

Finite Time Correlations and Compressibility Effects in the Three-Dimensional Kraichnan Model

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Abstract. Using the field theoretic renormalization group technique the simultaneous influence of the compressibility and finite time correlations of the non-solenoidal Gaussian velocity field on the advection of a passive scalar field is studied within the generalized Kraichnan model in three spatial dimensions up to the second-order approximation in the corresponding perturbative expansion. All possible infrared stable fixed points of the model, which drive the corresponding scaling regimes of the model, are identified and their regions of the infrared stability in the model parametric space are discussed. It is shown that, depending on the value of the parameter that drives the compressibility of the system, there exists a gap in the parametric space between the regions where the model with the frozen velocity field and the model with finite-time correlations of the velocity field are stable or there exists an overlap between them.

1 Introduction

Although the anomalous scaling in the fully developed turbulence, i.e., the existence of deviations from the predictions of the classical Kolmogorov-Obukhov (KO) [1, 2] observed in both natural experiments as well as numerical simulations [3, 4], still belongs to the open problems of the classical physics. Nevertheless, during the last few decades, great progress has been achieved in its understanding in the framework of the investigation of scaling properties of the passive scalar and vector quantities advected by synthetic turbulent environments with Gaussian statistics of the velocity field, namely, in the framework of the Kraichnan rapid-change model [5] and its various descendants (see, e.g., Refs. [6, 7] and references therein).

One of the most effective analytic approaches to the systematic investigation of the anomalous scaling properties of fully developed turbulent systems is based on the field-theoretic renormalization group (RG) method and the operator product expansion (OPE) technique [8, 9]. This technique was used for the first time in the systematic investigation of the anomalous scaling in the Kraichnan rapid-change model in Ref. [10] and, subsequently, various generalizations of the Kraichnan model were also analyzed [7]. One phenomenologically important generalization of the rapid-change-like models is the inclusion of the finite-time correlations of the velocity field [11]. In this respect, although a lot

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of properties of the Gaussian turbulent systems with finite-time correlations were obtained [12, 13], the influence the compressibility has on the anomalous scaling in the Kraichnan model with finite time correlations of the velocity field still remains an open problem. On one hand, the lowest (one-loop) approximation is not suitable for this purpose [12] and, on the other hand, the corresponding two-loop analysis is much more complicated [14] and is not yet fully developed.

Therefore, the aim of the present study is to perform the first part of the necessary analysis for solving the problem of the anomalous scaling in compressible turbulent systems with finite time velocity correlations, namely, to perform the two-loop field theoretic RG analysis of the compressible Kraichnan model with finite time correlations of the velocity field in the spatial dimension $d = 3$, i.e., to determine all possible scaling regimes of the model and to find the corresponding regions of their infrared (IR) stability.

2 Model description

In what follows the passive advection of a scalar tracer¹ field $\theta \equiv \theta(x) = \theta(t, \mathbf{x})$ via fluctuating part of a velocity field $\mathbf{v}(x)$ that is described by the stochastic equation

$$\partial_t \theta(x) + [\mathbf{v}(x) \cdot \nabla] \theta(x) = \nu_0 \Delta \theta(x) + f(x), \quad (1)$$

where $\partial_t \equiv \partial/\partial t$, ∇ is the nabla operator, $\Delta \equiv \nabla^2$, ν_0 represents the coefficient of molecular diffusivity, and $f = f(x)$ is a Gaussian random noise with zero mean, is to be investigated. The subscript 0 will hereafter denote bare parameters of the unrenormalized theory. The correlation function of the random force $f(x)$ maintains the steady state of the system but its detailed form is unessential here [11].

Furthermore, it is assumed that $\mathbf{v}(x)$ is a compressible velocity field with zero mean ($\langle \mathbf{v} \rangle = 0$) obeying Gaussian distribution with correlation function in the form

$$\langle v_i(x) v_j(x') \rangle = \int \frac{d\omega d^d \mathbf{k}}{(2\pi)^{d+1}} \frac{g_0 \nu_0^3 k^{4-d-2\varepsilon-\eta} R_{ij}(\mathbf{k}) \exp[-i\omega(t-t') + i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')] }{(i\omega + u_0 \nu_0 k^{2-\eta})(-i\omega + u_0 \nu_0 k^{2-\eta})},$$

where \mathbf{k} and ω correspond to the wave vector and frequency, respectively, and g_0 and u_0 take on the role of bare coupling constants (charges) of the model. Here, one considers $R_{ij}(\mathbf{k}) = \delta_{ij} + (\alpha - 1)k_i k_j / k^2$ in order to account for the geometric properties of the velocity field fluctuations, where $k = |\mathbf{k}|$, δ_{ij} is the ordinary Kronecker delta tensor, $\alpha \geq 0$ is a free parameter related to compressibility [the case of $\alpha = 0$ corresponds to the divergence-free (incompressible) advecting velocity field]. Finally, the exponents ε and η control the energy pumping and the form of the finite time correlations, respectively (see, e.g., Ref. [11] for details). Note that due to a different notation used in Ref. [11], in order to compare any formula with those presented here one must make a substitution $\varepsilon \rightarrow 2\varepsilon$.

It is also important to note that u_0 describes the magnitude of the finite time correlations. This can be seen by looking at its two noteworthy limits. Firstly, the case when $u_0 \rightarrow \infty$ while keeping the ratio $g'_0 \equiv g_0/u_0^2$ constant corresponds to the so-called rapid-change model which translates into delta correlations of the velocity field in time. On the other hand, when $u_0 \rightarrow 0$ and $g''_0 \equiv g_0/u_0$ is once again kept constant, one obtains the so-called frozen velocity field limit in which the velocity correlations are independent of time in the t representation.

General RG analysis of the model can be found in Refs. [11, 14].

¹The problem of a scalar density field will not be discussed here.

Table 1. Table of fixed points contained in the model.

	FPI	FPII	FPIII	FPIV	FPV
trivial	yes	no	yes	no	no
u_0	∞	∞	0	0	finite

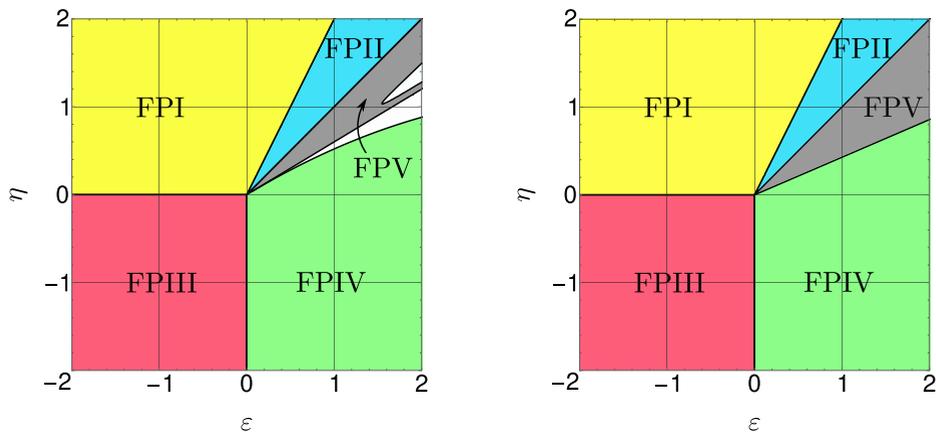


Figure 1. Phase diagram of IR stability regimes for all fixed points of the model under consideration, labeled FPI – FPV, with respect to ϵ and η for $\alpha = 0.8$ and $\alpha = 8/7$. Boundaries of the fixed points are depicted by thick curves. The white space corresponds to a region of instability that appears for any $0 < \alpha < 8/7$.

3 Results

As it turns out from a rather cumbersome analysis, there are five distinct fixed points present in the model whose basic classification is given in Table 1 (see also [14] for notation).

It has been found that the two-loop corrections play an important role when it comes to the stability, particularly due to the fact that if not for them, the stability regions of all fixed points would lay perfectly next to one another in the phase space of the parameters ϵ and η . Observations of such behavior can be made even when accounting for the higher corrections in the incompressible case ($\alpha = 0$). Here, the boundaries of FPII, FPIV, and FPV coincide with each other along the line $\eta = \epsilon$.

However, this is not the case when the compressibility is taken into account since the boundary of FPIV forms a parabola for $\alpha \neq 8/7$ or $\alpha \neq 0$, whilst, at least for $\alpha < 8/7$, the boundary of FPV is still in the form of a line. Such behavior necessarily leads to the formation of gaps of stability for $\alpha < 8/7$, when the boundary of FPIV is concave, i.e., for certain combinations of ϵ and η , no fixed point is stable, see Fig. 1, left. Moreover, with increasing values of α , one can notice a region of instability in a hyperbola-like form growing larger. The aforementioned region of instability is present for any value of $\alpha \neq 8/7$, however, its area increases proportionally to α . To be more specific, the tip of said region moves ever so closer to the origin.

When $\alpha = 8/7$, the situation is as follows: FPIV is governed purely by the one-loop contributions, the stability region of FPV is largest at this point and its boundary coincides with that of FPIV, see Fig. 1, right. This occurs due to the fact that any of the restrictions put on FPIV are indistinguishable for $\alpha = 8/7$.

Comparing these results with the ones obtained previously in [11, 12, 14] one can observe a complete agreement when it comes to the stability of FPI – FPIII as well as the one-loop results for

each of the fixed points from [12, 14]. Such behavior is to be expected as most of these results are in fact exact, i.e., they do not contain any correction of the order $O(\varepsilon^2)$ or higher [11]. The emergence of gaps in stability is specifically one of the reasons why the two-loop approximation is an important addition to the results from its one-loop counterpart.

It might be worth pointing out that the most physical point in the $\varepsilon - \eta$ plane, i.e., $\varepsilon = \eta = 4/3$, lays on the boundary between FPII and FPV for all values of α except for the incompressible case, in which FPIV is also relevant.

4 Conclusion

Influences of compressibility and finite time correlation of the velocity field obeying Gaussian statistics on IR scaling in a model of a passive scalar field advection were simultaneously investigated using the field theoretic RG approach at the two-loop level of approximation in the generalized Kraichnan model. Five distinct IR fixed points were found and regions of their IR stability were determined. It is shown the the two-loop corrections have nontrivial impact on the coordinates as well as on the IR stability of the nontrivial fixed point that describes the so-called frozen limit of the model as well as on the stability of the general fixed point that drives the model with finite time correlations of the turbulent velocity field.

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