EPJ Web of Conferences **258**, 01004 (2022) https://doi.org/10.1051/epjconf/202225801004
A Virtual Tribute to Quark Confinement and the Hadron Spectrum (vConf21)

Data-driven approaches to the evaluation of hadronic contributions to the $(g-2)_\mu$

Gilberto Colangelo[1,*]

[1]Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

Abstract. In this talk I reviewed the data-driven theoretical calculation of the hadronic contributions to the anomalous magnetic moment of the muon in the Standard Model mainly as it has been presented in the White Paper, but also including the most recent developments. All this is presented in the light of the new measurement of $(g-2)_\mu$ recently released by the Fermilab experiment, which led to an increase of the discrepancy with the Standard Model from 3.7 to 4.2σ.

1 Introduction

In April this year the Fermilab Muon $g-2$ Collaboration has announced their long-awaited first result of its series of measurements of the anomalous magnetic moment of the muon [1]. The comparison between the measurement and the Standard Model (SM) prediction, as it has been presented in the White Paper (WP) [2] showed a discrepancy at the level of 3.7σ with respect to the Brookhaven measurement [3]. The new experimental world average after the Fermilab result differs from the SM by 4.2σ. Table 1 summarizes the various contributions to the SM value for a_μ and shows that, as is well known, the two main sources of uncertainty are both hadronic and are the leading-order contribution, namely the hadronic vacuum polarization (HVP), and the new structure at next-to-leading order, namely hadronic light-by-light (HLbL).

Two further important news happened at the same time as the Fermilab announcement, both coming from the lattice: the BMW calculation of the HVP contribution, the first to reach sub-percent uncertainty, has been published [4] and a second lattice evaluation of the HLbL contribution has appeared on the arXiv [5]. Concerning the former, while the article has been on the arXiv since more than a year, the published version contained a slightly revised result which sits almost exactly in the middle between the experimental and the WP number and has a comparable uncertainty. This is an unsatisfactory situation which needs to be clarified. The second lattice result, on the other hand, agrees very well with both the first lattice calculation of the HLbL contribution as well as the data driven one, whose average is taken as the SM value for HLbL in the WP. Such a confirmation is of course very welcome and makes the perspective of further reductions in the final uncertainty for this contribution even more concrete.

*e-mail: gilberto@itp.unibe.ch

Table 1. Summary of the different contributions to a_μ in the Standard Model [2] and comparison to the present experimental world average.

Contribution	Value $\times 10^{11}$	References
Experiment (E821)	116 592 089(63)	Ref. [3]
Experiment (FNAL)	116 592 040(54)	Ref. [1]
Experiment (World-Average)	116 592 061(41)	
HVP LO (e^+e^-)	6931(40)	Refs. [6–11]
HVP NLO (e^+e^-)	−98.3(7)	Ref. [11]
HVP NNLO (e^+e^-)	12.4(1)	Ref. [12]
HVP LO (lattice, $udsc$)	7116(184)	Refs. [13–21]
HLbL (phenomenology)	92(19)	Refs. [22–34]
HLbL NLO (phenomenology)	2(1)	Ref. [35]
HLbL (lattice, uds)	79(35)	Ref. [36]
HLbL (phenomenology + lattice)	90(17)	
QED	116 584 718.931(104)	Refs. [37, 38]
Electroweak	153.6(1.0)	Refs. [39, 40]
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)	
HLbL (phenomenology + lattice + NLO)	92(18)	
Total SM Value	116 591 810(43)	
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	251(59)	

2 Hadronic vacuum polarization

The evaluation of the HVP contribution has a long history and mainly relies on the formula first discovered by Bouchiat and Michel sixty years ago [41], which expresses this contribution in terms of the cross section $e^+e^- \to$ hadrons. Many experimental measurements (see [2] for a complete list) have provided essential input for the application of this formula. In recent years mainly two groups have made a systematic and complete evaluation of this contribution and provided regular updates: the latest analyses of DHMZ [10] as well as KNT [11] constitute the basis for the SM number presented in the WP. But there is more than that: other analyses, like the one by Jegerlehner and collaborators [31] have also been considered and critically reviewed. Moreover, analyses of exclusive channels which make use of theoretical arguments (like analyticity and unitarity) to better constrain the data have also been used in the final average [8, 9, 42].

The method adopted to combine these analyses is the following: *1)* central values are obtained by simple averages (for each channel and mass range); *2)* it is always the largest experimental and systematic uncertainty of the analyses considered which is taken; *3)* half of the difference between analyes (or between data by BABAR [43] and KLOE [44–47] in the 2π channel, if this is larger) is added to the uncertainty. This led to the final result reported in Table 1, which has a final relative uncertainty of 0.6%. This is larger than could potentially be achieved in view of the precision of the data: but, as indicated, the combination procedure was aimed to err on the conservative side.

After the WP was published, new results by the SND collaboration [48] as well as an update by the BESIII collaboration [49] have been published. Their precision does not seem to be able to significantly impact the current estimate, but they will motivate updates of the a_μ^{HVP} evaluation. New results by CMD-3 [50] and BABAR are also expected in the near future.

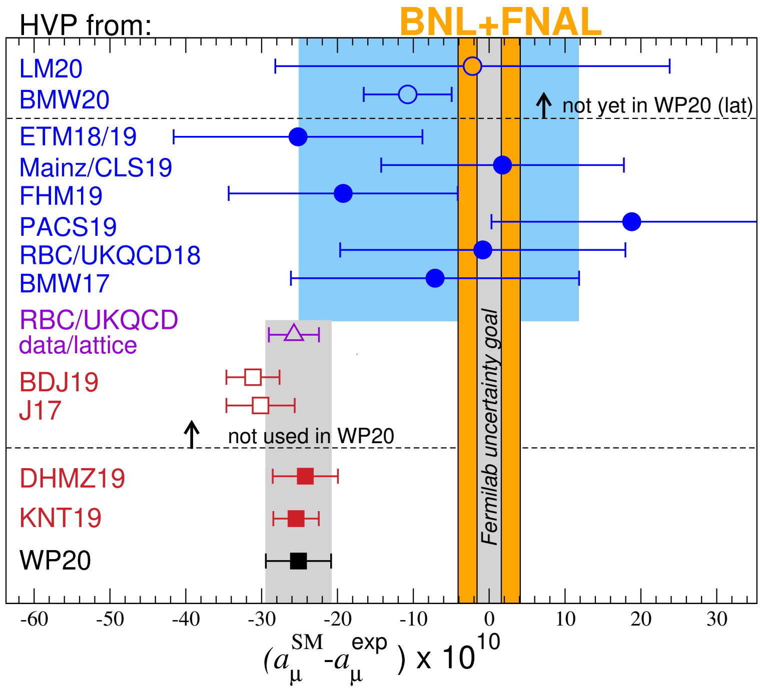

Figure 1. Comparison of different evaluations of the HVP contribution, both data driven (squares) as well as lattice (circles) and mixed (triangle) and of the corresponding total result for $a_\mu^{\rm SM} - a_\mu^{\rm exp}$. The evaluations above the lower dashed line have not been included in the WP average (grey band) and the lattice ones above the upper dashed line have not been included in the WP lattice average (light-blue band).

2.1 Lattice

Even though there has been a dedicated talk by Zoltan Fodor on the lattice perspective on the calculation of the HVP contribution to the $(g-2)_\mu$, I have to add a few comments on the most recent lattice results from the data-driven perspective. In particular because the recently published BMW calculation [4]:

$$10^{10} a_\mu^{\rm HVP,LO}({\rm BMW}) = 707.5(2.3)_{\rm stat}(5.0)_{\rm syst} = 707.5(5.5)\,,$$

is higher by about 2.1σ than the data-driven evaluation and, after adding all other contributions summarized in the WP, much closer to the experimental value of a_μ. The situation is illustrated in Fig. 1.

The discrepancy between the data-driven and the BMW result needs to be clarified. In view of the long history of the data-driven approach, the vast experimental database which is used in the calculation and all the independent checks which have been made over the years, I think that it is justified to consider this as the reference SM value for a_μ. The BMW result is the first complete lattice result with a precision comparable to the one of the data-driven approach. Given the relevance of this calculation and of a possible discrepancy with the data-driven approach it is important to first have a consensus lattice result based on different calculations relying on different discretizations and calculational settings. Several collaborations are working hard to reach this goal.

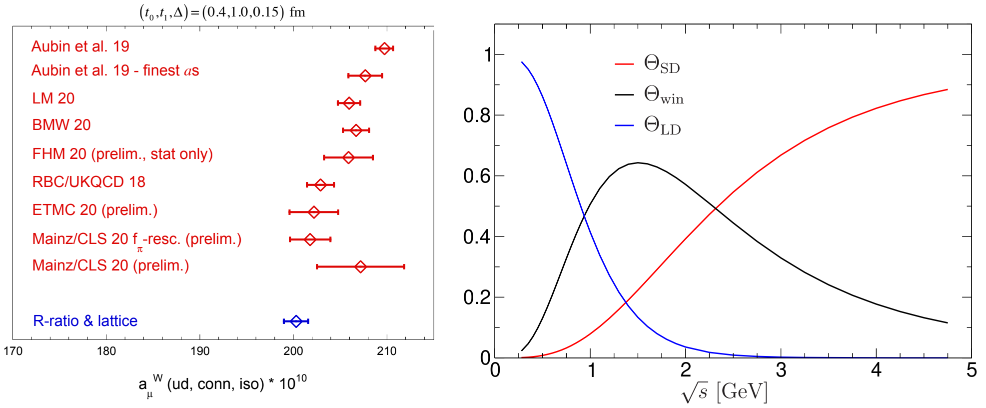

Figure 2. *Left:* Comparison of different lattice evaluations of the intermediate window quantity. Figure courtesy of Davide Giusti [51]. *Right:* Weight functions in s to be used in the time-like integral representation of $a_\mu^{\rm HVP}$ to obtain the window-quantities defined on the lattice. Figure courtesy of Martin Hoferichter.

An important aspect of the BMW result is the potential impact of the higher value of a_μ on the running of $\alpha_{\rm em}$ and on the value of $\alpha_{\rm em}(M_Z)$ [52]: a_μ is determined by an integral over the $e^+e^- \to$ hadrons cross section. The same cross section determines also the running $\alpha_{\rm em}$, though via an integral with a different kernel function: in particular $\alpha_{\rm em}(M_Z)$ is much more sensitive to the high-energy region than to the low-energy one, whereas for a_μ the opposite is true. Any increase in a_μ must be accompanied by an increase in $\Delta\alpha_{\rm em}^{\rm had}(M_Z)$: its size can only be estimated if one knows the energy distribution of the corresponding increase in the hadronic correction. Given the way the lattice calculation is made, this information is not available. Three different analyses [53–55] have estimated the possible impact under different sets of assumptions. They all reached a similar conclusion: unless one accepts to spoil the electroweak fit, the changes in the hadronic cross section have to happen below 2 GeV, the lower the better. Since the region below 1 GeV is dominated by the 2π channel, a specific analysis was dedicated to such a scenario [56]: we investigated the possible form and nature of the discrepancy with the data sets for this channel and pointed out correlations, not only with $\Delta\alpha_{\rm em}^{\rm had}(M_Z)$, but also with the pion charge radius.

Until other lattice collaborations will be able to reach the same level of precision for the complete physical quantity, it is possible to compare the so-called "intermediate window quantity", which is easier to calculate since it is less sensitive to systematic effects. A comparison of different lattice calculations for this window quantity (in the isospin limit) is displayed in Fig. 4 of Ref. [4] and shows a disagreement at the level of $\sim 2\sigma$ among lattice results of similar precision, in particular between the BMW and the one by RBC/UKQCD [36] (whereas BMW agrees well with Aubin et al. [20]). Several lattice collaborations are concentrating on this quantity and aiming to reach a consensus. The current situation has been discussed by Davide Giusti in his talk at this year's Lattice conference [51, 57] and is shown on the left panel of Fig. 2. Reaching a satisfactory consensus for this quantity is the most pressing goal of the lattice community interested in $a_\mu^{\rm HVP}$.

Interestingly, one can evaluate the same window quantity starting from e^+e^- data: the weight function originally defined in terms of Euclidean time for the lattice calculation can be translated in a weight function in s in the timelike region, as illustrated in the right panel of Fig. 2. The outcome of this calculation[1] is also shown in the left panel of Fig. 2: the data-driven result agrees with RBC/UKQCD and a few other preliminary results but is again lower than BMW and Aubin et al. While this may not seem surprising at first, one has to consider that the weight function in s suppresses the contribution from below 1 GeV. Such a disagreement seems to clash with the hypothesis made above that most of the discrepancy for the hadronic cross section occurs below 1 GeV. Together with Martin Hoferichter and Peter Stoffer we have investigated this point more quantitatively: in [56] we have obtained an explicit modification of the $\pi\pi$ cross section below 1 GeV which would give a value of $a_\mu^{\rm HVP}$ in agreement with that of BMW. Evaluating the window quantity with this modified cross section the blue point in Fig. 2 does move slightly to the right but not nearly enough to fill the gap with the BMW20 point. This implies that part of the modification of the hadronic cross section needed to explain the BMW result must happen above 1 GeV. With very minimal and reasonable assumptions about possible distributions of the change in the hadronic cross section we reached the following conclusion: of the $\sim 14\times 10^{-10}$ units of difference between the data-driven and BMW evaluations of $a_\mu^{\rm HVP}$, at least 5×10^{-10} have to originate from a change in the hadronic cross section above 1 GeV. This is a model-independent lower limit. Note that if all the shift is generated below 1 GeV it amounts to a $\sim 2.5\%$ relative change, whereas a shift of 5×10^{-10} generated between 1 and 2 GeV represents a $\sim 5\%$ relative change. Moreover the impact on the EW fit will be larger.

3 Hadronic light-by-light

The evaluation of the HLbL contribution in the WP has been much improved with respect to the time of the so-called "Glasgow consensus" [58] (see also [59, 60] for a somewhat different assessment of the situation circa 2009). This is mainly due to the formulation of a dispersive approach for HLbL [24, 61–63] which earlier had been deemed to be impossible. Table 2 illustrates well the improvements, in particular for what concerns the first three rows of the table, added up in the "subtotal" in the fourth row. As one can see by comparing the numbers in that row, the uncertainty reduction has been six- to five-fold with respect to 2009. The remaining rows contain smaller, but still relevant contributions for which the dispersive approach has not yet been applied to its full potential. The reason is that there are conceptual difficulties in including narrow resonances (beyond pseudoscalars) in such an approach: as it has been discussed in Ref. [24] the evaluation of the contribution of single poles in the different channels gives results which depend on the choice of the basis for the HLbL tensor, unless a set of sum rules are satisfied. This is automatically the case for pseudoscalars, but not for any other resonances. A recent discussion of this problem for the case of scalars can be found in Ref. [64], which also shows that progress in this direction is on-going. But it is important to stress that in the four central rows (scalars to short-distance) a superficial comparison of the numbers seems to indicate that uncertainties increased rather than decreased. This just reflects the fact that all possible sources have been accounted for and the explicit goal was to estimate them more conservatively. This is also seen in the way final uncertainties were added for this subset of contributions: linearly in the WP, whereas most previous analyses added them quadratically. For this reason the improvement in the uncertainty looks smaller than it actually is.

Among the remaining more uncertain contributions the axial vectors and the short-distance part are the most relevant ones. For these contributions there is on-going activity and what has been reported in the WP is a snapshot of an evolving situation. The issue of short-distance constraints (SDC) for HLbL has been first pointed out by Melnikov and Vainshtein

[1]Which also implies subtracting contributions from heavier valence quarks as well as isospin-breaking contributions—all done using lattice input—to make the comparison to the lattice meaningful.

Table 2. Comparison of different evaluations of the hadronic light-by-light contribution broken down into its different components identified by the relevant intermediate hadronic state. PdRV(09) is Ref. [58], N/JN(09) Refs. [59, 60] and J(17) Ref. [31].

Contribution	PdRV(09)	N/JN(09)	J(17)	WP(20)
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} − 1(3)
tensors	–	–	1.1(1)	}
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

(MV) in a seminal paper [22]: besides deriving these constraints, they also proposed a model for how to satisfy them. This essentially consisted in including only the lightest pseudoscalar poles in the HLbL tensor (in the limit of $(g-2)$ kinematics) to satisfy the longitudinal SDC, and the lightest axials to satisfy the SDC of the transverse components. The present precision requirements made it necessary to go beyond this model. Different attempts in this direction have been made recently: in our group we considered a tower of excited pseudoscalars to satisfy the longitudinal SDC [28, 65], whereas two different groups have addressed both the transverse as well as the longitudinal SDC by considering a tower of axial resonances within a model of holographic QCD [66–68]. While it is clear that the axials have to play a prominent role in satisfying the SDC because the contribution of excited pseudoscalars vanishes in the chiral limit, pseudoscalars have the unique advantage that for them the ambiguities mentioned above are absent. This means that each of the two approaches has drawbacks and only represents a step in the direction of a fully satisfactory solution of the SDC. For this reason it is particularly important to compare these two model-dependent solutions. This has been done in great detail in [69] and summarized in the left panel of Fig. 3 which shows the contribution to a_μ of the states responsible for satisfying the longitudinal SDC as a function of a lower cutoff $Q_{\rm min}$ on all three photon momenta. The figure shows that the solution in terms of excited pseudoscalars agrees well with different variants of the holographic model of QCD, and that both give a contribution significantly lower than was predicted by the original MV model: keeping only the pion pole contribution for $g-2$ kinematics is well justified in the high-q^2 region but is a bad approximation at low q^2. Taking into account also the transverse components led to the estimate shown in Table 2 for the contribution of short distance. The figure also shows the curve obtained by Lüdtke and Procura [70] who analyzed the longitudinal SDC on the basis of a set of interpolants and confirmed the estimate in the WP.

Other important developments concern the calculations performed by H. Bijnens and his group of perturbative [71] and non-perturbative [72] corrections to the leading order OPE, which is essentially given by the quark loop [22, 27]. These corrections allow for further reductions of the uncertainties in the evaluation of this contribution, as discussed in [69]. Other theoretical aspects, in particular concerning axial mesons, have been further discussed in [73–76].

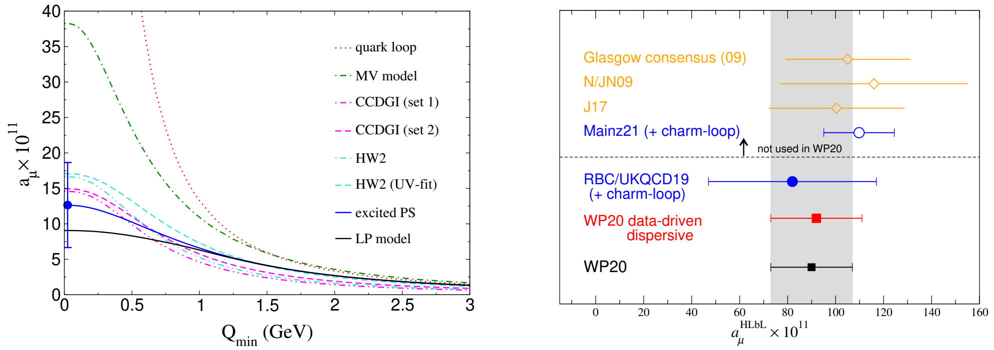

Figure 3. *Left*: contribution to a_μ due to the longitudinal SDC for different solutions to the latter as a function of a lower cutoff on the modulus of photon momenta (from [69]). *Right*: comparison of different evaluations of the HLbL contribution. The two results below the dashed line have been averaged to yield the result presented in the WP. Above the dashed line there is the recent lattice evaluation by the Mainz collaboration and the three other phenomenological ones also shown in Table 2.

3.1 Lattice

Compared to the two-point function which is relevant for HVP, evaluating the four-point function which enters the HLbL contribution is significantly more difficult on the lattice (and not only). In this case, on the other hand, the precision requirements are much less severe. Attempts at calculating the HLbL contribution have started much later than those for HVP, and much of the early (but still recent) work was devoted to developing a calculational strategy [77–81]. These efforts, which were carried out mainly by two lattice groups, RBC/UKQCD and Mainz, have culminated in the first two complete lattice calculations of this contribution: first by the RBC/UKQCD collaboration [36], a result which was early enough to be considered in the WP and which in fact was averaged with the data-driven one and included in the SM prediction for a_μ, see Table 1. The Mainz collaboration first published a result in the SU(3) limit [82], and only very recently completed the calculation for physical quark masses [5], thereby confirming both the RBC/UKQCD result as well as the data-driven one. Both results are shown in Fig. 3.

4 Conclusions

I have briefly reviewed the current status of the Standard Model evaluation of the muon anomalous magnetic moment concentrating in particular on the two main hadronic contributions, HVP and HLbL, and their data-driven as well as lattice evaluations. The comparison with the current experimental world average after the recent Fermilab result shows a 4.2σ discrepancy, which makes the muon $g-2$ one of the most interesting quantities in the search for deviations from the Standard Model and the quest for new physics. The present picture isn't as sharp as it could be because of the lattice calculation of the HVP contribution by the BMW collaboration, which shows a discrepancy with the calculation based on the data-driven approach and, if confirmed, would move the SM value of $(g-2)_\mu$ closer to the measurement. With lattice calculations it is always important to make universality tests, namely to show that different lattice formulations of QCD (in particular for what concerns fermion discretizations) lead to the same result in the continuum limit. This will have to wait until other lattice collaborations will produce results of similar precision as the one by BMW. Work in this direction, in particular for the simpler window quantity discussed above, is on-going and will hopefully soon lead to a full clarification of the situation.

Acknowledgments

I thank the colleagues of the Muon $g-2$ Theory Initiative and in particular of the Steering Committee for the pleasant and fruitful collaboration which produced the White Paper. I am grateful to Martin Hoferichter and Peter Stoffer for the longstanding collaboration on many of the topics described here. This work has been partially supported by the Swiss National Science Foundation.

References

[1] B. Abi et al. (Muon $g-2$), Phys. Rev. Lett. **126**, 141801 (2021), 2104.03281
[2] T. Aoyama et al., Phys. Rept. **887**, 1 (2020), 2006.04822
[3] G.W. Bennett et al. (Muon $g-2$), Phys. Rev. D **73**, 072003 (2006), hep-ex/0602035
[4] S. Borsanyi et al., Nature (2021), 2002.12347
[5] E.H. Chao, R.J. Hudspith, A. Gérardin, J.R. Green, H.B. Meyer, K. Ottnad (2021), 2104.02632
[6] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, Eur. Phys. J. C **77**, 827 (2017), 1706.09436
[7] A. Keshavarzi, D. Nomura, T. Teubner, Phys. Rev. D **97**, 114025 (2018), 1802.02995
[8] G. Colangelo, M. Hoferichter, P. Stoffer, JHEP **02**, 006 (2019), 1810.00007
[9] M. Hoferichter, B.L. Hoid, B. Kubis, JHEP **08**, 137 (2019), 1907.01556
[10] M. Davier, A. Hoecker, B. Malaescu, Z. Zhang, Eur. Phys. J. C **80**, 241 (2020), [Erratum: Eur. Phys. J. C **80**, 410 (2020)], 1908.00921
[11] A. Keshavarzi, D. Nomura, T. Teubner, Phys. Rev. D **101**, 014029 (2020), 1911.00367
[12] A. Kurz, T. Liu, P. Marquard, M. Steinhauser, Phys. Lett. B **734**, 144 (2014), 1403.6400
[13] B. Chakraborty et al. (Fermilab Lattice, LATTICE-HPQCD, MILC), Phys. Rev. Lett. **120**, 152001 (2018), 1710.11212
[14] S. Borsanyi et al. (Budapest-Marseille-Wuppertal), Phys. Rev. Lett. **121**, 022002 (2018), 1711.04980
[15] T. Blum, P.A. Boyle, V. Gülpers, T. Izubuchi, L. Jin, C. Jung, A. Jüttner, C. Lehner, A. Portelli, J.T. Tsang (RBC, UKQCD), Phys. Rev. Lett. **121**, 022003 (2018), 1801.07224
[16] D. Giusti, V. Lubicz, G. Martinelli, F. Sanfilippo, S. Simula (ETM), Phys. Rev. D **99**, 114502 (2019), 1901.10462
[17] E. Shintani, Y. Kuramashi, Phys. Rev. D **100**, 034517 (2019), 1902.00885
[18] C.T.H. Davies et al. (Fermilab Lattice, LATTICE-HPQCD, MILC), Phys. Rev. D **101**, 034512 (2020), 1902.04223
[19] A. Gérardin, M. Cè, G. von Hippel, B. Hörz, H.B. Meyer, D. Mohler, K. Ottnad, J. Wilhelm, H. Wittig, Phys. Rev. D **100**, 014510 (2019), 1904.03120
[20] C. Aubin, T. Blum, C. Tu, M. Golterman, C. Jung, S. Peris, Phys. Rev. D **101**, 014503 (2020), 1905.09307
[21] D. Giusti, S. Simula, PoS **LATTICE2019**, 104 (2019), 1910.03874
[22] K. Melnikov, A. Vainshtein, Phys. Rev. D **70**, 113006 (2004), hep-ph/0312226

[23] P. Masjuan, P. Sánchez-Puertas, Phys. Rev. D **95**, 054026 (2017), 1701.05829
[24] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP **04**, 161 (2017), 1702.07347
[25] M. Hoferichter, B.L. Hoid, B. Kubis, S. Leupold, S.P. Schneider, JHEP **10**, 141 (2018), 1808.04823
[26] A. Gérardin, H.B. Meyer, A. Nyffeler, Phys. Rev. D **100**, 034520 (2019), 1903.09471
[27] J. Bijnens, N. Hermansson-Truedsson, A. Rodríguez-Sánchez, Phys. Lett. B **798**, 134994 (2019), 1908.03331
[28] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, P. Stoffer, JHEP **03**, 101 (2020), 1910.13432
[29] V. Pauk, M. Vanderhaeghen, Eur. Phys. J. C **74**, 3008 (2014), 1401.0832
[30] I. Danilkin, M. Vanderhaeghen, Phys. Rev. D **95**, 014019 (2017), 1611.04646
[31] F. Jegerlehner, Springer Tracts Mod. Phys. **274**, 1 (2017)
[32] M. Knecht, S. Narison, A. Rabemananjara, D. Rabetiarivony, Phys. Lett. B **787**, 111 (2018), 1808.03848
[33] G. Eichmann, C.S. Fischer, R. Williams, Phys. Rev. D **101**, 054015 (2020), 1910.06795
[34] P. Roig, P. Sánchez-Puertas, Phys. Rev. D **101**, 074019 (2020), 1910.02881
[35] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera, P. Stoffer, Phys. Lett. B **735**, 90 (2014), 1403.7512
[36] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, Phys. Rev. Lett. **124**, 132002 (2020), 1911.08123
[37] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, Phys. Rev. Lett. **109**, 111808 (2012), 1205.5370
[38] T. Aoyama, T. Kinoshita, M. Nio, Atoms **7**, 28 (2019)
[39] A. Czarnecki, W.J. Marciano, A. Vainshtein, Phys. Rev. D **67**, 073006 (2003), [Erratum: Phys. Rev. D **73**, 119901 (2006)], hep-ph/0212229
[40] C. Gnendiger, D. Stöckinger, H. Stöckinger-Kim, Phys. Rev. D **88**, 053005 (2013), 1306.5546
[41] C. Bouchiat, L. Michel, J. Phys. Radium **22**, 121 (1961)
[42] B.L. Hoid, M. Hoferichter, B. Kubis, Eur. Phys. J. C **80**, 988 (2020), 2007.12696
[43] B. Aubert et al. (BaBar), Phys. Rev. Lett. **103**, 231801 (2009), 0908.3589
[44] F. Ambrosino et al. (KLOE), Phys. Lett. B **670**, 285 (2009), 0809.3950
[45] F. Ambrosino et al. (KLOE), Phys. Lett. B **700**, 102 (2011), 1006.5313
[46] D. Babusci et al. (KLOE), Phys. Lett. B **720**, 336 (2013), 1212.4524
[47] A. Anastasi et al. (KLOE-2), JHEP **03**, 173 (2018), 1711.03085
[48] M.N. Achasov et al. (SND), JHEP **01**, 113 (2021), 2004.00263
[49] M. Ablikim et al. (BESIII) (2020), 2009.05011
[50] F. Ignatov et al. (CMD-3), EPJ Web Conf. **212**, 04001 (2019)
[51] D. Giusti (????)
[52] M. Passera, W. Marciano, A. Sirlin, Phys. Rev. D **78**, 013009 (2008), 0804.1142
[53] A. Crivellin, M. Hoferichter, C.A. Manzari, M. Montull, Phys. Rev. Lett. **125**, 091801 (2020), 2003.04886
[54] A. Keshavarzi, W.J. Marciano, M. Passera, A. Sirlin, Phys. Rev. D **102**, 033002 (2020), 2006.12666
[55] B. Malaescu, M. Schott, Eur. Phys. J. C **81**, 46 (2021), 2008.08107

[56] G. Colangelo, M. Hoferichter, P. Stoffer, Phys. Lett. B **814**, 136073 (2021), 2010.07943
[57] D. Giusti, S. Simula, PoS, **LATTICE2021** p. 189 (????)
[58] J. Prades, E. de Rafael, A. Vainshtein, Adv. Ser. Direct. High Energy Phys. **20**, 303 (2009), 0901.0306
[59] F. Jegerlehner, A. Nyffeler, Phys. Rept. **477**, 1 (2009), 0902.3360
[60] A. Nyffeler, Phys. Rev. D **79**, 073012 (2009), 0901.1172
[61] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP **09**, 091 (2014), 1402.7081
[62] G. Colangelo, M. Hoferichter, B. Kubis, M. Procura, P. Stoffer, Phys. Lett. B **738**, 6 (2014), 1408.2517
[63] G. Colangelo, M. Hoferichter, M. Procura, P. Stoffer, JHEP **09**, 074 (2015), 1506.01386
[64] I. Danilkin, M. Hoferichter, P. Stoffer, Phys. Lett. B **820**, 136502 (2021), 2105.01666
[65] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, P. Stoffer, Phys. Rev. D **101**, 051501 (2020), 1910.11881
[66] J. Leutgeb, A. Rebhan, Phys. Rev. D **101**, 114015 (2020), 1912.01596
[67] L. Cappiello, O. Catà, G. D'Ambrosio, D. Greynat, A. Iyer, Phys. Rev. D **102**, 016009 (2020), 1912.02779
[68] J. Leutgeb, J. Mager, A. Rebhan (2021), 2110.07458
[69] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub, P. Stoffer, Eur. Phys. J. C **81**, 702 (2021), 2106.13222
[70] J. Lüdtke, M. Procura, Eur. Phys. J. C **80**, 1108 (2020), 2006.00007
[71] J. Bijnens, N. Hermansson-Truedsson, L. Laub, A. Rodríguez-Sánchez, JHEP **04**, 240 (2021), 2101.09169
[72] J. Bijnens, N. Hermansson-Truedsson, L. Laub, A. Rodríguez-Sánchez, JHEP **10**, 203 (2020), 2008.13487
[73] P. Masjuan, P. Roig, P. Sánchez-Puertas (2020), 2005.11761
[74] M. Knecht, JHEP **08**, 056 (2020), 2005.09929
[75] M. Zanke, M. Hoferichter, B. Kubis, JHEP **07**, 106 (2021), 2103.09829
[76] M. Hoferichter, P. Stoffer, JHEP **05**, 159 (2020), 2004.06127
[77] J. Green, O. Gryniuk, G. von Hippel, H.B. Meyer, V. Pascalutsa, Phys. Rev. Lett. **115**, 222003 (2015), 1507.01577
[78] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Lehner, Phys. Rev. D **93**, 014503 (2016), 1510.07100
[79] T. Blum, P.A. Boyle, T. Izubuchi, L. Jin, A. Jüttner, C. Lehner, K. Maltman, M. Marinkovic, A. Portelli, M. Spraggs, Phys. Rev. Lett. **116**, 232002 (2016), 1512.09054
[80] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, Phys. Rev. Lett. **118**, 022005 (2017), 1610.04603
[81] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner, Phys. Rev. D **96**, 034515 (2017), 1705.01067
[82] E.H. Chao, A. Gérardin, J.R. Green, R.J. Hudspith, H.B. Meyer, Eur. Phys. J. C **80**, 869 (2020), 2006.16224