EPJ Web of Conferences **238**, 02001 (2020) https://doi.org/10.1051/epjconf/202023802001
EOSAM 2020

Numerically solving generated Jacobian equations in freeform optical design

Lotte B. Romijn[1,*], *Martijn J.H.* Anthonissen[1,**], *Jan H.M.* ten Thije Boonkkamp[1,***], and *Wilbert L.* IJzerman[1,2,****]

[1]Eindhoven University of Technology (TU/e), Department of Mathematics and Computer Science, PO Box 513, 5600 MB Eindhoven, The Netherlands
[2]Signify Research, High Tech Campus 7, 5656 AE Eindhoven, The Netherlands

Abstract. We present an efficient numerical algorithm that can be used to solve the generalized Monge-Ampère equations for a single freeform reflector and lens surface. These equations are instances of so-called 'generated Jacobian equations' which are characterized by associated generating functions. The algorithm has a wide applicability to any optical system that can be described by a smooth generating function.

1 Introduction

We consider two optical systems: (System 1) a lens surface for a point source and far-field target intensity, and (System 2) a reflector surface for a parallel incoming beam and near-field target illuminance. For System 1 the source emits light radially outward and we consider a source emittance in spherical coordinates $f_1(\phi, \theta)$ in [lm/sr]. The target intensity distribution $g_1(\psi, \chi)$ in [lm/sr] in the far field is expressed with respect to a different set of spherical coordinates (ψ, χ) with origin the lens surface approximated as a point in space (far-field approximation). The first surface of the lens is spherical and the second surface is freeform. For System 2 we are given an emittance of the source domain $f_2(\mathtt{x})$ in [lm/m^2] and a target illuminance in the near field given by $g_2(\mathtt{y})$ in [lm/m^2], where $\mathtt{x}$ and $\mathtt{y}$ are the local Cartesian coordinates of the source and target planes, respectively. In figure 1 a schematic representation is given for System 2. Our goal is to compute the optical surfaces that define the mappings $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ transforming f_1 into g_1 and f_2 into g_2, respectively.

2 Mathematical model

In previous work on System 1 [1], we used optimal transport theory to derive a relation of the form $u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$, where u_1 specifies the location of the freeform optical surface and $c(\boldsymbol{x}, \boldsymbol{y})$ is a logarithmic cost function in optimal transport theory with $(\boldsymbol{x}, \boldsymbol{y})$ denoting the stereographic coordinates obtained from performing coordinate transformations on the source and target domains. Subsequently, we could derive an optical map $\boldsymbol{m}$ and derive the generalized Monge-Ampère equation as

$$\det(\mathrm{D}\boldsymbol{m}) = \frac{\det(\boldsymbol{P}(\boldsymbol{x}))}{\det(\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x})))} = \frac{f_1(\boldsymbol{x})\,(1 + |\boldsymbol{m}(\boldsymbol{x})|^2)^2}{g_1(\boldsymbol{m}(\boldsymbol{x}))\,(1 + |\boldsymbol{x}|^2)^2},$$

where $\mathrm{D}\boldsymbol{m}$ is the Jacobi matrix of $\boldsymbol{m}$, $\boldsymbol{P}$ is a symmetric positive-definite (SPD) matrix and $\boldsymbol{C} = \mathrm{D}_{\boldsymbol{xy}} c = (c_{x_i y_j})$ is the matrix of mixed second-order partial derivatives with respect to the stereographic coordinates of the source and target.

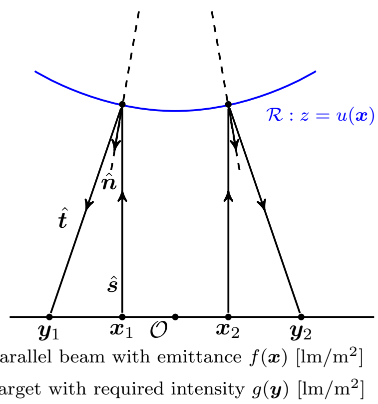

Figure 1: Schematic representation of a parallel beam of light rays, reflected by the reflector surface onto a target in the near field.

However, for some optical systems such as System 2 a relation of the form $u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$ does not exist in any coordinate system. In this paper, we generalize the cost-function formulation in optimal transport theory to a generating-function formulation by defining $u(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{y}, z(\boldsymbol{y}))$ and $z(\boldsymbol{y}) = H(\boldsymbol{x}, \boldsymbol{y}, u(\boldsymbol{x}))$, where u is the unknown location of the optical surface, G and H are smooth functions and $G(\boldsymbol{x}, \boldsymbol{y}, \cdot)$ and $H(\boldsymbol{x}, \boldsymbol{y}, \cdot)$ are each other's inverses. For both systems, we can reformulate the generalized Monge-Ampère equation as a generated Jaco-

*e-mail: l.b.romijn@tue.nl
**e-mail: m.j.h.anthonissen@tue.nl
***e-mail: j.h.m.tenthijeboonkkamp@tue.nl
****e-mail: wilbert.ijzerman@signify.com

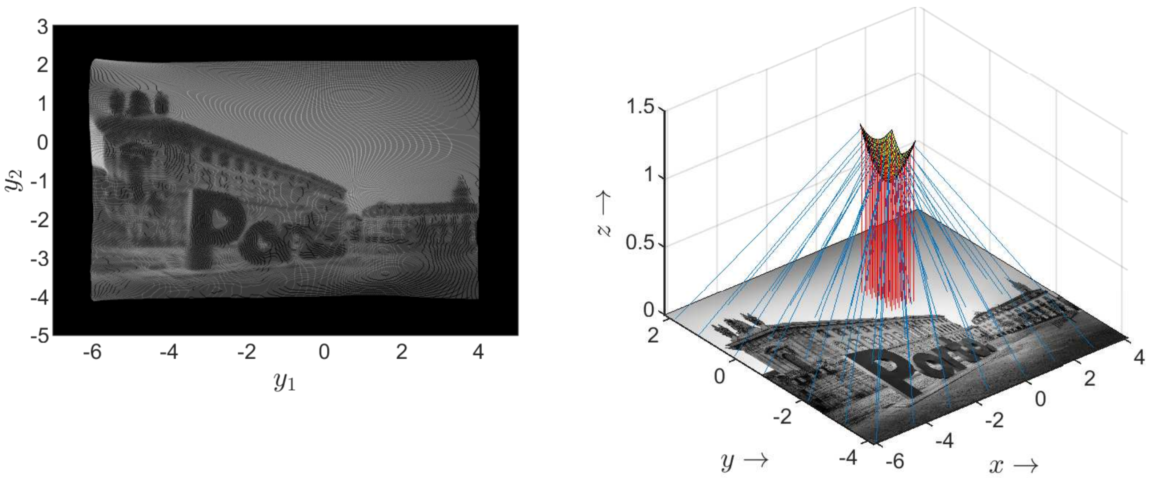

Figure 2: (Left) Ray traced image of Porto projected on a screen in the near field. (Right) Cartoon of a single freeform reflector that converts a small parallel beam (in red) into an output beam (in blue) that produces a picture of Porto on a screen in the near field. Only a small subset of the total rays traced (approximately 100.000) has been plotted.

bian equation [2]

$$\det(\mathrm{D}\boldsymbol{m}) = \frac{\det(\boldsymbol{P}(\boldsymbol{x}))}{\det(\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}), u(\boldsymbol{x})))} = \frac{F(\boldsymbol{x})}{G(\boldsymbol{m}(\boldsymbol{x}))},$$

where $\boldsymbol{P}$ is a symmetric positive-definite (SPD) matrix and we redefine $\boldsymbol{C}$ as $\boldsymbol{C} = \mathrm{D}_{\boldsymbol{xy}}H(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x}), u(\boldsymbol{x})) = (H_{x_i y_j})$. Note the added dependency of $\boldsymbol{C}$ on the surface u. For System 1, we consider the stereographic coordinates $(\boldsymbol{x}, \boldsymbol{y})$ and have $F(\boldsymbol{x}) = f_1(\boldsymbol{x})/(1 + |\boldsymbol{x}|^2)^2$ and $G(\boldsymbol{m}(\boldsymbol{x})) = g_1(\boldsymbol{m}(\boldsymbol{x}))/(1 + |\boldsymbol{m}(\boldsymbol{x})|^2)^2$. For System 2, we take Cartesian coordinates $\boldsymbol{x} = \mathtt{x}$ and $\boldsymbol{y} = \mathtt{y}$, and derive $F(\boldsymbol{x}) = f_2(\mathtt{x})$ and $G(\boldsymbol{m}(\boldsymbol{x})) = g_2(\boldsymbol{m}(\mathtt{x}))$.

For System 1, we use Hamilton's angular characteristic and derive $H(\boldsymbol{x}, \boldsymbol{y}, u(x)) = u(\boldsymbol{x})\left(n - 1 + \frac{2\,|\boldsymbol{x}-\boldsymbol{y}|^2}{(1+|\boldsymbol{x}|^2)\,(1+|\boldsymbol{y}|^2)}\right)$. Inverting this equation gives the generating function

$$G(\boldsymbol{x}, \boldsymbol{y}, z(\boldsymbol{y})) = z(\boldsymbol{y})\left(n - 1 + \frac{2\,|\boldsymbol{x} - \boldsymbol{y}|^2}{(1 + |\boldsymbol{x}|^2)\,(1 + |\boldsymbol{y}|^2)}\right)^{-1}.$$

For System 2, the function H is Hamilton's point characteristic $H(\boldsymbol{x}, \boldsymbol{y}, u(\boldsymbol{x})) = u(\boldsymbol{x}) + \sqrt{|\boldsymbol{y} - \boldsymbol{x}|^2 + u(\boldsymbol{x})^2}$ which is equal to the optical path length from a point $\boldsymbol{x}$ on the source domain to a point $\boldsymbol{y}$ on the target domain, see figure 1. Inverting this equation gives the generating function $G(\boldsymbol{x}, \boldsymbol{y}, z(\boldsymbol{y})) = u$ with $z(\boldsymbol{y}) = 1/H$ as derived in [2, 3]

$$G(\boldsymbol{x}, \boldsymbol{y}, z) = \frac{1}{2\,z} - \frac{z}{2}|\boldsymbol{x} - \boldsymbol{y}|^2.$$

3 Numerical method

Using a generalized least-squares approach [4] we iteratively update the mapping $\boldsymbol{m}$. We write the generalized Monge-Ampère equation as the matrix equation $\boldsymbol{C}\mathrm{D}\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x})$, where $\boldsymbol{P}(\boldsymbol{x})$ satisfies $\det(\boldsymbol{P}(\boldsymbol{x})) = F(\boldsymbol{x})/G(\boldsymbol{m}(\boldsymbol{x}))$, cf. [5]. An iterative procedure is used to find the mapping, which involves an efficient procedure to find the numerical solution of a constrained minimization problem for each grid point to compute $\boldsymbol{P}$ and the solution of a linear elliptic boundary value problem to compute $\boldsymbol{m}$. In order to update the matrix $\boldsymbol{C}$ during the iterative procedure, the location of the optical surface is calculated from the mapping also in a least-squares sense.

The numerical method can be used to compute reflector and lens surfaces for various examples. For instance, we can compute the reflector surface that transforms light of a parallel beam into a near-field picture of Porto, as shown in figure 2. For System 1 we will compare the results of the cost-function approach previously presented in [1] to the new generating-function approach.

By using a generating-function approach we are able to extend the applicability of the least-squares procedure originally presented in [5] to any optical system that can be described by a smooth generating function.

References

[1] L. Romijn, J. ten Thije Boonkkamp, W. IJzerman, JOSA A **36**, 1926 (2019)

[2] N.S. Trudinger, Discrete Contin. Dyn. Syst. **34**, 1663 (2012)

[3] F. Jiang, N. Trudinger, N.S. Arch. Ration. Mech. Anal **229**, 547 (2018)

[4] L.B. Romijn, J.H. ten Thije Boonkkamp, W.L. IJzerman, J. Comput. Phys. **408**, 109283 (2020)

[5] C. Prins, Ph.D. thesis, Eindhoven University of Technology, Eindhoven (2014)