the local basis V is an octahedron and a dodecahedron. For the simplest example, we give a more detailed (with inevitable lacunae) output and only brief information in another case.



V = *octahedron*
$F(V) \cong \mathsf{S}_4$



X = *icosahedron*
$G(X) \cong \mathsf{A}_5$



V = *dodecahedron*
$F(V) \cong \mathsf{A}_5$

1. Wreath product $\mathsf{S}_4(\textit{octahedron}) \wr \mathsf{A}_5(\textit{icosahedron})$

`Representation dimension:` 2 176 782 336 `Rank:` 122 776
`Number of different suborbit lengths:` 46
`Wreath suborbit lengths:`
$1^{35}, 2^{249}, 3^{11}, 4^{258}, 5^{16}, \dots, 2097152^{18}, 4194304^{16}, 5242880^{2}, 6291456^{24}, 16777216^{7}$
`Checksum =` 2176782336 `Maximum multiplicity =` 12006
`Local centralizer algebra basis:`

$$A_1 = \mathbb{1}_6,\ A_2 = \begin{pmatrix} \mathbb{0}_3 & \mathbb{1}_3 \\ \mathbb{1}_3 & \mathbb{0}_3 \end{pmatrix},\ A_3 = \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix},\ Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

```
Wreath centralizer algebra basis:
```

$$\widetilde{A}_1 = A_1^{\otimes 12}$$
$$\widetilde{A}_2 = A_1^{\otimes 5} \otimes A_2 \otimes A_1^{\otimes 2} \otimes A_2 \otimes A_1^{\otimes 3}$$
$$\vdots$$
$$\widetilde{A}_{122775} = A_3^{\otimes 4} \otimes A_2 \otimes A_3^{\otimes 7} + A_3^{\otimes 5} \otimes A_2 \otimes A_3^{\otimes 6} + A_3^{\otimes 8} \otimes A_2 \otimes A_3^{\otimes 3} + A_3^{\otimes 9} \otimes A_2 \otimes A_3^{\otimes 2}$$
$$\widetilde{A}_{122776} = A_2 \otimes A_3^{\otimes 11} + A_3 \otimes A_2 \otimes A_3^{\otimes 10} + A_3^{\otimes 6} \otimes A_2 \otimes A_3^{\otimes 5} + A_3^{\otimes 7} \otimes A_2 \otimes A_3^{\otimes 4}$$

```
Wreath product decomposition is multiplicity free
Number of irreducible components:
```
122 776
```
Number of different dimensions:
```
134
```
Irreducible dimensions:
```
$1, 4^6, 6^3, 8^6, 9^3, 12^{15}, \ldots, 531441, 629856^{62}, 708588^{15}, 944784^{26}, 1180980, 1417176^9$

`Checksum =` 2176782336 `Maximum number of equal dimensions =` 6966

```
Local irreducible projectors:
```
$B_1 = \frac{1}{6}(A_1 + A_2 + A_3)$, $B_2 = \frac{1}{3}\left(A_1 + A_2 - \frac{1}{2}A_3\right)$, $B_3 = \frac{1}{2}(A_1 - A_2)$
```
Wreath irreducible projectors:
```

$$\widetilde{B}_1 = B_1^{\otimes 12}$$
$$\widetilde{B}_2 = B_1^{\otimes 3} \otimes B_2 \otimes B_1^{\otimes 6} \otimes B_2 \otimes B_1$$
$$\vdots$$
$$\widetilde{B}_{122775} = B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 7} + B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 6} + B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3^{\otimes 3} + B_3^{\otimes 5} \otimes B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3 + B_3^{\otimes 8} \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 + B_3^{\otimes 9} \otimes B_2 \otimes B_3 \otimes B_2$$
$$\widetilde{B}_{122776} = B_3^{\otimes 3} \otimes B_2^{\otimes 2} \otimes B_3^{\otimes 7} + B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3^{\otimes 6} + B_3 \otimes B_2 \otimes B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 6} + B_3^{\otimes 6} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 3} + B_3^{\otimes 7} \otimes B_2^{\otimes 2} \otimes B_3^{\otimes 3} + B_3^{\otimes 9} \otimes B_2^{\otimes 2} \otimes B_3$$

```
Time:  0.58 sec    Maximum number of tensor monomials:  531441
```

2. Wreath product $\mathsf{A}_5(\textit{dodecahedron}) \wr \mathsf{A}_5(\textit{icosahedron})$

`Representation dimension:` 4 096 000 000 000 000 `Rank:` > 502 985 717
```
Wreath product decomposition has non-trivial multiplicities
Number of irreducible components:
```
502 985 717
```
Number of different dimensions:
```
1065
```
Time:  26 min 46.16 sec    Maximum number of tensor monomials:  2176782336
```

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