# The Oscillation Numbers and the Abramov Method of Spectral Counting for Linear Hamiltonian Systems

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**Abstract.** In this paper we consider linear Hamiltonian differential systems which depend in general nonlinearly on the spectral parameter and with Dirichlet boundary conditions. For the Hamiltonian problems we do not assume any controllability and strict normality assumptions which guarantee that the classical eigenvalues of the problems are isolated. We also omit the Legendre condition for their Hamiltonians. We show that the Abramov method of spectral counting can be modified for the more general case of finite eigenvalues of the Hamiltonian problems and then the constructive ideas of the Abramov method can be used for stable calculations of the oscillation numbers and finite eigenvalues of the Hamiltonian problems.

## 1 Introduction

In this paper we consider the spectral and oscillation theory for the linear Hamiltonian systems

$$y' = JH(t,\lambda)y,\ t\in[a,b],\ y=\begin{pmatrix}x(t,\lambda)\\ u(t,\lambda)\end{pmatrix},\ \lambda\in R \quad (1)$$

with the Dirichlet boundary conditions

$$x(a,\lambda)=x(b,\lambda), \quad (2)$$

where

$$H(t,\lambda)=H^T(t,\lambda),\ J=\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix},\ \frac{d}{d\lambda}H(t,\lambda)\ge 0. \quad (3)$$

Here $A\ge 0$ means that the symmetric matrix $A$ is nonnegative definite, and $I,0$ denote the identity and zero matrices of appropriate dimensions. The Hamiltonian $H(t,\lambda)\in R^{2n\times 2n}$ and it's derivative in (3) are piece-wise continuous matrix functions with respect to $t\in[a,b]$ and $\lambda\in R$.

The linear Hamiltonian system (1) covers as a special case scalar and matrix second order Sturm–Liouville differential equations, as well as Sturm–Liouville differential equations of arbitrary even order, other self-conjugate differential equations and systems [3-4].

Based on the consideration in [5] we do not suppose any conditions connected with strict monotonicity of some matrix functions associated with (1). Such conditions will guarantee that the classical eigenvalues of (1), (2) are isolated (see [1-3]). For example, in [3] problem (1), (2) is considered under the strict normality assumptions with respect to $t\in[a,b]$ and $\lambda\in R$. In particular, the strict normality supposes that solutions of (1) are not "degenerate" with respect to change in $\lambda\in R$, i.e., if $y(t,\lambda)$ solves system (1) for different $\lambda_1,\lambda_2\in R$ on some non-degenerate interval of $[a,b]$, then necessary $y(t,\lambda)=0,\ t\in[a,b]$.

For a special case of the second order Sturm–Liouville differential equation

$$(r(t,\lambda)x')'+q(t,\lambda)x=0,\ t\in[a,b]$$

condition (3) means that $r(t,\lambda)\ne 0$ is nonincreasing while $q(t,\lambda)$ is nondecreasing in λ, but strict monotonicity of $q(t,\lambda)$ is not required in this paper. This corresponds to removing the strict normality assumption compared to [3, Assumption (8.3.7), p. 245].

The spectral and oscillation theory for the Hamiltonian systems without the strict normality and the controllability assumptions are developed in [5-7] (see also the references given therein). In [5] the notion of a finite eigenvalue of (1), (2) was introduced. Let $Y(t,\lambda)=(X^T(t,\lambda),U^T(t,\lambda))^T$ be $2n\times n$ matrix solutions of (1) with the conditions

$$rank\,Y(t,\lambda)=n,\ Y^T(t,\lambda)JY(t,\lambda)=0 \quad (4)$$

(the so-called conjoined bases). Consider a conjoined basis $Y(t,\lambda)$ with the initial condition $Y(a,\lambda)=(0\ I)^T$ (the principal solution of (1) at $t=a$). Then, under assumption (3) $rank\,X(t,\lambda)$ is piecewise constant in $\lambda\in R$ (see [3]) and $\lambda_0\in R$ is called a left (right) finite eigenvalue of (1),(2) with the multiplicity $\theta^-(\lambda_0)$ (resp. $\theta^+(\lambda_0)$) provided

$$\theta^{\pm}(\lambda_0)=rank\,X(b,\lambda_0^{\pm})-rank\,X(b,\lambda_0)\ge 1 \quad (5)$$

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Here $rank\, X(b,\lambda_0^{\pm})$ denote the left-hand (the right-hand) limits of $rank\, X(b,\lambda)$ at $\lambda_0$. Under the strict normality assumption the matrix $X(t,\lambda)$ is invertible except at isolated values of $\lambda \in R$ and then left (right) finite eigenvalues reduce to the classical ones which are determined by the condition $\det X(b,\lambda_0)=0$ with the multiplicities

$$\theta^-(\lambda_0)=\theta^+(\lambda_0)=n-rank\, X(b,\lambda_0)\,. \tag{6}$$

In general the multiplicities $\theta^-(\lambda_0)$ and $\theta^+(\lambda_0)$ are connected by the equality

$$\theta^+(\lambda_0)-\theta^-(\lambda_0)=rank\, X(b,\lambda_0^+)-rank\, X(b,\lambda_0^-)\,.$$

The global oscillation theorem (see Theorem 3.5 in [5]) relates the number of left finite eigenvalues of (1),(2) in the interval $(-\infty,\beta]$ with the number of the so-called proper focal points of the principal solution of system (1) in $(a,b]$ evaluated for $\lambda=\beta$. This result is derived under the additional monotonicity assumption

$$B(t,\lambda)=(0\,I)H(t,\lambda)(0\,I)^T\geq 0 \tag{7}$$

for the Hamiltonian $H(t,\lambda)$ (the Legendre condition). In [6-7], following the main ideas in [1-2] we generalized Theorem 3.5 in [3] introducing the so-called oscillation numbers which calculate the number of left finite eigenvalues in the interval $(\alpha,\beta], \alpha<\beta$ without assumption (7). In the recent paper (J. Elyseeva, *Relative oscillation theory for linear Hamiltonian systems with nonlinear dependence on the spectral parameter*, submitted) we introduced the dual oscillation numbers which make possible to evaluate right finite eigenvalues of (1),(2) in $[\alpha,\beta), \alpha<\beta$.

The main results of this paper (see Theorems 2,4) relate the spectral count introduced by A. Abramov in [1] with the oscillation numbers in [6-7]. The Abramov method of spectral counting [1-2] is developed for the case when classical eigenvalues are isolated and their multiplicities are defined by (6). In this paper we show how the approach in [1] can be modified for the more general case of finite eigenvalues with the multiplicities (5). The practical value of these results is in possible implementations of the outstanding ideas of [1-2] for stable calculations of the oscillation numbers and finite eigenvalues of (1), (2).

## 2 Oscillation numbers and the Abramov spectral count

In this section we recall the main notions used in the paper such that the comparative index (see [8] and Chapter 3 in [9]), the oscillation numbers [6-7], as well as the Abramov method of spectral counting in [1].

### 2.1. The comparative index

According to the definition of the comparative index in [8-9] (see also the references given therein) we consider $2n\times n$ matrices $Y=(X^TU^T)^T, \hat{Y}=(\hat{X}^T\hat{U}^T)^T$ with condition (4) using the notation $M=(I-XX^\dagger)\hat{X}$, $T=I-M^\dagger M$, where $A^\dagger$ denotes Moore-Penrose pseudoinverse of $A$. We also define the symmetric matrix

$$P=T(\hat{X}^T(\hat{Q}-Q)\hat{X})T\,, \tag{8}$$

where the symmetric matrices $Q,\hat{Q}$ solve the matrix equations

$$X^TQX=X^TU,\ \hat{X}^T\hat{Q}\hat{X}=\hat{X}^T\hat{U}. \tag{9}$$

The comparative index is defined by

$$\mu(Y,\hat{Y})=\mu_1(Y,\hat{Y})+\mu_2(Y,\hat{Y}),$$

where

$$\mu_1(Y,\hat{Y})=rank\, M,\ \mu_2(Y,\hat{Y})=ind\, P\,.$$

Here $ind\, P$ denotes the number of negative eigenvalues of the symmetric matrix $P$. For the case $\det X\neq 0$ we have $\mu_1(Y,\hat{Y})=0$ and $Q=UX^{-1}$. If, additionally, $\det\hat{X}\neq 0$ we have

$$\mu(Y,\hat{Y})=ind(\hat{Q}-Q)\,.$$

We also define the dual comparative index

$$\mu^*(Y,\hat{Y})=\mu_1(Y,\hat{Y})+\mu_2^*(Y,\hat{Y}),$$

where

$\mu_2^*(Y,\hat{Y})=ind(-P)$. For $\mu(Y,\hat{Y})$ and $\mu^*(Y,\hat{Y})$ we have the following connections

$$\mu^*(Y,\hat{Y})-\mu(Y,\hat{Y})=\mu_2^*(Y,\hat{Y})-\mu_2(Y,\hat{Y})=i(P)\,, \tag{10}$$

where $i(P)=ind(-P)-ind(P)$ denotes the inertia of $P=P^T$. We also have (combining Properties 5,6 of the comparative index in [8])

$$\mu^*(Y,\hat{Y})+\mu(Y,\hat{Y})=rank\,\hat{X}-rank\, X \\ +rank\ w(Y,\hat{Y}), \tag{11}$$

where we use the Wronskian according to

$$w(Y,\hat{Y})=Y^TJ\hat{Y}.$$

### 2.2. The oscillation numbers

Consider the definition of the oscillation numbers for conjoined bases $Y(t,\lambda)$ of (1) according to [6-7], where $\lambda \in R$ is a fixed number. Introduce a partition of the interval $[a,b]$:

$$\Pi : a = t_0 < t_1 < \ldots < t_{p-1} < t_p = b\,, \qquad (12)$$

that for any $[t_i, t_{i+1}] \subset [a,b], i = 0,1,\ldots, p-1$ there exists a constant symplectic matrix $R_i(\lambda) = \begin{pmatrix} K_i(\lambda) & M_i(\lambda) \\ L_i(\lambda) & S_i(\lambda) \end{pmatrix}$, $R_i^T(\lambda) J R_i(\lambda) = J$ with the condition

$$\det \tilde{X}_i(t,\lambda) \neq 0, t \in [t_i, t_{i+1}], i = 0,1,\ldots, p-1, \qquad (13)$$

where

$$\tilde{Y}_i(t,\lambda) = R_i^{-1}(\lambda) Y(t,\lambda) = (\tilde{X}_i^T(t,\lambda)\, \tilde{U}_i^T(t,\lambda)\,)^T.$$

Then we define the oscillation number

$$N(Y(\lambda),[a,b]) = \sum_{i=0}^{p-1} \mu(Y(t,\lambda), R_i(\lambda)(0\, I)^T)\Big|_{t_i}^{t_{i+1}}$$
$$= -\sum_{i=0}^{p-1} \mu(\tilde{Y}(t,\lambda), R_i^{-1}(\lambda)(0\, I)^T)\Big|_{t_i}^{t_{i+1}}, \qquad (14)$$

where we use the notation for the substitution $A(t)\Big|_x^y = A(y) - A(x)$. According to Section 2.1 and (13) the comparative index $\mu(\tilde{Y}(t,\lambda), R_i^{-1}(\lambda)(0\, I)^T)$ in (14) takes the form

$$\begin{aligned} &\mu(\tilde{Y}(t,\lambda), R_i^{-1}(\lambda)(0\, I)^T) = ind\, P(t,\lambda), \\ &P(t,\lambda) = M_i(\lambda)(Q_i(\lambda) - \tilde{Q}_i(t,\lambda)) M_i^T(\lambda), \end{aligned} \qquad (15)$$

where

$$\begin{aligned} &M_i(\lambda) Q_i(\lambda) M_i^T(\lambda) = -M_i(\lambda) K_i^T(\lambda), \\ &\tilde{Q}_i(t,\lambda) = \tilde{U}_i(t,\lambda) \tilde{X}_i^{-1}(t,\lambda). \end{aligned}$$

According to the main results in [6-7] oscillation numbers (14) are invariant with respect to the choice of partitions and transformation matrices $R_i(\lambda)$.

Moreover, under the Legendre condition (7) the oscillation numbers are nonnegative and coincide with the total number of left proper focal points of $Y(t,\lambda)$ in $(a,b]$

$$N(Y(\lambda),[a,b]) = \sum_{t_0 \in (a,b]} m^-(t_0,\lambda) \geq 0,$$
$$m^-(t_0,\lambda) = rank\, X(t_0^-,\lambda) - rank\, X(t_0,\lambda).$$

Under the additional assumptions that (1) is controllable for $t \in [a,b]$ we have by analogy with (6) that the points $t_0 \in [a,b]$ such that $\det X(t_0,\lambda) = 0$ are isolated and $m^-(t_0,\lambda) = n - rank\, X(t_0,\lambda)$.

Assume (3), then for the principal solution $Y(t,\lambda)$ of (1) we have (see Theorem 3.1 in [6])

$$\begin{aligned} &N(Y(\lambda_0^+)\,,[a,b]) = N((Y(\lambda_0)\,,[a,b]), \\ &N((Y(\lambda_0)\,,[a,b]) - N(Y(\lambda_0^-)\,,[a,b]) = \theta^-(\lambda_0), \end{aligned} \qquad (16)$$

where the multiplicity of a left eigenvalue of (1),(2) $\theta^-(\lambda_0)$ is defined by (5). Then for arbitrary

$$\alpha < \beta, \alpha, \beta \in R$$

$$\begin{aligned} &N(Y(\beta),[a,b]) - N(Y(\alpha),[a,b]) \\ &= \sum_{\lambda_0 \in (\alpha,\beta]} \theta^-(\lambda_0) \end{aligned} \qquad (17)$$

(see Theorem 3.2 in [6]). Remark that the sum in the right-hand side of (17) is finite because $rank\, X(t,\lambda)$ is piecewise constant in $\lambda \in R$ (see [5]).

### 2.3 The Abramov spectral count

Here we recall the construction of the spectral count in [1] for problem (1),(2) when the left boundary condition $x(a,\lambda) = 0$ is transferred to the right point $t = b$. The general case described in [1] can be considered analogously.

For a conjoined basis $Y(t,\lambda)$ of (1) with a fixed $\lambda$ consider a partition

$$\Pi : a = s_0 < s_1 < \ldots < s_{l-1} < t_l = b$$

and the symplectic orthogonal transformation matrices

$$R_{\beta_k} = \begin{pmatrix} \cos(\beta_k) I & -\sin(\beta_k) I \\ \sin(\beta_k) I & \cos(\beta_k) I \end{pmatrix}, \beta_k(\lambda) \neq \frac{\pi m}{2}, m \in Z$$

such that

$$\det \breve{U}_k(t,\lambda) \neq 0, t \in [s_k, s_{k+1}], k = 0,1,\ldots, l-1,$$

where

$$\breve{Y}_k(t,\lambda) = R_{\beta_k}^{-1}(\lambda) Y(t,\lambda) = (\breve{X}_k^T(t,\lambda)\, \breve{U}_k^T(t,\lambda)\,)^T.$$

Then there exist the symmetric matrices

$$\breve{Q}_k(t,\lambda) = \breve{X}_k(t,\lambda) \breve{U}_k^{-1}(t,\lambda), t \in [s_k, s_{k+1}], \qquad (18)$$

and

$$\sigma_k(t,\lambda) = \cos\beta_k (\cos\beta_k \breve{Q}_k(t,\lambda) - \sin\beta_k I), t \in [s_k, s_{k+1}], \qquad (19)$$

for all $k = 0,1,\ldots, l-1$.

Let $Y(t,\lambda)$ be the principal solution of (1). Consider problem (1), (2) under assumptions which guarantee that all classical eigenvalues which are zeros of $\det X(b,\lambda)$

are isolated and their multiplicities in (5) are defined by (6). For the case $\det X(b,\lambda) \neq 0$ define the number

$$N_A(Y(\lambda),[a,b]) = \sum_{k=0}^{l-1} i(\sigma_k(t,\lambda))\Big|_{s_k}^{s_{k+1}}, \qquad (20)$$

where $i(\sigma_k(t,\lambda_0))$ is the inertia of $\sigma_k(t,\lambda_0)$ given by (19).

Then, by Theorem 5 in [1]

$$N_A(Y(\lambda_0^+),[a,b]) - N_A(Y(\lambda_0^-),[a,b]) = 2\theta(\lambda_0)$$

and by Theorem 6 in [1] for any $\alpha < \beta, \alpha, \beta \in R$ such that $\det X(b,\alpha) \neq 0, \det X(b,\beta) \neq 0$

$$N(Y(\beta),[a,b]) - N(Y(\alpha),[a,b]) = 2 \sum_{\lambda_0 \in (\alpha,\beta)} \theta(\lambda_0).$$

Remark that the symmetric matrices $\tilde{Q}_i(t,\lambda)$ and $\breve{Q}_k(t,\lambda)$ in (15) and (18) obey transformed matrix Riccati equations for all $t \in [t_i, t_{i+1}]$ and $t \in [s_k, s_{k+1}]$ with coefficients associated with the transformed Hamiltonians

$$R_i^T(\lambda)H(t,\lambda)R_i(\lambda) \quad \text{and} \quad R_{\beta_k}^T(\lambda)H(t,\lambda)R_{\beta_k}(\lambda),$$

respectively. Moreover, the angles $\beta_k(\lambda) \neq \frac{\pi m}{2}, m \in Z$ can be chosen in such a way that for $\breve{Q}_k(t,\lambda)$ defined by (18) we have

$$\left\|\breve{Q}_k(s_k,\lambda)\right\| \leq C(n),$$

where $C(n)$ depends only on the dimension of the problem. A similar estimate can be derived for the matrices $\tilde{Q}_i(t,\lambda)$ in (15) for the special case of symplectic orthogonal transformation matrices in form $R_{\beta_k}$.

## 3 Main results

In this section we prove main connections between the oscillation numbers (14) and the Abramov spectral function (20).

**Lemma 1.** For arbitrary conjoined basis $Y(t,\lambda)$ of (1) we have the following representation for oscillation number (14)

$$\begin{aligned} &N(Y(\lambda),[a,b]) \\ &= -\frac{1}{2}\left(\sum_{i=0}^{p-1} i(P_i(t,\lambda))\Big|_{t_i}^{t_{i+1}} + rank\, X(t,\lambda)\Big|_a^b\right) \qquad (21) \\ &= \frac{1}{2}\left(\sum_{i=0}^{p-1} i(\tilde{P}_i(t,\lambda))\Big|_{t_i}^{t_{i+1}} - rank\, X(t,\lambda)\Big|_a^b\right), \end{aligned}$$

where $P_i(t,\lambda)$ and $\tilde{P}_i(t,\lambda)$ are the symmetric matrices defined by (8) for the comparative indices $\mu(Y(t,\lambda), R_i(\lambda)(0I)^T)$ and $\mu(\tilde{Y}(t,\lambda), R_i^{-1}(\lambda)(0I)^T)$ in (14), respectively.

**Proof.** The proof is based on the identity

$$\mu(Y,\hat{Y}) = \frac{1}{2}(rank\,\hat{X} - rank\,X + rank\; w(Y,\hat{Y}) - i(P)), \qquad (22)$$

which follows from (10), (11). Indeed, by the definition of $\mu(Y,\hat{Y})$ and $\mu^*(Y,\hat{Y})$ we have in the left-hand side of (11)

$$\mu(Y,\hat{Y}) + \mu^*(Y,\hat{Y}) = 2\mu_1(Y,\hat{Y}) + 2\mu_2(Y,\hat{Y}) + \mu_2^*(Y,\hat{Y}) - \mu_2(Y,\hat{Y}).$$

Then, by (10) it follows (22).

Applying (22) for the case $Y := Y(t,\lambda), \hat{Y} := R_i(\lambda)(0I)^T$ (see the first representation of the oscillation number in (14)) we have

$$\mu(Y(t,\lambda), R_i(0\,I)^T) = \frac{1}{2}(rank\, M_i - rank\, X(t,\lambda) + rank\;\tilde{X}_i(t,\lambda) - i(P_i(t,\lambda))),$$

then by (13)

$$\mu(Y(t,\lambda), R_i(0\,I)^T)\Big|_{t_i}^{t_{i+1}} = -\frac{1}{2}(rank\, X(t,\lambda) + i(P_i(t,\lambda)))\Big|_{t_i}^{t_{i+1}}.$$

Substituting the last representation into (14) we derive the first representation in (21). The second one follows from (22) for the case $Y := \tilde{Y}_i(t,\lambda), \hat{Y} := R_i^{-1}(\lambda)(0I)^T$. For this case we have instead of (22)

$$\mu(\tilde{Y}_i(t,\lambda), R_i^{-1}(0\,I)^T) = \frac{1}{2}(rank\,(-M_i^T) - rank\,\tilde{X}_i(t,\lambda) + rank\;(-X^T(t,\lambda)) - i(\tilde{P}_i(t,\lambda))),$$

then, by (13)

$$\mu(\tilde{Y}_i(t,\lambda), R_i^{-1}(0\,I)^T)\Big|_{t_i}^{t_{i+1}} = \frac{1}{2}(rank\;(X^T(t,\lambda)) - i(\tilde{P}_i(t,\lambda)))\Big|_{t_i}^{t_{i+1}}.$$

Sustituting the last representation into (14) we derive the second representation in (21). The proof is completed.

From Lemma 1 we derive the main result of the paper.

**Theorem 2.** Consider problem (1),(2) under condition (3) coupled with the assumption that all zeros of $\det X(b,\lambda)$ are isolated for $\lambda \in R$, where $X(b,\lambda)$ is the upper block of the principal solution $Y(t,\lambda)$ of (1) at $t = a$ . Then, for any $\tilde{\lambda} \in R$ such that $\det X(b,\tilde{\lambda}) \neq 0$ we have

$$N_A(Y(\tilde{\lambda}),[a,b]) = 2N(Y(\tilde{\lambda}),[a,b]) + n, \qquad (23)$$

where $N_A(Y(\lambda),[a,b])$ is the Abramov spectral count given by (20) and $N(Y(\lambda),[a,b])$ is the oscillation number (14).

**Proof.** According to the definition of $N_A(Y(\lambda),[a,b])$ in Section 2.3, we have the following representation

$$\breve{Y}_k(t,\lambda) = \begin{pmatrix} \breve{Q}_k(t,\lambda) \\ I \end{pmatrix} \breve{U}_k(t,\lambda) = R^{-1}_{\beta_k}(\lambda) Y(t,\lambda),$$
$$\det \breve{U}^T_k(t,\lambda) \neq 0, \quad t \in [s_k, s_{k+1}].$$

Then the matrix

$$R_{\beta_k}(\lambda) J^T = \begin{pmatrix} \sin(\beta_k) I & \cos(\beta_k) I \\ -\cos(\beta_k) I & \sin(\beta_k) I \end{pmatrix},$$
$$\beta_k(\lambda) \neq \frac{\pi m}{2}, m \in Z$$

can be used in the definition of the oscillation numbers (14) for the same partition as for $N_A(Y(\lambda),[a,b])$ . Rewriting (14) for the partition

$$\Pi : a = s_0 < s_1 < \ldots < s_{l-1} < t_l = b$$

we see that the comparative index in the second representation takes the form

$$\mu(J\breve{Y}_k(t,\lambda), JR^{-1}_{\beta_k}(0\, I)^T) = ind\, \sigma_k(t,\lambda) ,$$

then the second formula in (21) can be rewritten as follows

$$N(Y(\lambda),[a,b]) = \frac{1}{2}\left( \sum_{k=0}^{l-1} i(\sigma_k(t,\lambda))\Big|_{t_i}^{t_{i+1}} - rank\, X(t,\lambda)\Big|_a^b \right).$$

For the special case when $Y(t,\lambda)$ is the principal solution of $(1): X(a,\lambda) = 0$ and for $\lambda := \tilde{\lambda}$ with $\det X(b,\tilde{\lambda}) \neq 0$ the last formula tranfers into

$$N(Y(\tilde{\lambda}),[a,b]) = \frac{1}{2}\left( \sum_{k=0}^{l-1} i(\sigma_k(t,\tilde{\lambda}))\Big|_{t_i}^{t_{i+1}} - n \right)$$
$$= \frac{1}{2}\left( N_A(Y(\tilde{\lambda}),[a,b]) - n \right).$$

The proof of (23) is completed.

### 3.1 Modifications of the oscillation numbers

In this section we present different forms of representation of the oscillation numbers (14). The theoretical ground of these representations is the following separation theorem for the oscillation numbers (see Theorem 4.1 in [5]).

**Theorem 2.** Let $Y(t,\lambda)$ and $\hat{Y}(t,\lambda)$ be conjoined bases of (1), then for arbitrary $[\tau_0, \tau_1] \subset [a,b]$

$$N(Y(\lambda),[\tau_0,\tau_1]) - N(\hat{Y}(\lambda),[\tau_0,\tau_1])$$
$$= \mu(Y(\tau_1,\lambda),\hat{Y}(\tau_1,\lambda)) - \mu(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda)). \qquad (24)$$

Remark that in (24) $Y(t,\lambda)$ and $\hat{Y}(t,\lambda)$ are considered for the same fixed value of $\lambda$ . We present the following modification of the oscillation number which can be used as a spectral count for (1), (2).

**Theorem 3.** Let $Y(t,\lambda)$ and $\hat{Y}(t,\lambda)$ be the principal solutions of (1) at $t = a$ and $t = b$, i.e., $Y(a,\lambda) = \hat{Y}(b,\lambda) = (0\; I)^T$ . Then, under assumption (3) the oscillation number $N(Y(\lambda),[a,b])$ in formulas (16), (17) can be replaced according to

$$N(Y(\lambda),[a,b]) = N(Y(\lambda),[a,\tau_0]) + N(\hat{Y}(\lambda),[\tau_0,b])$$
$$-\mu(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda)),\ \tau_0 \in [a,b]. \qquad (25)$$

**Proof.** For the case $\tau_1 := b$ and $\hat{Y}(b,\lambda) = (0\; I)^T$ (in this case $\hat{Y}(t,\lambda)$ is called the principal solution of (1) at $t = b$ ) formula (24) takes the form

$$N(Y(\lambda),[\tau_0,b]) = N(\hat{Y}(\lambda),[\tau_0,b]) - \mu(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda)).$$

Finally, using the additivity property of the oscillation numbers [6] for $\tau_0 \in [a,b]$

$$N(Y(\lambda),[a,b]) = N(Y(\lambda),[a,\tau_0]) + N(Y(\lambda),[\tau_0,b])$$

we complete the proof of (25).

Based on (25) one can present a complete analogue of the results in [1] related to the transferring of the left and right boundary conditions of (1),(2) to arbitrary point $\tau_0 \in [a,b]$. Introduce partitions of the intervals $[a,\tau_0]$ and $[\tau_0,b]$ according to (12), (13) for $Y(t,\lambda)$ and $\hat{Y}(t,\lambda)$, respectively. Then introduce unified numbering of the points of the partitions and transformation matrices $R_i$ in such a way, that condition (13) holds for $Y(t,\lambda)$ when $i = 0,1,\ldots,r-1$ (so we have $\tau_0 := t_r$ ) and similarly, (13) holds for $\hat{Y}(t,\lambda)$ when $i = r, r+1,\ldots, p-1$. Under this agreement we prove the following result.

**Theorem 4.** Under the assumptions of Theorem 3 formula (25) can we rewritten in form

$$N(Y(\lambda),[a,b]) = \frac{1}{2}\left( \sum_{i=0}^{r-1} i(\tilde{P}_i(t,\lambda))\Big|_{t_i}^{t_{i+1}} \right.$$
$$\left. - \sum_{i=r}^{p-1} i(\hat{P}_i(t,\lambda))\Big|_{t_{i+1}}^{t_i} + i(P(\tau_0,\lambda)) - rank\, X(b,\lambda) \right), \qquad (26)$$

where the symmetric matrices $\tilde{P}_i(t,\lambda)$, $\hat{P}_i(t,\lambda)$, and $P(\tau_0,\lambda)$ are defined by (8) for the comparative indices

$$\mu(R_i^{-1}(\lambda)Y(t,\lambda),R_i^{-1}(\lambda)(0I)^T),\mu(R_i^{-1}(\lambda)\hat{Y}(t,\lambda),R_i^{-1}(\lambda)(0I)^T)$$

in (14) and for $\mu(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda))$ in (25).

**Proof.** According to the second representation in (21) we have

$$N(Y(\lambda),[a,\tau_0])=\frac{1}{2}\left(\sum_{i=0}^{r-1} i(\tilde{P}_i(t,\lambda))\Big|_{t_i}^{t_{i+1}} - rank\, X(\tau_0,\lambda)\right)$$

$$N(\hat{Y}(\lambda),[\tau_0,b])=\frac{1}{2}\left(-\sum_{i=r}^{p-1} i(\hat{P}_i(t,\lambda))\Big|_{t_{i+1}}^{t_i} + rank\, \hat{X}(\tau_0,\lambda)\right)$$

and by (22)

$$\mu(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda))=\frac{1}{2}(rank\,\hat{X}(\tau_0,\lambda)-rank\,X(\tau_0,\lambda)$$
$$+rank\;\; w(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda))-i(P(\tau_0,\lambda))).$$

Remark that the Wronskian $w(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda))$ is constant

$$w(Y(\tau_0,\lambda),\hat{Y}(\tau_0,\lambda))=w(Y(b,\lambda),\hat{Y}(b,\lambda))=X^T(b,\lambda).$$

Substituting the derived above representations into (25) and cancelling the same terms we prove (26). The proof of Theorem 4 is completed.

**Remark.** Observe that the matrix $P(\tau_0,\lambda)$ in (26) corresponds to the symmetric matrix $\omega$ defined by (3.3), (3.4) in [1] for $\tilde{\lambda}$ such that $\det X(b,\tilde{\lambda})\neq 0$. One can proof using the properties of the comparative index (see Properties 1,3,4 p. 448 in [8]) that $P(\tau_0,\tilde{\lambda})=\sigma_{r-1}(\tau_0,\tilde{\lambda})(\sigma_{r-1}(\tau_0,\tilde{\lambda})-\hat{\sigma}_r(\tau_0,\tilde{\lambda}))^{-1}\hat{\sigma}_r(\tau_0,\tilde{\lambda})$, where $\sigma_{r-1}(\tau_0,\tilde{\lambda})$ is defined by (19) for $Y(t,\lambda)$ and $\hat{\sigma}_r(\tau_0,\tilde{\lambda})$ is defined similarly for $\hat{Y}(t,\lambda)$. Moreover, by analogy with [1] in the proof we assume that the angles in the definitions of these matrices obey the condition $\beta_{r-1}=\beta_r$.

## 4 Conclusions

In this paper we show that the results in [1] can be extended in several directions. First of all, in this paper we consider a generalization of classical eigenvalues to the case of left finite eigenvalues with the multiplicities (5). Comparing with [1], in the main construction of the oscillation numbers (14) we do not use terms which are not defined at the eigenvalues of the Hamiltonian problems, moreover, oscillation numbers are defined for arbitrary constant symplectic transformations with condition (13).

The results in [1] concern general separated boundary conditions and boundary conditions with joint end points which include the Dirichlet boundary conditions considered in this paper as a special case. However, the results in this paper can be extended to such separated (and even jointly varying) endpoints by a standard method, which is based on adding two isolated points to the interval $[a,b]$, see [3-4].

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