

Theoretical Progress at the Frontiers of Small- x Physics

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Abstract. In recent years, the theoretical foundations of small- x physics have made significant advances in two frontiers: higher-order (NLO) corrections and power-suppressed (sub-eikonal) corrections. Among the former are the NLO calculations of the linear (BFKL) and nonlinear (BK-JIMWLK) evolution equations, as well as cross sections for various processes. Among the latter are corrections to the whole framework of high-energy QCD, including new contributions from quarks and spin asymmetries. One common element to both of these frontiers is the appearance of collinear logarithms beyond the leading-order framework. The proper treatment of these logarithms is a major challenge in obtaining physical cross sections at NLO, and they lead to a new double-logarithmic resummation parameter which governs spin at small x . In this paper, I will focus on the role of these collinear logarithms in both frontiers of small- x physics, as well as give a brief sample of other recent advances in its theoretical foundations.

1 Introduction: Small- x Physics at Leading Order

In the limit of high energies and fixed transverse momenta, QCD enters what is known as the “small- x regime,” with $x \sim p_T^2/s$ where p_T is a particle’s transverse momentum and s is the center-of-mass energy squared of the collision (see [1] and references therein for a review). When $x \ll \alpha_s \ll 1$, one has the onset of “small- x kinematics” or “Regge kinematics.” In this regime, processes which can be higher order in α_s but leading in x dominate. One example is the transition of Deep Inelastic Scattering from a “knockout” process, in which a virtual photon ejects a quark from the proton, to a dipole process in which the virtual photon fluctuates into a $q\bar{q}$ pair and scatters hadronically on the proton [2–4]. The features of small- x kinematics, like boost invariance, are ubiquitous at top RHIC and LHC energies.

Features of small- x kinematics, however, must be distinguished from signals of small- x evolution. Quantum evolution is triggered when systematic large logarithms of $1/x$ begin to spoil the pQCD perturbation series; this occurs when $\alpha_s \ln \frac{1}{x} \gtrsim 1$, or equivalently $x \lesssim \exp[-1/\alpha_s]$. The deviation from pQCD is reflected in the tension of pure DGLAP fits to the proton structure functions at HERA; these can be considered experimental signs of the onset of *linear* small- x evolution [5]. The linear small- x evolution leads to a power-law divergence of the F_2 structure function as $x \rightarrow 0$ which is irreconcilable with the unitarity of QCD. This contradiction is indicative of a third regime at even smaller x , when the saturation momentum $Q_s(x)$ induced by the small- x evolution has grown to become a semi-hard scale, $Q_s(x) \gg \Lambda_{QCD}$. This high-energy, high-density regime of QCD is characterized by gluon

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saturation: the softening of the power-law divergence as $x \rightarrow 0$ due to a maximum gluon number density. As a consequence of gluon saturation, this regime admits a powerful semi-classical field theoretic description, known as the color-glass condensate (CGC).

Various formulations of the CGC framework exist, employing different languages to describe the same physics. These include the “background field” or “rapidity factorization” methods [6]; light-front Hamiltonian methods [7]; and functional integral methods [8]. While the languages used to describe small- x physics are very different, these methods are carefully calibrated to reproduce the known results at leading order. Important unanswered questions remain, however, about these descriptions beyond leading order.

2 Active Frontiers of Small- x Theory

Much of the recent progress in small- x theory focuses on corrections to this LO picture. The calculation of small- x evolution and specific processes at next-to-leading order (NLO) in the strong coupling α_s is picking up speed, and it is of vital importance for the quantitative comparison between theory and experiment at a future electron-ion collider (EIC). More challenging is extending the small- x framework beyond leading *power* in x as $x \rightarrow 0$. Since $x \sim p_T^2/s$, this is equivalent to keeping terms which are suppressed by powers of $1/s$. These power-suppressed corrections are often referred to as “sub-eikonal corrections” in this context, with the “eikonal approximation” referring to keeping only leading powers $O(x^0) \sim O(s^0)$. Another important frontier of progress consists not of improving the underlying small- x framework, but using it to compute many-body correlations.

2.1 Small- x Theory at NLO

NLO corrections are nominally suppressed by a factor of α_s compared to the leading order. But when combined with small- x evolution equations which resum powers of $\alpha_s \ln \frac{1}{x} \sim O(1)$, a suppression by α_s is equivalent to a suppression by one logarithm: $O(\frac{1}{\ln(1/x)})$. These corrections are therefore much more numerically important at high energy / low x than fully $O(x^1)$ power-suppressed corrections. NLO corrections generally preserve the structure of the small- x framework, but can introduce new physical effects. Perhaps the most well-known of these is the running coupling, as implemented in the rcBK code [9]. The running coupling is of course just one of many NLO corrections to the small- x evolution equations, which have been fully computed to NLO now for both the linear and non-linear evolution equations. These NLO evolution equations must then be coupled to specific physical processes computed to NLO accuracy, which has been performed for a handful of observables (e.g., [10]).

Another new physical channel which opens up at NLO involves the QCD odderon [11]. Many small- x observables are expressed in terms of the dipole amplitude

$$D_{xy} = \frac{1}{N_c} \text{tr} [V_x V_y^\dagger], \quad (1)$$

where N_c is the number of quark colors and V_x is a quark Wilson line at transverse position \vec{x}_\perp . In general, one can decompose the dipole amplitude into a piece which is symmetric and antisymmetric under exchange of the quark/antiquark coordinates:

$$D_{xy} = \frac{1}{2} [D_{xy} + D_{yx}] + \frac{1}{2} [D_{xy} - D_{yx}] \equiv S_{xy} + iO_{xy}. \quad (2)$$

The symmetric part S_{xy} of the dipole amplitude is known as the Pomeron, while the antisymmetric part O_{xy} is known as the odderon. The Pomeron possesses all the quantum numbers of

the vacuum (even under C , P , and T), while the odderon changes sign under C and P . At lowest order in pQCD, the Pomeron consists of two gluons in a color-singlet configuration, while the odderon consists of *three* gluons. The odderon is, therefore, a type of $O(\alpha_s)$ NLO contribution opening up new antisymmetric scattering channels. All of this describes the low- p_T regime $p_T \lesssim Q_s(x)$ where the semihard scale $Q_s(x)$ underlies a perturbative description. At higher $p_T \gg Q_s(x)$ the odderon enters instead as a higher-twist (twist-3) correction instead.

2.2 Sub-Eikonal Corrections to Small- x Theory

The incorporation of $O(1/s)$ sub-eikonal corrections begins to fundamentally change the structure of the small- x formalism. In eikonal (Regge) kinematics $s \rightarrow \infty$, a hadronic state is infinitely Lorentz-contracted (i.e., a delta function). But for finite s , the collision changes from being an instantaneous shockwave to occurring over a finite time. In the case of heavy-ion collisions, one can no longer simply compute the “pre-hydrodynamic” phase of the collision using the classical Yang-Mills equations of small- x theory and then switch suddenly to a hydrodynamic description. Instead, one needs a dynamical picture in which hydrodynamics is initiated continuously as the colliding ions pass through each other [12].

Another consequence of a finite-duration collision is that in-medium radiation can occur. In the eikonal approximation, all partonic branchings occur in vacuum before or after the collision. But at sub-eikonal accuracy, branchings which occur *during* the propagation of a high-energy particle through the medium begin to contribute. The resulting medium-induced radiation is given the well-studied non-Abelian Landau-Pomeranchuk-Migdal (LPM) effect for the radiative energy loss of jets [13–17]. An extension of small- x theory to include $O(1/s)$ sub-eikonal corrections will thus make contact with the physics of jet energy loss.

Aside from these *kinematic* changes to particle production at small x , the interactions themselves can change as well. At eikonal accuracy, a high-energy particle propagates through a background field of gluons, with one component of the gluon field A^μ leading to a total color rotation of the traversing particle. For a particle moving along the $+z$ axis through a background field of gluons, this leads to the standard eikonal Wilson line $V_x \equiv \mathcal{P} \left[ig \int dz^+ A^-(z^+, 0^-, \vec{x}_\perp) \right]$, where $v^\pm \equiv \frac{1}{\sqrt{2}}(v^0 \pm v^3)$ are the standard light-front coordinates. This expression is a consequence of the dominance of the background field A^- in, say, Feynman gauge. Beyond eikonal accuracy, other components of the gluon field A^μ begin to enter, which change the structure of the Wilson line. The new field components (notably \vec{A}_\perp) transfer different information to the target, including the exchange of spin [18, 19].

Even more challenging is the complete breakdown of the “background field” concept beyond eikonal accuracy. Gluons dominate in the regime of small x , but at $O(1/s)$, quark-mediated interactions play an important role as well. The operators for quark exchange differ significantly from standard Wilson lines, allowing a high-energy parton to change its identity (e.g. between a quark and a gluon) by exchanging quarks with the medium [20, 21]. These complications also directly reflect the fact that baryon stopping, the transfer of valence quarks from the forward direction to mid-rapidity, arises at $O(1/s)$. As such, sub-eikonal corrections also capture the substantial qualitative changes in the theory of heavy-ion collisions from top collider energies down to the RHIC Beam Energy Scan, where all of these difficult sub-eikonal effects are large and non-negligible [22].

2.3 Multiparticle Correlations Within Small- x Theory

Many-body correlations such as the multiparticle anisotropic flow cumulants $v_n\{m\}$ in heavy-ion collisions are quite challenging to compute within the CGC framework. One approach is

to use a toy “CGC parton model” [23]. This approach resembles the “hybrid factorization” formalism for particle production at forward rapidities [24], although the CGC parton model is generally applied to mid-rapidity. Valence partons from a dilute projectile become correlated due to scattering in the same correlated color fields. While far from realistic, the CGC parton model does have the virtue of singling out the impact of color correlations alone.

In a complete CGC calculation, particle production at mid-rapidity is dominated by gluons. Full dilute/dense calculations with many particles in the final state are challenging; at present two- and three-gluon correlations exist in the literature [25, 26]. In the simplifying dilute/dilute limit, this has been extended up to four-gluon correlations [27]. Even the structure of two-gluon correlations has yielded new insights recently. While the leading two-gluon correlation function can describe features of the elliptic flow cumulant $v_2\{2\}$, it possesses a mirror symmetry such that it cannot generate triangular flow $v_3\{2\}$ [25]. This feature of the dilute/dense calculation was puzzling, since this mirror symmetry was observed to be broken numerically in “dense/dense” classical Yang-Mills simulations [28]. Recently, it was shown analytically that the first high-density correction to the projectile (“next-to-dilute / dense”) breaks the symmetry which of the dilute/dense case and does generate $v_3\{2\}$ [29]. Moreover, the mechanism by which this symmetry is broken – through a relative complex phase arising from rescattering – is analogous to the breaking of PT symmetry in the Siverson function [30].

Finally, while gluons dominate the particle production cross section at mid-rapidity, quark-antiquark pairs can be produced with only a relative penalty of α_s . These quark correlations differ from the gluon correlations in three significant ways. First, because the quarks exist in the fundamental representation of the gauge group $SU(3)$, their dipole amplitude (1) can be complex and lead to a nontrivial odderon contribution (2). Second, because the quarks are fermions, quark correlations will exhibit “Pauli blocking” due to their quantum statistics [31] in contrast to the well-known Bose enhancement of two-gluon correlations [32, 33]. Finally, because quarks carry a variety of conserved charges, correlations among (anti)quarks are responsible for the spatial distribution of conserved charge fluctuations. This feature has recently been implemented in a new Monte Carlo event generator for the initial conditions of conserved charges in ultrarelativistic heavy-ion collisions [34, 35].

3 A Common Thread: New Collinear Logarithms

Many CGC cross sections at NLO suffer from a negativity problem. This problem was first encountered in the context of “hybrid factorization,” which describes forward particle production in $p + A$ dilute/dense collisions [24]. While the LO hybrid factorization cross section is well-behaved, the NLO cross section inevitably becomes negative (and unphysical) for large transverse momentum [36]. A closely-related problem arises in the small- x evolution equations themselves: the rate of growth of the dipole amplitude (1) with energy starts out initially positive, but at NLO turns over and goes negative [37]. The source of the negativity was identified to be terms containing an additional large collinear logarithm [38]:

$$\frac{dS_{xy}}{dY} \supset -\frac{\bar{\alpha}_s^2}{4\pi} \int d^2z \frac{(x-y)_\perp^2}{(x-z)_\perp^2 (z-y)_\perp^2} (S_{xz}S_{zy} - S_{xy}) \ln \frac{(x-z)_\perp^2}{(x-y)_\perp^2} \ln \frac{(y-z)_\perp^2}{(x-y)_\perp^2}. \quad (3)$$

These numerically large logarithms spoil the stability of the perturbation series and lead to unphysical results. Clearly, to stabilize the perturbation series, these problematic logarithms need to be resummed, but the scheme for how to do so is not unique. One proposal involves completing the first two terms of the perturbation series to a resummed function which matches at NLO, akin to replacing $1 + x \approx e^x$ at NLO accuracy. An alternative scheme modifies the evolution equations to be non-local instead. All of these proposals are equivalent

at NLO and differ only by higher-order terms. The result of these schemes are observables which remain physical, but are subject to a scheme dependence on the final results [38].

Physically, the new large collinear logarithms at NLO arise from a corner of phase space associated with lifetime ordering (or angular ordering) of the gluon cascade. While the LO evolution equations do not need to explicitly enforce lifetime ordering at that order, this “kinematic constraint” cannot be ignored at NLO. This same problem also arises for longitudinal spin asymmetries at small x [20]. Because helicity dependence vanishes in the strict eikonal limit, longitudinal spin asymmetries are directly sensitive to sub-eikonal physics. In the small- x evolution of polarized observables, the lifetime ordering condition leads to a second parametrically large logarithm. While the unpolarized small- x evolution is leading-logarithmic (resumming $\alpha_s \ln \frac{1}{x} \sim 1$), polarized small- x evolution is double-logarithmic (resumming $\alpha_s \ln^2 \frac{1}{x} \sim 1$) [39]. Thus lifetime ordering, which enters only at NLO in the unpolarized (eikonal) sector, is a leading-order effect in the polarized (sub-eikonal) sector.

The important difference between these cases is that polarized evolution is governed by a resummation parameter. In the polarized case, the logarithm that results is $\ln \frac{1}{x}$, and the evolution equations must systematically resum all double logarithms of $1/x$. But in the NLO unpolarized case, the logarithms do not correspond to a large external parameter like $1/x$; they are simply numerically unstable large logarithms. As a result, the NLO unpolarized evolution suffers from scheme dependence associated with the non-unique ways of resumming the problematic logarithms. The LO polarized evolution, on the other hand, is uniquely specified by the double-logarithmic parameter $\alpha_s \ln^2 \frac{1}{x} \sim 1$.

The similarities between the NLO unpolarized evolution and the LO polarized evolution go further. In both cases, the ordering of the cascade in longitudinal momentum k^+ makes the lifetime-ordering condition unwieldy. For polarized evolution, this leads to a messy dependence on an auxiliary “neighbor dipole amplitude” even in the large- N_c limit [20]. It is interesting that, in the NLO unpolarized case, simpler equations arise from using an inverse ordering variable k^- ; perhaps this could simplify the polarized evolution equations as well. In any case, the common origins of the large double logarithms suggests possible deep connections between the NLO frontier of small- x theory and the sub-eikonal frontier. Perhaps exploring these connections further will lead to better control of both processes.

4 Conclusions

The theory of particle production at small x , long established at leading order and leading power, is now making significant progress on three active fronts: the generalization to NLO, the incorporation of “sub-eikonal” power-suppressed corrections, and multiparticle correlations. While there are a range of very distinct formalisms expressing the theory of small- x physics that all agree at LO, it remains an open question whether these formalisms will continue to agree at NLO. More generally, the program of computing corrections to the LO picture of small- x physics is an essential exercise to establish the validity and convergence of the small- x picture. If this regime of QCD can really be understood as an effective theory at high energies and high densities, then it is crucial to establish that its approximations can be systematically improved and compared to data.

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