Transseries gradient expansion of Yang-Mills plasma

Jakub Jankowski[1,⋆]

[1] *Faculty of Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland*

Abstract. We discuss a resurgence framework in which both hydrodynamical and transient degrees of freedom of $\mathcal{N} = 4$ SYM plasma are incorporated on an equal footing. In consequence, at late times, all information about initial conditions is encoded in infinitely many exponentially damped modes, controlled by transseries parameters.

1 Introduction

In the last few years the meaning of the hydrodynamic expansion attracted a lot of attention, motivated by applications of hydrodynamics to the phenomenology of Heavy Ion Collisions as well as by various theoretical developments [1]. In particular, much of insight came from the AdS/CFT direction, where it was realised that hydrodynamisation is in general a fast process [2, 3], and that the gradient expansion is generically a divergent series [4]. A tentative reason for both effects is the presence of exponentially damped modes, which are not explicitly captured by the gradient expansion. It turns out that a possible theoretical framework incorporating both sets of degrees of freedom is the theory of resurgence [5], which usually finds its place in theoretical physics in connection with the topic of perturbation theory in quantum mechanics and quantum field theory [5]. Surprisingly it has also been useful in applications to plasma dynamics, as was recently demonstrated in Ref. [6].

2 Hydrodynamization and Bjorken flow

We will start by briefly reviewing the meaning of fast hydrodynamization, which is a process by which system approaches state described by hydrodynamic equations. Physically, fast hydrodynamization is a consequence of exponential damping of initial conditions information, and it is well supported by numerical simulations. To make the problem approachable it is useful to adopt Bjorken symmetry conditions which can be implemented by adopting the coordinate system defined by $ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2$ and imposing the requirement of no dependence on the y and $x_\perp$ coordinates [7]. As a consequence, the energy momentum tensor has the following form

$$\langle T_{\mu\nu}(\tau)\rangle = \mathrm{diag}\{\epsilon(\tau), P_L(\tau), P_T(\tau), P_T(\tau)\}\ . \tag{1}$$

Due to energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, and conformal invariance, $T^\mu_\mu = 0$, longitudinal and transversal pressures take are expressed as

$$P_L = -\epsilon - \tau\dot{\epsilon}\ , \qquad P_T = \epsilon + \frac{1}{2}\tau\dot{\epsilon}\ , \tag{2}$$

⋆e-mail: Jakub.Jankowski@fuw.edu.pl

© The Authors, published by EDP Sciences. This is an open access article distributed under the terms of the Creative Commons Attribution License 4.0 (http://creativecommons.org/licenses/by/4.0/).

and the dynamics of the system is determined by a single function of proper-time $\epsilon(\tau)$, *i.e.*, the energy density at mid-rapidity.

A holographic description of boost-invariant dynamics of SYM plasma requires a suitable class of asymptotically-AdS geometries [2, 3]. Once the geometry is numerically determined one can apply the AdS/CFT dictionary to calculate $\epsilon(\tau)$. It is convenient to adopt the notion of effective temperature $T(\tau)$, which is defined by the relation

$$\epsilon(\tau) = \frac{3}{8}\pi^2 N^2 T^4(\tau) \,. \tag{3}$$

It is important to note that $T(\tau)$ is not a temperature in strict thermodynamic sense, yet due to high symmetry constraints, it is a useful physical quantity.

At late time it is possible to calculate $\epsilon(\tau)$ as an asymptotic series which can be expressed as an expansion of the effective temperature

$$T(\tau) = \frac{\Lambda}{(\Lambda\tau)^{1/3}}\left(1 - \frac{1}{6\pi}\frac{1}{(\Lambda\tau)^{2/3}} + \frac{\log 2 - 1}{36\pi^2}\frac{1}{(\Lambda\tau)^{4/3}} + \ldots\right) . \tag{4}$$

The integration constant Λ appearing here is the only trace of the initial condition, which survives in late time limit. The above expansion is closely related to the gradient expansion of hydrodynamics.

The dependence on Λ suggests that it is useful to introduce the dimensionless pressure anisotropy defined by

$$\mathcal{A}(w) = \frac{P_T(w) - P_L(w)}{P(w)} \,, \tag{5}$$

where $\epsilon(w) = 3P(w)$. Everything is then expressed as a function of the dimensionless quantity $w = \tau T(\tau)$, the proper time measured in units of the relaxation time $\tau_0 \sim 1/T$. The pressure anisotropy is a good measure of the deviation from hydrodynamics, and at late times it is independent of initial conditions. Indeed, numerical simulations show that for generic initial configuration $\mathcal{A}(w)$ follows universal time evolution determined by hydrodynamic behaviour (see Fig. 1). [1] This reflects a universal approach to equilibrium, in which initial state information is dissipated exponentially at rather early time.

3 Transseries gradient expansion

Here we will briefly review applications of resurgence theory to the late time expansion of energy density of $\mathcal{N} = 4$ SYM. For details we refer the reader to Ref. [6].

3.1 Multiparameter transseries

The series of Eq. (4) turns out to be a divergent one [4], and furthermore – as expected based on physical grounds – is supplemented with non-perturbative corrections of exponential nature. Exponentially suppressed terms form an infinite set, and are determined by quasinormal mode (QNM) frequencies of the dual black hole. The asymptotic series of Eq. (4) can be computed to very high order utilizing the AdS/CFT correspondence [4], and takes the following form

$$\varepsilon_{\text{hydro}}(\tau) \sim \frac{\Lambda}{(\Lambda\tau)^{4/3}} \sum_{n=0}^{\infty} \varepsilon_n^{(0)} (\Lambda\tau)^{-2n/3} \,, \tag{6}$$

[1] Different initial conditions are labelled by different entropies as measured by the area of the apparent horizon determined by means of Bekenstein-Hawking relatio [3].

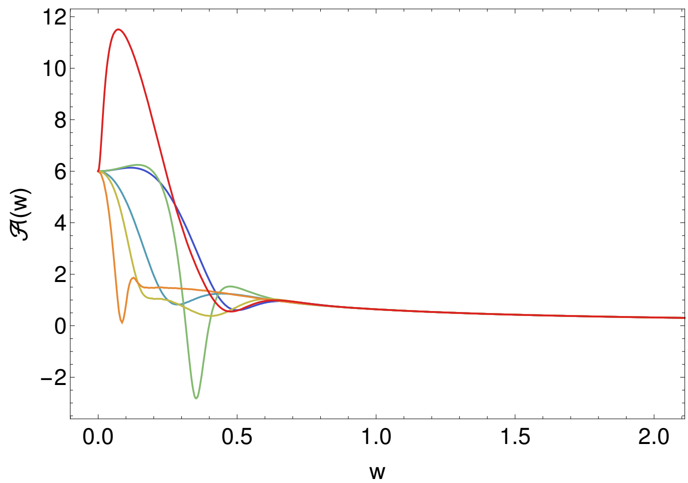

Figure 1. The pressure anisotropy $\mathcal{A}(w)$ for six different initial configurations. The initial conditions all have the same value of the pressure anisotropy, but evolve differently, which reflects the richness of the initial states. However at late times, for $w > 0.7$, all the curves behave identically and match the first order hydro prediction [2, 3].

where coefficients $\varepsilon_n^{(0)}$ diverge factorially, *i.e.*, $\varepsilon_n^{(0)} \sim \Gamma(n+\beta)A_1^{-n-\beta}$for $n \gg 1$. The full expansion of $\epsilon(\tau)$ takes form of multi-parameter transseries with infinitely many transseries parameters [6]. For simplicity let us only show one sample contribution of the least-damped, transient mode determined by quasi-normal mode (QNM) frequency $A_1 \in \mathbb{C}$ [2]

$$\epsilon(\tau) \sim \underbrace{\frac{\Lambda}{(\Lambda\tau)^{4/3}} \sum_{n=0}^{\infty} \varepsilon_n^{(0)} (\Lambda\tau)^{-2n/3}}_{\varepsilon_{\text{hydro}}(\tau)} + \frac{\Lambda\sigma_1}{(\Lambda\tau)^{4/3}} \sum_{n=0}^{\infty} \varepsilon_n^{(1)} (\Lambda\tau)^{-2n/3} e^{-A_1(\Lambda\tau)^{2/3}} + \cdots \tag{7}$$

The transseries parameter associated with the A_1 mode, belonging to an infinite set, $\sigma_1 \in \mathbb{C}$ is free and encodes information about the initial configuration. The original series $\varepsilon_n^{(0)} \sim \Gamma(n+\beta)A_1^{-n-\beta}$ for $n \gg 1$ diverge factorially, and so does the first sector $\varepsilon_n^{(1)} \sim \Gamma(n+\beta_2)A_2^{-n-\beta_2}$ for $n \gg 1$. The transseries expansion of Eq. (7) can be determined using AdS/CFT duality. The parameter $A_2 \in \mathbb{C}$, present in the $\varepsilon_n^{(1)}$ coefficients, is second to most damped QNM mode in the sense that Im $A_1 <$ Im A_2.

3.2 Borel transform

In order to make sense of factorially divergent asymptotic series one can perform a Borel transform. In the case at hand it is convenient to adopt a slightly non-standard definition. For example in the original hydrodynamic series it reads

$$\mathcal{B}\left[\varepsilon_{\text{hydro}}\right](\xi) = \xi^{\beta-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\varepsilon_n^{(0)}}{\Gamma\left(n+\beta+\frac{1}{2}\right)} \xi^n \,. \tag{8}$$

[2]Imaginary part of A_1 and A_2 is proportional to relaxation time.

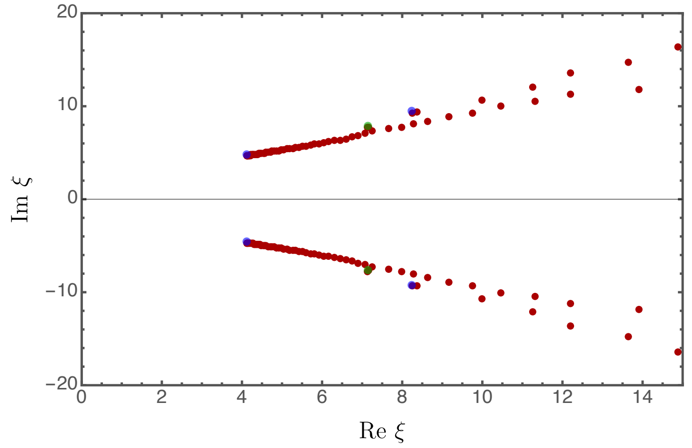

Figure 2. Poles of the Borel-Padé approximant $\mathrm{BP}_{189}\left[\epsilon_{\mathrm{hydro}}\right]$, in the complex ξ−plane. In addition in colour: $\xi = A_1, \overline{A_1}, 2A_1, 2\overline{A_1}$ (blue dots), $\xi = A_2, \overline{A_2}$ (green dots).

The key point now is that series $\mathcal{B}\left[\varepsilon_{\mathrm{hydro}}\right](\xi)$ has a finite radius of convergence determined by the exponent A_1, and can be analytically continued to a function defined in the whole complex plane, called the Borel plane. In the case at hand, due to limited number of terms given, we used diagonal Pade approximant to perform the continuation. To be more specific one approximates the sought after function by a rational function

$$\mathrm{BP}_N\left[\Phi\right](\xi) = \frac{P_N(\xi)}{Q_N(\xi)}\,, \tag{9}$$

where $P_N(\xi)$ and $Q_N(\xi)$ are polynomials of order N, fitting their coefficients to match a number of consecutive terms in the series of Eq. (8). This ensures the correct radius of convergence of the approximating function. Poles of the Borel-Pade approximant condense on cuts of the underlying function, while numerical artefacts are suppressed by large order of used polynomials. In short, poles of the approximant resemble the singularity structure of the exact Borel transform.

The singularity structure of the analytically-continued Borel transform carries information about non-perturbative degrees of freedom present in the system and is shown in Fig. 2. In particular one can identify there the second least-damped frequency $A_2 \in \mathbb{C}$, corresponding to the second QNM of the dual black hole, as well as non linear terms present in (7), though not shown explicitly.

3.3 Large order relations

One important consequence of the analytic structure mentioned above are the so-called *large order relations* [5], which relate coefficients from the $\varepsilon_n^{(0)}$ sector for $n \gg 1$ with those of $\varepsilon_n^{(1)}$ sector for $n \sim 1$. In our particular example the relation reads

$$\varepsilon_n^{(0)} \sim -\frac{S_{0\to1}}{2\pi\mathrm{i}}\frac{\Gamma(n+\beta)}{A_1^{n+\beta}}\left(\varepsilon_0^{(1)} + \frac{A_1\varepsilon_1^{(1)}}{n+\beta-1} + \frac{A_1^2\varepsilon_2^{(1)}}{(n+\beta-1)(n+\beta-2)} + \cdots\right) + \mathrm{c.c.} + \cdots \tag{10}$$

where $S_{0\to1}$ is the Stokes constant, determined numerically to be $S_{0\to1} = 0.01113\cdots - \mathrm{i}0.03050\cdots$. Each sector comes with its own independent Stokes parameter that characterizes underlying system.

To compare coefficients obtained from holographic computation, and those predicted by large-order relation (10) we make use of the following quantity

$$\Delta_{\mathbf{0}}\varepsilon_m^{(\mathbf{e}_1)} \equiv \frac{\varepsilon_m^{(1)}\,|_{\mathbf{0}-\text{predicted}} - \varepsilon_m^{(1)}\,|_{\text{numerical}}}{\varepsilon_m^{(1)}\,|_{\text{numerical}}}\,, \qquad m \geq 1\,. \tag{11}$$

Here, coefficients $\varepsilon_m^{(1)}\,|_{\text{numerical}}$ are coefficients of the expansion in the first transsector (corresponding to A_1), computed with AdS/CFT, and $\varepsilon_m^{(1)}\,|_{\mathbf{0}-\text{predicted}}$ are those obtained from $\varepsilon_m^{(0)}$ assuming relation (10) holds. In Fig. 3 we plot $\Delta_{\mathbf{0}}\varepsilon_m^{(\mathbf{e}_1)}$ for a few first values of consecutive number m. The agreement between both computations is of the order of 10^{-50}, which provides strong evidence of the resurgence properties of the gradient expansion in strongly coupled Yang-Mills plasma.

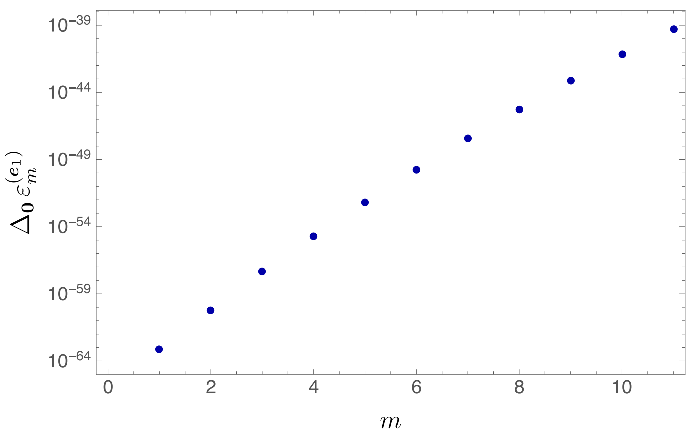

Figure 3. Differences of the values of series coefficients as computed from holography and predicted by large order relations, defined in Eq. (11) .

4 Conclusions

In this short note we have reviewed results of Ref. [6] which provide a unified description of hydrodynamic and transient degrees of freedom in strongly coupled $\mathcal{N} = 4$ SYM theory. It is also the first example demonstrating resurgence property in a first principle computation of quantum field theory. The energy density, as a function of proper time, turns out to be a multi-parameter transseries, where the original gradient expansion is supplemented by exponentially damped corrections, each appearing with an independent transseries parameter. Taking into account more and more of those modes allows one to extend ansatz (7) towards earlier times, however never reaching strictly times close to $\tau = 0$. In consequence, information about the initial configuration is present in the system at late times in the form of an infinite number of exponentially damped modes, fast dissipation of which is responsible for short hydrodynamisation time. A few possible extensions of that computation would be to relax restrictive symmetry assumptions or study the Bjorken flow in a non-conformal theory.

Acknowledgements

I would like to thank Michał Spaliński and Mauricio Martinez for useful discussions. This research was supported by the Polish National Science Centre (NCN) grant 2016/23/D/ST2/03125.

References

[1] W. Florkowski, M. P. Heller and M. Spalinski, Rept. Prog. Phys. **81**, no. 4, 046001 (2018)
[2] M. P. Heller, R. A. Janik and P. Witaszczyk, Phys. Rev. Lett. **108**, 201602 (2012)
[3] J. Jankowski, G. Plewa and M. Spaliński, JHEP **1412**, 105 (2014)
[4] M. P. Heller, R. A. Janik and P. Witaszczyk, Phys. Rev. Lett. **110**, no. 21, 211602 (2013)
[5] I. Aniceto, G. Basar and R. Schiappa, Phys. Rept. **809**, 1 (2019)
[6] I. Aniceto, B. Meiring, J. Jankowski and M. Spaliński, JHEP **1902**, 073 (2019)
[7] J. D. Bjorken, Phys. Rev. D **27**, 140 (1983)