The 4-loop slope of the Dirac form factor

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Abstract. The 4-loop contribution to the slope of the Dirac form factor in QED has been evaluated with 1100 digits of precision. The value is

$$m^2 F_1^{(4)\prime}(0) = 0.886545673946443145836821730610315359390424032660064745\ldots \left(\frac{\alpha}{\pi}\right)^4 .$$

We have also obtained a semi-analytical fit to the numerical value. The expression contains harmonic polylogarithms of argument $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$, one-dimensional integrals of products of complete elliptic integrals and six finite parts of master integrals, evaluated up to 4800 digits. We show the correction to the energy levels of the hydrogen atom due to the slope.

1 The slope of the Dirac form factor

In QED the vertex can be written

$$\bar{u}(p_1)\left(\gamma_\mu F_1(t) + \frac{\sigma_{\mu\nu}}{2m} q_\nu F_2(t)\right) u(p_2) , \qquad (1)$$

where m is the electron mass, $F_1(t)$ and $F_2(t)$ are the Dirac and Pauli form factors. At $t = 0$, the charge conservation implies that

$$F_1(0) = 1 , \qquad (2)$$

whereas the value of the Pauli form factor is the g-2

$$F_2(0) = \frac{g-2}{2} . \qquad (3)$$

The quantity $\left.\frac{d}{dt}F_1(t)\right|_{t=0} = F_1'(0)$ is the *slope* of the Dirac form factor.

1.1 Theoretical expression

We expand perturbatively the slope in powers of $\left(\frac{\alpha}{\pi}\right)$

$$m^2 F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots . \qquad (4)$$

The coefficients known in analytical form are [2–4]

$$A_1 = -\frac{1}{8} - \frac{1}{6\epsilon} . \qquad (5)$$

$$A_2 = -\frac{4819}{5184} - \frac{49}{432}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3)$$

$$= 0.469\,941\,487\,459\,992\ldots , \qquad (6)$$

$$A_3 = -\frac{17}{24}\pi^2\zeta(3) + \frac{25}{8}\zeta(5) - \frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{103}{1080}\pi^2\ln^2 2 + \frac{3899}{25920}\pi^4 - \frac{2929}{288}\zeta(3) + \frac{41671}{2160}\pi^2\ln 2 - \frac{454979}{38880}\pi^2 - \frac{77513}{186624}$$
$$= 0.171\,720\,018\,909\,775\ldots . \qquad (7)$$

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0.886545673946443145836821730610315359390424032660064745368805
5909320840316465628927454836486324177336869351275874721830799
6875923974888466826147611753011917584831446774752672980326917
4027192146515393255198447931004950196245313721193729467160800
6342998095842536958494506068383665985141387321894210012394882
7595153823786537220388349644856007568985761687756410271977960
3910290276615122356406105399227905150277608224592369504332757
0361335093525176476399251682267935964524928545665821844102867
4547644077579921118603788315350119800677785150747802126742479
0405222247330295021831074290199029916276829160228905899116426
4634498789876307270828483643587434780024554153724340089695147
1683115538642559188352093478066512674887503345902599182245563
6131251241198806154155376213371122848462776848674219282896865
6811548030353727600787303621093059264752959892234017835732828
9717496239918335278488413242436969926422136403200684400061242
3529815833966332566753158241741448217616597381276692161976675
0950507406493095613619589880245645116354567571623094417388481
1565020098334847940590188785421700667378220853053541953188378
610075518116338519220...
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Table 1. First 1100 digits of A_4.

In this paper we present the result of the calculation of A_4 with a precision of 1100 digits. The first digits of the result are

$$A_4 = 0.886545673946443145836821730610315\ldots . \qquad (8)$$

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loop	$F_1'(0)$	$F_2(0)$
1	∞	0.5
2	0.469941487459	−0.328478965579
3	0.171720018909	1.181241456587
4	0.886545673946	−1.912245764926
5		6.737(159)
	positive	alternating signs

Table 2. Values of the known contributions to $F_1'(0)$ and $F_2(0)$

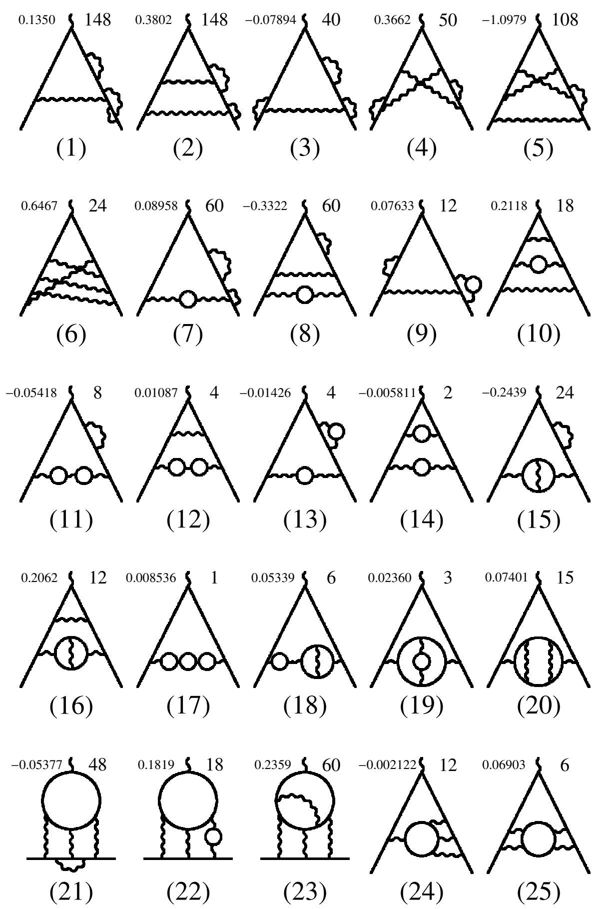

Figure 1. Typical representative diagrams of gauge-invariant sets. For each set only one diagrams is shown. Top left: contribution of the set to $F_1'(0)$; top right: number of diagrams of the set

The full-precision result is shown in table 1. In table 2 we have listed the known values of the slope and g-2; we see that A_2, A_3 and A_4 are all positive, in contrast with the alternating signs observed in the g-2.

1.2 Shift to the hydrogen levels

Let us now consider the shift to the hydrogen energy levels due to A_4. We express the energy shift in terms of the frequency shift $\Delta f = \Delta E/h$. For the level nS the frequency shift is [5, 6]

1	0.1350531726346435372674724541103838371038
2	0.3802929165240844585555252829884357965837 1
3	- 0.0789488893676831608109628366941799823079
4	0.3662786736588470044584250527325325702299
5	- 1.0979832148317652705103820073196531832520
6	0.6467871429585372084492391789800382619165
7	0.0895891170440342216099366534902414320652
8	- 0.3322086225106643608126657791889571079890
9	0.0763376479373933425961220467893817339605
10	0.2118669010888818123786340161652003594809
11	- 0.0541837571893361764657206136746826299854
12	0.0108761535582321058694530867351119912448
13	- 0.0142646608196830116628021692409901716905
14	- 0.0058117416010420357833143542203438251011
15	- 0.2439068506475319592123409557076293747890
16	0.2062012570841125786262218639260170000956
17	0.0085366428673036656037790352019835488011
18	0.0533927095302949341276880145918233326838
19	0.0236058911191014021135877461122766184082
20	0.0740163162205724051338179043210727390276
21	- 0.0537711607064956999082765338567906834199
22	0.1819474273966664016975772159395176159307
23	0.2359289294543601921365690660148707901595
24	- 0.0021225895319909487365222280699442649666
25	0.0690362620755704991160330435886767859471

Table 3. Contribution to A_4 of the 25 gauge-invariant sets of Fig.1.

$$\Delta f_{\text{slope}}(n\text{S}, \text{4-loop}) = \frac{4(Z\alpha)^4 mc^2}{h\, n^3}\left(\frac{m_r}{m}\right)^3\left[\left(\frac{\alpha}{\pi}\right)^4 A_4\right], \quad (9)$$

where m_r is the reduced mass $m_r = mM/(m+M)$ and M is the proton mass. Inserting the values of m, M, c, h and $Z = 1$, the correction due to A_4 is

$$\Delta f_{\text{slope}}(n\text{S}, \text{4-loop}) = \frac{36.11}{n^3}\ \text{Hz}, \quad (10)$$

and is comparable with the experimental error of the extremely precise measurement of 1S − 2S transition[7]

$$f(1\text{S} - 2\text{S}) = 2466\,061\,413\,187\,018 \pm 11\ \text{Hz}. \quad (11)$$

Eq.(10) is the first calculated four-loop correction to the energy levels, of the kind $\left(\frac{\alpha}{\pi}\right)^4 (Z\alpha)^4$.

2 Gauge-invariant sets

There are 891 vertex diagrams contributing to A_4. These vertex diagrams can be arranged in 25 gauge-invariant sets (Fig.1). The sets are classified according to the number of photon corrections on the same side of the main electron line and the insertions of electron loops (see Ref.[8]). The numerical contributions of each set, truncated to 40 digits, are listed in the table 3. The separate contributions to the slope from diagrams without or with internal loops are listed in table 4.

loop	$F_1'(0)$ no elec. loop	$F_1'(0)$ only elec. loop
2	0.438352514986	0.031588972473
3	−0.002437444568	0.174157463478
4	0.351479801576	0.535065872369

Table 4. Separate contributions to the slope from diagrams without and with electron loops.

3 The analytical fit

By building systems of integration-by-parts identities[9, 10] and solving them[11], the contributions to A_4 of all the diagrams are expressed as linear combinations of 334 master integrals, the same ones as appeared in the calculation of 4-loop g-2 [1]. In Ref.[1] these master integrals were calculated numerically with precision ranging from 1100 to 9600 digits; analytical expressions were fit to all these master integrals (single or in particular combinations) by using the PSLQ algorithm[12, 13]. We use those results here.

Therefore, the analytical expression of A_4 contains the same transcendentals appeared in the g-2 result: values of harmonic polylogarithms[14] with argument 1, $\frac{1}{2}$, $e^{\frac{i\pi}{3}}$, $e^{\frac{2i\pi}{3}}$, $e^{\frac{i\pi}{2}}$ [15, 16] , a family of one-dimensional integrals of products of elliptic integrals, and the finite terms of the ϵ−expansions of six master integrals belonging to topologies 24 and 25 of Fig.1. The expression of the analytical fit is written as follows:

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U \,, \tag{12}$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) \\
&+\frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+\frac{4572662443}{12247200}\ln^2 2\zeta(2) - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) \\
&+\frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&-\frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) \\
&-\frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 + \frac{18215}{27}a_5\ln 2 \\
&-\frac{18215}{27}\zeta(5)\ln 2 + \frac{402152509}{189000}a_4\zeta(2) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 \\
&-\frac{328317209}{302400}\zeta(4)\ln^2 2 - \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 \\
&-\frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) \\
&-\frac{427145}{504}a_4\zeta(3) - \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) \\
&+\frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 \\
&+\frac{607282}{189}a_6\ln 2 - \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 \\
&+\frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 \\
&+\frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 \\
&+\frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 \,,
\end{aligned} \tag{13}$$

$$\begin{aligned}
W =& -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) \\
&-118\Big(4\mathrm{Re}H_{0,1,0,1,1}\,(i)\,\zeta(2) + 4\mathrm{Im}H_{0,1,1}\,(i)\,\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\
&-2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Big) \,,
\end{aligned} \tag{14}$$

$$\begin{aligned}
V_a =& -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) \\
&+\frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi \\
&+\frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&+\frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) \\
&-\frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 \\
&-\frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 \\
&-\frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 + \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 \,,
\end{aligned} \tag{15}$$

$$\begin{aligned}
V_b =& \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
&-\frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&-507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&-\frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&-\frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
&-\frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
&-\frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
&-\frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \,,
\end{aligned} \tag{16}$$

$$\begin{aligned}
E_a =& \pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) \\
&-\frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\Big(\frac{751}{972}\ln 2\; f_2(0,0,1) \\
&-\frac{365478661}{24494400}f_2(0,2,0) + \frac{119022487}{5443200}f_2(0,1,1) \\
&-\frac{119022487}{14515200}f_2(0,0,2)\Big) - \frac{751}{729}\zeta(2)f_1(0,0,1) \\
&+\pi\Big(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1) + \frac{1105}{108}\ln 2\; f_2(0,0,2) \\
&-\frac{2210}{81}\ln 2\; f_2(0,1,1) + \frac{4420}{243}\ln 2\; f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) \\
&+\frac{272833}{41472}f_2(0,1,2) - \frac{4011005}{497664}f_2(0,2,1) + \frac{8417635}{2239488}f_2(0,3,0) \\
&+\frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0) \\
&-\frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\Big) \,,
\end{aligned} \tag{17}$$

T	- 2191.23546965751178841316292285882885509
$\sqrt{3}V_a$	- 648.74441479274053140037234999290048941
V_b	- 400.66449515766079257160481868291283752
W	1539.32919916681645350981276108756905937
$\sqrt{3}E_a$	- 266.54091710106238111286732183079933994
E_b	1928.22253648844241548541655379066123429
U	40.52010672766306764861492021782154366

Table 5. Numerical values of the addends appearing in Eq.12.

$$E_b = \zeta(2)\Big(-\frac{4629335}{165888} f_1(0,0,2) + \frac{112357}{1536} f_1(0,1,1) - \frac{99731}{1944} f_1(0,2,0) + \frac{157753}{41472} f_1(1,0,1)\Big)\,, \tag{18}$$

$$U = \frac{174623}{288000} C_{81a} + \frac{29479}{7200} C_{81b} - \frac{43}{6} C_{81c} + \frac{10871}{14400} C_{83a} - \frac{157}{1620} C_{83b} - \frac{95}{24} C_{83c}\,. \tag{19}$$

In the above expressions $\zeta(n) = \sum_{i=1}^{\infty} i^{-n}$, $a_n = \sum_{i=1}^{\infty} 2^{-i}\ i^{-n}$, $b_6 = H_{0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $b_7 = H_{0,0,0,0,0,1,1}\left(\frac{1}{2}\right)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$, $\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(e^{i\theta})$. $H_{i_1,i_2,\ldots}(x)$ are the harmonic polylogarithms. The integrals f_j are defined as follows:

$$\begin{aligned} f_m(i,j,k) = &\int_1^9 ds\, D_1(s) \mathrm{Re}\left(\sqrt{3}^{m-1} D_m(s)\right) \times \\ &\left(s - \frac{9}{5}\right) \ln^i(9-s) \ln^j(s-1) \ln^k(s)\,, \\ D_m(s) = &\frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}} \times \\ &K\left(m - 1 - (2m-3)\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right)\,; \end{aligned} \tag{20}$$

$K(x)$ is the complete elliptic integral of the first kind. The constants B_3 and C_3 have the following hypergeometric representations [1, 17]:

$$\begin{aligned} B_3 = &\frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right) - {}_4\tilde{F}_3\left({\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right)\right]\,, \\ C_3 = &\frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left({\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right) - {}_4\tilde{F}_3\left({-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2} \atop -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}};1\right)\right]\,, \end{aligned} \tag{21}$$

$${}_4\tilde{F}_3\left({a_1\ a_2\ a_3\ a_4 \atop b_1\ b_2\ b_3};x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\, {}_4F_3\left({a_1\ a_2\ a_3\ a_4 \atop b_1\ b_2\ b_3};x\right). \tag{22}$$

The numerical values of the constants appearing in Eq.(12) are listed in Table 5. Note the strong numerical cancellations in Eq.(12): the largest term is $-\frac{2749470791}{387072}\zeta(2)\zeta(5) = -12115.862$.

4 Method of calculation

We sketch the method used to obtain A_4. It is the same used in Ref.[1].

1. Generation of 891 vertex diagrams (*C* program) from 104 self-mass diagrams. These are the same of the 4-loop g-2 calculation.
2. Extraction of the contribution to A_4 from the amplitude of each diagram by using projectors [18, 19] with a `FORM` program[20, 21].
3. Algebraic reduction to master integrals, obtained by building and solving *large* systems of integration-by-parts identities[9, 10] by using the program `SYS`[11].
4. For the sake of checks we generate a different system for each group of vertex diagrams obtained from the same self-mass diagram.
5. The smallest system contains 10^8 identities, with size of 90GB. The system with the largest number of identities contains 5×10^8, with a size of 170GB. The largest system has 3×10^8 identities with a size of 1.2TB.
6. The ratio between number of independent identities and total number of generated identities is in the range $0.2-0.3$. The dependent identities become trivial zeroes when substituted into the system, and have been used to check the reliability of hardware and software. No hardware errors were detected. Instead, software errors have been detected in this way (frequency: one every 2-3 weeks), caused by a bug in the OpenMPI message passing library used with the highest level of threads support.
7. We algebraically check that the contribution from a diagram is invariant to the changes in the particular internal routing of the momentum of the external photon.
8. The renormalization is carried out by subtracting suitable counterterms, which are generated with C and `FORM` programs and calculated numerically with `SYS`.

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References

[1] S. Laporta, Phys. Lett. B **772** (2017) 232
[2] R. Barbieri, J. A. Mignaco and E. Remiddi, Nuovo Cim. A **11** (1972) 824.
[3] R. Barbieri, J. A. Mignaco and E. Remiddi, Nuovo Cim. A **11** (1972) 865.
[4] K. Melnikov and T. van Ritbergen, Phys. Rev. Lett. **84** (2000) 1673.
[5] G. W. Erickson and D. R. Yennie, Annals Phys. **35**, (1965) 271.
[6] G. W. Erickson and D. R. Yennie, Annals Phys. **35**, (1965) 447.
[7] A. Matveev *et al.*, Phys. Rev. Lett. **110** (2013) no.23, 230801.
[8] P. Cvitanovic, Nucl. Phys. B **127** (1977) 176.
[9] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.
[10] F. V. Tkachov, Phys. Lett. **100B** (1981) 65.
[11] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087
[12] H. R. P. Ferguson and D. H. Bailey, RNR Technical Report RNR-91-032.
[13] D. H. Bailey and D. J. Broadhurst, Math. Comput. **70** (2001) 1719
[14] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) 725.
[15] S. Laporta, PoS LL **2018** (2018) 073. doi:10.22323/1.303.0073
[16] J. Ablinger, J. Blumlein and C. Schneider, J. Math. Phys. **52** (2011) 102301
[17] Y. Zhou, Ramanujan J. (2019). `https://doi.org/10.1007/s11139-018-0090-z`
[18] R. Barbieri, M. Caffo, E. Remiddi, S. Turrini and D. Oury, Nucl. Phys. B **144** (1978) 329.
[19] S. Laporta and E. Remiddi, Adv. Ser. Direct. High Energy Phys. **20** (2009) 119.
[20] J. A. M. Vermaseren, arXiv:math-ph/0010025
[21] J. Kuipers, T. Ueda, J. A. M. Vermaseren and J. Vollinga, Comput. Phys. Commun. **184** (2013) 1453