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Fractals, metamorphoses and symmetries in quantum field theory

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Abstract. The mechanism of spontaneous breakdown of symmetry in quantum field theory is reviewed, the Goldstone theorem derived and the formation of ordered patterns is discussed. Metamorphoses and fractal self-similarity are described in terms of the dynamical rearrangement of symmetry in spontaneously broken symmetry theories. The isomorphism between fractal self-similarity and deformed coherent states is exhibited. It is also discussed how the missing of infrared terms in physical observations is responsible of the metamorphosis processes. The dynamical generation of squeezed, entangled coherent states plays a crucial role in the formation of ordered patterns (morphogenesis). The discussion on metamorphoses and other results extend to condensed matter physics, elementary particles, biological systems and brain studies.

1 Introduction

Metamorphoses processes have attracted ever great attention in many disciplines, from botanic studies to biology, theory of evolution, genetics, morphogenesis studies, etc.. In this report I will review how metamorphoses may be described in QFT in terms of observable manifestations of symmetry properties of the underlying field dynamics.

Spontaneous breakdown of symmetry (SBS) in quantum field theory (QFT) is the physical mechanism underlying the Standard Model of elementary particle physics and many phenomena in condensed matter physics.

Suppose that the interacting field equations are invariant under a symmetry group G of continuous field transformation. SBS occurs when the ground state (the vacuum state) of the system is not symmetric under all the transformations belonging to the group G, but it might be symmetric under one of the subgroups of G (called the vacuum stability group) [1–7]. The word "spontaneous" means that the vacuum is dynamically singled out among a number of possibilities so that the its stability is guaranteed.

When SBS occurs, the equations for the free fields in terms of which physical observations are described are invariant under a group $G' \neq G$. The vacuum stability group is a subgroup also of G'. The process leading from G to G' is called the *dynamical rearrangement of symmetry* and it implies the formation of long-range correlations among the elementary components of the system with the formation of observable ordered patterns. The specificity and the degree of ordering are quantified by a measurable classical field called *order parameter*. Its space-time variations denote changes in the ordered patterns, namely in the "forms" through which the system appears to our observations. SBS, the dynamical rearrangement of symmetry, and the space-time evolution of the order parameter thus provide the dynamics of the *metamorphoses* possibly occurring during the system evolution [8, 9].

The description of the dynamical rearrangement of the symmetry and the example of the ferromagnet are given, by closely following refs. [5, 9–14], in sections 2 and 3, the algebraic structure underlying the formation of ordered patterns is presented in section 4. The effects due to localization of observations and to temperature are discussed in sections 5 and 6. In section 7, an isomorphism is shown to exist between fractals and deformed coherent states, so that fractals appear to be the observable manifestation of the coherence of long-range correlations responsible for the formation of ordered patterns [15–18]. Section 8 is devoted to conclusions. Details of the mathematical formalism are given in the Appendices.

2 Dynamical rearrangement of symmetry

The von Neumann theorem in quantum mechanics (QM) states that for systems with a finite number of degrees of freedom all the representations of the canonical commutation relations (CCR) (for bosons, or anticommutation relations (CAR) for fermions) are unitarily equivalent [19] (unitary equivalence or non-equivalence means physical equivalence or non-equivalence, respectively). Fields carry by definition an infinite number of degrees of freedom, thus the von Neumann theorem does not hold in QFT, where, therefore, infinitely many unitarily inequivalent representations (uir) of the CCR (or CAR) exist [1–7].

The meaning of the existence of infinitely many uir is that the system may have different dynamical regimes, or *phases*, each phase being identified by a specific value of the order parameter (common examples are non-ferromagnetic and ferromagnetic phases, superconductive and non-superconductive ones, the crystal and the amorphous phases, etc.). Transitions among these phase cannot be induced by unitary operators. The processes of

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phase transitions thus present "singularities" and "criticality". Each phase is however well defined and represented by states belonging to a given Hilbert space H_M. The label M denotes a specific value, including zero, of the order parameter (e.g. the magnetization in the case of ferromagnets), each representation H_M being orthogonal to the other ones (with different label values M', M", etc.). States belonging to H_M thus cannot be written as superposition of states belonging to the other representation $H_{M'}$, $H_{M''}$, etc.. Phase transitions are described by the evolution of the *order parameter* M in terms of trajectories in the space of the uir [20].

QFT develops on a double level of language. One level is the one of the field operators undergoing interactions, called the Heisenberg or interacting fields, here generically denoted by $\psi(x)$, defined on some Hilbert space $\mathcal{H}$. The Lagrangian L describes the dynamics and we assume that the system of interest is a closed system, so that canonical formalism can be used. We write symbolically the field equations derived from L as

$$\Lambda(\partial)\psi(x) = J(\psi(x)) \,, \tag{1}$$

where $\Lambda(\partial)$ denotes the operator kinematic terms and J the interaction non-linear terms in the $\psi(x)$ fields. Eq. (1) is understood to be an equality between expectation values on states belonging to $\mathcal{H}$, it is a "weak" equality.

Our observations are allowed, however, only on *physical or asymptotic* states, not belonging to $\mathcal{H}$ but to other spaces H_M, where physical or asymptotic free field operators $\phi(x)$ are defined. They satisfy free field equations:

$$K(\partial)\phi(x) = 0. \tag{2}$$

The operator $K(\partial)$ is a projector operator on the on-shell dispersion relation. We have thus another level of language, expressed in terms of the free $\phi(x)$ fields, the level of the *manifestation* to our observations of the underlying interaction dynamics. Also Eq. (2) has to be understood in a "weak" sense, as an equation for matrix elements in H_M. This implies that a choice is required of the state space H_M among the set $\{H_M, \forall\, M\}$ of the infinitely many uir.

In the Lehmann-Symanzik-Zimmerman (LSZ) formalism, the fields $\phi(x)$ are often denoted as $\phi^{in}(x)$ and $\phi^{out}(x)$, incoming fields (to the interaction region) and outgoing fields (from the interaction region), respectively. For notational simplicity, they are here denoted as $\phi(x)$.

The fact that we do not have access to $\mathcal{H}$ means that the mapping between $\psi(x)$ and $\phi(x)$ field operators (the dynamical map or the Haag expansion) must be considered:

$$\psi(x) = \Psi(\phi(x)) \,, \tag{3}$$

to be understood, again, as a weak equality in H_M. Through Eq. (3) one can compute the $\psi(x)$ field matrix elements in H_M, although the $\psi(x)$ are originally defined to operate on the space $\mathcal{H}$.

The sets of ψ and ϕ fields are irreducible sets of fields. However, the numbers of ψ and ϕ fields need not to be the same; e.g., a bound state might be created out of the interaction of ψ fields.

The Haag theorem states that $\mathcal{H}$ and H_M are unitarily inequivalent spaces [1, 2, 6, 21]. Therefore for computational tasks perturbation theory methods cannot be applied; non-perturbative analysis is necessary. The Haag theorem and the Haag expansion also mean that our knowledge is characterized by an intrinsic "opacity" of the $\psi(x)$ field basic dynamics [22]. This can be known *only* in the language of the free physical fields $\phi(x)$ and physical states belonging to one of the (infinitely) many spaces H_M.

As mentioned in the Introduction, SBS may occur and the free field equations are then invariant under the symmetry group $G' \neq G$ * [3–5, 10, 12–14, 23, 24].

Let by $g \in G$ a transformation of the Heisenberg field $\psi(x)$,

$$\psi(x) \to \psi'(x) = g[\psi(x)] \,, \tag{4}$$

and $h \in G'$ a transformation of the fields ϕ

$$\phi(x) \to \phi'(x) = h[\phi(x)] \,, \tag{5}$$

the map (3) then implies that there exist an $\tilde{h} \in G'$ such that

$$\tilde{g}[\psi(x)] = \Psi(\tilde{h}[\phi(x)]), \quad \text{with } \tilde{g} \in G. \tag{6}$$

Eq. (6) for $G' \neq G$ expresses the dynamical rearrangement of symmetry $G \to G'$.

As we will see, the symmetry group G' controls the formation of the observable ordered patterns. The conclusion is that the basic dynamics (defined by equations (1)) "unfolds" into a multiplicity of observable realizations $\{H_M\}$; it "appears" in different *forms*, characterized by different ordered patterns, *dynamical rearrangements* or *metamorphoses* of the same basic symmetry G.

In the next section, the example of the ferromagnet is discussed explicitly. The results, with convenient modifications, essentially hold for systems with Lagrangians invariant under compact continuous transformation groups and SBS.

3 The example of the ferromagnet

In the case of ferromagnets, the Lagrangian and the Heisenberg field equations are invariant under the continuous $SU(2)$ group of transformations (spherical rotations in the spin space). The ferromagnet ground state (the vacuum) $|0\rangle$ is however symmetric only under rotations around the direction of the magnetization M, e.g., the 3rd axis in the spin-space ($U(1)$ subgroup of $SU(2)$). We thus have SBS, triggered for example by a weak coupling ϵ with an external agent, but persisting also when such a coupling vanishes, $\epsilon \to 0$ [13, 14, 25]. The role of the trigger is thus to select the appropriate H_M, for a given non-zero value of the order parameter M.

Due to the invariance, the three $SU(2)$ generators commute with the Hamiltonian. The corresponding conservation laws survive the SBS mechanism. Indeed, the physical field equations are found to be invariant under three parameter $E(2)$ group, consisting of the $U(1)$ rotation group

*The case of the "explicit" breakdown of symmetry, due to the introduction of a symmetry breaking term in the Lagrangian, is not considered in this work.

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around the magnetization direction and two field translation generators (see below). These last ones induce transitions among the "phases" H_M's (phase transitions), with corresponding changes in the order parameter values.

Consider the itinerant electron model of ferromagnets [13, 14]. The $SU(2)$ transformation for the Heisenberg electron fields, which have usual equal-time CAR, is

$$\psi(x) \to \psi^{'}(x) = \exp(i\theta_i\lambda_i)\psi(x), \qquad i = 1, 2, 3, \qquad (7)$$

$\lambda_i = \sigma_i/2$, with σ_i the Pauli matrices. The rotation angles θ_i in the spin-space denote the real continuous group parameters. The $\psi(x)$ electron field is

$$\psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix}, \qquad (8)$$

where ↑ and ↓ denote spin up and spin down, respectively.

In the functional integration formalism (cf. Appendix A), it is not necessary to give the explicit expressions for the Lagrangian $\mathcal{L}$ and for the $SU(2)$ spin density operators $S^{(i)}_\psi(x), \ \ i = 1, 2, 3$. $\mathcal{L}$ is only required to be $SU(2)$ symmetric under the transformations (7), $\mathcal{L}[\psi(x)] = \mathcal{L}[\psi^{'}(x)]$, and one possible form for $S^{(i)}_\psi$ is $S^{(i)}_\psi(x) = \psi^\dagger(x)(\sigma_i/2)\psi(x)$, with $SU(2)$ algebra

$$[S^{(i)}_\psi(x), S^{(j)}_\psi(x)] = i\epsilon_{ijk}S^{(k)}_\psi(x)\ . \qquad (9)$$

The magnetization is given by $g\mu_B\langle S^{(3)}_\psi(x)\rangle_\epsilon$, where μ_B denotes the Bohr magneton. The expectation value of the spin density in the third direction is $\langle S^{(3)}_\psi(x)\rangle_\epsilon \equiv M(\epsilon)$, given in Eqs. (A.8) and (A.9), which leads to

$$M = \lim_{\epsilon\to 0} M(\epsilon) = \lim_{\epsilon\to 0} i\epsilon\Delta_i(\epsilon, 0)\ ,\ i = 1, 2\ , \qquad (10)$$

where

$$\Delta_i(\epsilon, p) = \rho_{(i)}(p)\left(\frac{1}{p_0 - \omega_p + i\epsilon a_i} - \frac{1}{p_0 + \omega_p - i\epsilon a_i}\right). \qquad (11)$$

Eq. (10) shows that non-zero M (i.e. SBS) is possible only provided that $\omega_p = 0$ at $p = 0$. We have $M = 2\rho$ (see Appendix A).

The Goldstone theorem has been thus proved [26]. It states indeed that SBS in QFT requires the existence of massless (gapless) particles, the so called Nambu-Goldstone (NG) boson particles or modes, as in fact shown by Eq. (10). In the ferromagnetic system, the quanta of the magnon fields $\chi(x)$, i.e the spin waves quanta, are gapless bound states of electrons (with vanishingly small mass m). Their field equation is given by $\Sigma(\overrightarrow{\partial})\chi^\dagger(x) = 0$.

The irreducible set of free fields is now made by the $\{\phi(x), \chi(x)\}$ fields and the dynamical map becomes

$$\psi(x) = \Psi(\phi(x), \chi(x))\ . \qquad (12)$$

Eq. (10) shows that the possibility to have a non-zero order parameter (i.e. SBS) is due to the dynamical formation of long-range correlations described by the two-point (Green's) functions in Eqs. (A.8) - (A.10). The NG bosons (the magnons) are responsible of these long-range spin correlations over (large) distances of the order of $R \propto 1/m$, and therefore of the ferromagnetic ordered patterns. The structure of the vacuum state is the one of a coherent condensate of NG modes (see section 4).

4 A dynamical process: from symmetry to ordered patterns

In order to clarify the process of the symmetry rearrangement $SU(2) \to E(2)$ ($G \to G'$), it is necessary to consider the S-matrix $\mathcal{S}$ and the spin densities $S^{(i)}(x)$. Use of the LSZ formula leads to

$$\mathcal{S}(\phi, \phi^\dagger, \chi, \chi^\dagger) = \langle : \exp[-iA(\phi, \phi^\dagger, \chi, \chi^\dagger)] : \rangle \qquad (13)$$

$$S^{(i)}(\phi, \phi^\dagger, \chi, \chi^\dagger) = \langle S^{(i)}_\psi(x) : \exp[-iA(\phi, \phi^\dagger, \chi, \chi^\dagger)] : \rangle, \qquad (14)$$

where $i = 1, 2, 3$, the symbols : ... : and $\langle ... \rangle$ denote normal ordering and functional average, respectively. $A(\phi, \phi^\dagger, \chi, \chi^\dagger)$ is given in the Appendix B.

One finds the field transformations (cf. Appendix B)

$$\begin{aligned} \phi(x) &\to \phi_\theta(x) = \phi(x)\ , \\ \chi(x) &\to \chi_\theta(x) = \chi(x) + i\theta_1\,(M/2)^{1/2}\ , \end{aligned} \qquad (15)$$

for $\theta_2 = \theta_3 = 0$,

$$\begin{aligned} \phi(x) &\to \phi_\theta(x) = \phi(x)\ , \\ \chi(x) &\to \chi_\theta(x) = \chi(x) - \theta_2\,(M/2)^{1/2}\ , \end{aligned} \qquad (16)$$

for $\theta_1 = \theta_3 = 0$. The (unbroken) $U(1)$ third axis rotation is

$$\phi(x) \to \phi_\theta(x) = e^{i\theta_3\lambda_3}\phi(x)\ , \qquad (17)$$

$$\chi(x) \to \chi_\theta(x) = e^{-i\theta_3}\chi(x), \qquad (18)$$

for $\theta_1 = \theta_2 = 0$. These, and their hermitian conjugates, are canonical transformations and belong to the $E(2)$ group, which is known to be the Inönü-Wigner $SU(2)$ group contraction [10, 27–29].

The general theorem can be proved, stating that the dynamical rearrangement of continuous compact symmetry group $G \to G'$ is controlled by the mathematical structure of group contraction [10, 30, 31]. What actually appears in our physical observations is not the basic symmetry G, but its *physical manifestation* expressed by the group contraction G' ($E(2)$ in the present case). This includes fields translations such as those in Eqs. (15) and (16) (to be compared with the rotations (7)) which induce the NG condensation in the vacuum state. ...A metamorphosis process.

The electromagnetic field has not been considered in the above discussion since it does not affect the rearrangement $G \to G'$ (see [3, 6, 32–34] for details). Moreover, it has been assumed that the system temperature T is below the critical temperature T_C (the Curie temperature). Magnetization changes as T changes and it vanishes for $T > T_C$ ($SU(2)$ symmetry"restoration"); accordingly the system ordered patterns evolve through different dynamical regimes (phases transitions); they undergo transitions through different *forms* (metamorphoses).

Consider now the generators of the transformations (translations) of the magnon $\chi(x)$ in Eqs. (15) and (16). These transformations are not unitarily implementable and their generators need to be regularized by introducing a volume cut-off (square integrable) function $f(x)$ [2, 3, 21], which also needs to be solution of the equations for the magnon fields $\chi(x)$ in order to preserve their invariance under these transformations. These are thus to be understood to be the limit for $f(x) \to 1$ of $\chi(x) \to \chi_\theta(x) = \lim_{f(x)\to 1}(\chi(x) + f(x)c_j)$, with $c_j = i^j\theta_j\,(M/2)^{1/2}$, $j = 1, 2$, and h.c..

The generators of (15) - (18) (where θ_i is replaced by $f(x)\theta_i$) are

$$s_f^{(1)} = \left(\frac{M}{2}\right)^{1/2} \int d^3x[\chi(x)f(x) + \chi^\dagger(x)f^*(x)] \quad (19)$$

$$s_f^{(2)} = -i\left(\frac{M}{2}\right)^{1/2} \int d^3x[\chi(x)f(x) - \chi^\dagger(x)f^*(x)] \quad (20)$$

$$s_f^{(3)} = \int d^3x[\phi^\dagger(x)\lambda_3\phi(x) - \chi^\dagger(x)\chi(x)] \quad (21)$$

with their (projective) $e(2)$ algebra:

$$[s_f^{(1)}, s_f^{(2)}] = iM \int d^3x|f(x)|^2 = (const.)\mathbf{1}, \quad (22)$$

$$[s_f^{(3)}, s_f^{(i)}] = i\epsilon_{3ij}s_f^{(j)}.$$

Note that since we always consider weak equalities, expectation values in physical states imply naturally the introduction of square integrable functions, because, by their nature, observable states are states localized in a finite volume. Regularization means to take indeed integrations over finite volume V and then the limit $V \to \infty$ (i.e. $f(x) \to 1$) at the end of computation.

Also note that $s_f^{(\pm)} = (1/2)(s_f^{(1)} \pm i s_f^{(2)})$ are generators of coherent states [35, 36], which shows that the ferromagnet ground state is a coherent state of magnons. Ordered patterns, the "forms" in which the dynamics manifests itself at the observation level, are generated by the coherent condensation of magnons (NG modes). Moreover, since quantum fluctuations $\langle \Delta n \rangle$ over the condensate number $\langle n \rangle$ of particles goes as the inverse of the coherence strength [10, 25, 36], the system macroscopic behaviour is characterized by the order parameter M, which is indeed a classical field, i.e. independent of quantum fluctuations due to coherence (except of course in critical processes of phase transitions and when the classical limit cannot be achieved [37]). The system appears then as a *macroscopic quantum system*, meaning that its macroscopic (classical) behaviour cannot be derived without recourse to the quantum dynamics of its elementary components. In the following we will see that fractal self-similarity is isomorph to deformed coherent states.

So far, the limit $f(x) \to 1$ has been considered to be performed at the end of the computations and then, in that limit, one has homogeneous condensation. However, one may also consider non-homogeneous condensation processes by not taking that limit. The function $f(x)$ can be regular (divergenceless, Fourier integrable), or a singular function with divergences or topological singularities. In these cases, it will play the role of a "form factor" and describe condensation of topologically non-trivial extended objects, e.g. vortices, solitons, monopoles, etc.. [3, 4, 6, 15, 34, 38]. Singularities in $f(x)$ are allowed only for gapless (massless) bosons and for such a reason there may be formation of topologically non-trivial extended objects only during the phase transition processes in the presence of ordered patterns (i.e. of gapless NG modes).

The high stability of these extended object solutions is due to the conservation of their topological charge (they cannot decay to the vacuum or to other solutions with zero topological charge). Moreover, they manifest as collective motion solutions with macroscopic classical behaviour (thus solutions of classical field equations) due to their coherent structures [6, 15, 38–40], and they depend on powers of the inverse of the self-coupling constant, say λ, of the elementary constituents, so that their formation is favoured for low λ. This accounts for the possibility of formation of macroscopic very stable (and energetic) collective phenomena; for example the formation of solitary waves in water and hurricanes [41, 42] is possible exactly because of the low coupling among water molecules.

Much interesting is the spontaneous breakdown of external symmetries, e.g. breakdown of space continuous translational symmetry generating periodic lattice structures (crystal lattices, space discreteness) [3], also related to Bloch periodic functions and the q-deformed Weyl-Heisenberg algebra [43, 44].

Finally, it is remarkable that the NG field (the magnon $\chi(x)$) appears as a phase in the exponents of Eqs. (13) and (14) (see also the Appendix B) and that it is the one that undergoes non-trivial transformations, in contrast with the matter field $\phi(x)$ which is "frozen", as shown in Eqs. (15) and (16). Since the NG bosons are responsible of the generation of long-range correlations, these are "in-phase" correlations. Ordering thus results from "in-phase" correlation dynamics (collective dynamics), not from the exchange among the elementary components of an intermediate boson (a force). Of course, as usual in a quantum theory, to the "correlation wave" (the spin wave in the ferromagnetic example) one can associate the corresponding (de Broglie) quanta, which in the SBS are the NG boson quanta.

In conclusion, the invariance of the *same* dynamics may manifest into homogeneous and non-homogeneous coherent boson condensations (the boson transformation theorem) [3, 4]. In such a morphogenesis dynamics, many different forms are thus produced and may evolve through metamorphosis processes.

5 Local observations and metamorphoses

As mentioned above, physical states and observables are always localized in finite space and time. This means that terms of the order of $1/V$, so called infrared terms, do not contribute to them, and this originates the rearrangement of the symmetry.

The magnon boson field $\chi(x)$ (cf. Eq. (C.1)) can be split into the "hard" part, $\chi_t(x)$, containing momenta larger than a given small value η, and the "soft" or infrared part $\chi_\eta(x)$, containing momenta smaller than η,

$$\chi(x) = \chi_t(x) + \chi_\eta(x)\,. \quad (23)$$

$\chi_\eta(x)$ goes as η, and for $\eta \to 0$ it is independent of x, cf. Appendix C, where it is shown that the generators depend on $\chi_\eta(x)$ as:

$$S_f^{(1)} = s_f^{(1)} + (1/2M)^{1/2}\,(\chi_\eta + \chi_\eta^\dagger) : s_t^{(3)} : , \quad (24)$$

$$S_f^{(2)} = s_f^{(2)} - i\,(1/2M)^{1/2}\,(\chi_\eta - \chi_\eta^\dagger) : s_t^{(3)} : , \quad (25)$$

$$S_f^{(3)} = s_t^{(3)} + (1/2M)^{1/2}\left[i(\chi_\eta - \chi_\eta^\dagger)s_t^{(2)} - (\chi_\eta + \chi_\eta^\dagger)s_t^{(1)}\right], \quad (26)$$

and we see that the $su(2)$ algebra is recovered by

$$\lim_{\eta\to 0}\lim_{\bar\eta\to 0}\lim_{f\to 1}[S_f^{(i)}, S_f^{(j)}] = i\epsilon_{ijk}S^{(k)} . \quad (27)$$

Here η and $\bar\eta$ are used for the two rotations in the commutator. Instead, if the infrared contributions are missed, one gets the (projective) $e(2)$ algebra:

$$\lim_{f\to 1}\lim_{\tilde\eta\to 0}[S_f^{(1)}, S_f^{(2)}] = iM\lim_{f\to 1}\int d^3x|f(x)|^2 = (const)\mathbf{1} , \quad (28)$$

$$\lim_{f\to 1}\lim_{\tilde\eta\to 0}[S_f^{(3)}, S_f^{(i)}] = \lim_{f\to 1}\lim_{\tilde\eta\to 0} i\epsilon_{3ij}S^{(j)} , \quad (29)$$

where $\tilde\eta$ denotes the minimum of η and $\bar\eta$.

The conclusion is that $f \to 1$ and $\bar\eta \to 0$ are non-commuting limits. The algebra $e(2)$ is obtained when the generators are expressed in terms of physical fields where infrared terms are missing. Instead, the $S^{(i)}$ commutators expressed in terms of the Heisenberg field (recovered by including the whole volume in the $f \to 1$ limit and adding up all the infrared contributions) give the $su(2)$ algebra: the localized nature of physical states and observables is at the origin of the metamorphosis $SU(2) \to E(2)$. This result is general and *exact*, not obtained in a linear approximation, e.g. in the Holstein-Primakoff representation [45].

Notice that the locality and causality principle has as a prerequisite the localization of physical states and observables. Indeed, the *locality* of a theory means that the system A and the system B, which is separate from A at some distance δ, may communicate through the exchange of a quantum travelling at a speed not larger than the light speed c. This is required by relativity and guaranties that the causality principle is also satisfied, namely, that the effect necessarily follows in time the cause (at a $\Delta t \geq \delta/c$)). The basic requirement for the locality is therefore that A is distinguishable from B, i.e. that they are localized in finite, *separate*, non-overlapping space-time regions, thus defined by a set of finite support space-time functions $\{f(x)\}$ (further details in [9, 46]. See also Appendix D.) .

6 Finite volume and temperature effects in spontaneous breakdown of symmetry

On the basis of the previous discussion, we may expect that the finiteness of the system volume may produce boundary effects on the range of the ordering correlation, thus on the size of ordered patterns and/or their distortions and fragmentation. One might also consider the possibility that the system boundaries might not be imposed by an external action, but be dynamically created, as for example in the case of non-homogeneous condensation.

Considering Eqs. (A.8) , (A.9) and (A.10), it is thus interesting to limit the integration to finite volume $V \equiv \eta^{-3}$. By using

$$\delta_\eta(p) = \frac{1}{2\pi}\int_{-\frac{1}{\eta}}^{\frac{1}{\eta}} dx\, e^{ipx} = \frac{1}{\pi p}\,\sin\frac{p}{\eta} , \quad (30)$$

and $\lim_{\eta\to 0}\delta_\eta(p) = \delta(p)$, we have

$$\lim_{\eta\to 0}\int dp\,\delta_\eta(p)\,f(p) = f(0) = \lim_{\eta\to 0}\int dp\,\delta(p-\eta)\,f(p) . \quad (31)$$

Since $\delta_\eta(p) \simeq \delta(p-\eta)$ for small η, we obtain (cf. Eq. (11))

$$M(\epsilon, \eta, \mathbf{y}) = i\epsilon e^{-i\eta\cdot\vec{y}}\,\Delta_i(\epsilon, \eta, p_0 = 0) , \; i = 1, 2 , \quad (32)$$

Thus, $\lim_{\epsilon\to 0}\lim_{\eta\to 0} M(\epsilon, \eta, \mathbf{y}) \neq 0$, provided $\omega_{\vec p=\eta} = 0$ for $\eta \to 0$. The Goldstone theorem is recovered in the infinite volume limit ($\eta \to 0$). However, at finite volume, i.e. for non-zero η, it is $\lim_{\epsilon\to 0} M(\epsilon, \eta, \mathbf{y}) = 0$; the χ bosons behave as having an "effective mass" $\propto \omega_{\vec p=\eta} \neq 0$ due to “boundary effects” ($\eta \neq 0$). Then, $M \neq 0$ only if $\epsilon \neq 0$. ϵ plays the role of the pump supplying energy to keep $M \neq 0$.

This is a general result [25]. Boundary effects manifest as an effective mass m_{eff} for the correlation quanta and the ordered pattern linear size is $\xi \propto 1/m_{eff}$. Conversely, if some external agent (e.g. thermal effects) generates an effective mass for the NG bosons, then the ordered patterns rearrange within boundaries dynamically generated, of linear size $\propto 1/m_{eff}$ [47] (fragmentation of ordered patterns).

Thermalization produces symmetry restoration ($\xi \to 0$) when temperature T becomes larger than the critical temperature T_C, with $m_{eff} \propto \sqrt{|T - T_C|/T_C}$, and fluctuations around T_C may produce size fluctuations $\propto 1/m_{eff}$. Conversely, fluctuations in size may manifest as thermal fluctuations.

As a final comment on the stability of the ordered patterns, note that the S-matrix is invariant under c-number translations of $\chi(x)$ if it depends on $\chi(x)$ through its derivative, $\partial_\mu\chi(x)$ is in fact invariant under $\chi(x) \to \chi(x) + const.$. Then the S-matrix is independent of low momentum $\chi(x)$ fields because $\partial_\mu\chi(x) \to 0$ for $p_\mu \to 0$, which is the Dyson low-energy theorem for magnons (the Adler theorem in particle physics, the soft boson limit of current algebra theory) [3, 4, 6, 48, 49]. The excitation of low momenta NG modes (long wave-length quanta, i.e. $p_\mu \approx 0$) therefore do not affect the stability of the ordered patterns. In the case of ordered domains of size $\propto 1/m_{eff}$ (non-homogeneous condensation), the S-matrix invariance requires its dependence on $\chi(x)$ through its field equation on-shell projector operator. Then, stability is under excitation of quanta of wave-length $\lambda \propto 1/m_{eff}$, i.e. $p_\mu \propto m_{eff}$ fixes the stability threshold.

7 Coherent states and fractal-self-similarity

Fractals are widely observed in many phenomena in natural sciences [50, 51]. Fractal studies in solid state and condensed matter physics include amorphous structures, percolation and quasicrystal research [52–55]. Dislocations formed at low temperature in a crystal submitted to deforming actions are observed [56] to form fractal patterns and suggest that a relation may exists with phonon condensation in coherent states.

It is known [57–59] that dissipative systems, where the *arrow of time* appears, namely time-reversal symmetry is broken, are described by generalized $SU(1,1)$ coherent states. In this section, the existence of an isomorphism between fractal self-similarity and coherent states with dissipative character is reviewed [15–18].

Consider the Koch curve (Fig. 1) and denote the starting stage by u_0, put $u_0 = 1$. The n-stage $u_{n,q}(\alpha)$, with $\alpha = 4$ and $q = 1/(3^d)$, is found to be $u_{n,q}(\alpha) = (q\alpha)^n = 1$ for any n, and the fractal dimension, or self-similarity dimension, is $d = \ln 4/\ln 3 \approx 1.2619$. Similar relations, with different q and α, can be obtained for other deterministic and random (iterative) fractals, for example the random Sierpinski carpet (see e.g. [54]). Notice that self-similarity is properly defined only in the $n \to \infty$ limit.

In the complex α-plane, and using $q = e^{-d\theta}$, the self-similarity relation $(q\alpha)^n = 1$ is written as $d\theta = \ln\alpha$. Note now that, apart the normalization factor $1/\sqrt{n!}$, the $u_{n,q}(\alpha)$s are the restriction to real $q\alpha$ of the functions $u_{n,q}(\alpha) = (q\alpha)^n/\sqrt{n!}$ for $n \in N^+$, $q\alpha \in C$. These form a basis in the space F of the entire analytic functions. Therefore, we can study the fractal properties in F [17]. Notice that F is the Fock–Bargmann representation of the Weyl–Heisenberg algebra describing the (Glauber) coherent states [36].

By applying q^N, called the fractal operator, to the coherent state $|\alpha\rangle$, $a|\alpha\rangle = \alpha|\alpha\rangle$, with a the annihilator operator, and $N \equiv \alpha d/d\alpha$ (as in the Foch-Bargmann representation [36]), one has the q-deformed coherent state

$$q^N|\alpha\rangle = |q\alpha\rangle = \exp\left(-\frac{|q\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(q\alpha)^n}{\sqrt{n!}}|n\rangle,$$

which is a squeezed coherent state, with $\zeta = \ln q$ the squeezing parameter. The expectation value of $(a)^n$ on $|q\alpha\rangle$, restricting then to real $q\alpha$, gives the n-th iteration stage of the fractal:

$$\langle q\alpha|(a)^n|q\alpha\rangle = (q\alpha)^n = u_{n,q}(\alpha), \quad q\alpha \to Re(q\alpha).$$

The operator $(a)^n$ acts as a "magnifying" lens watching inside the $|q\alpha\rangle$ expansion. The n-th term in this expansion is in a one-to-one correspondence with the fractal n-th stage of iteration, $n = 0, 1, 2, ..., \infty$.

Consider, as another example, the logarithmic spiral. In polar coordinates (r, θ), it is given by

$$r = r_0\, e^{d\,\theta}\,, \tag{33}$$

with $r_0 > 0$ and d real constants. Eq. (33) is represented in a log-log plot of coordinates $(\theta, \ln(r/r_0))$ by the straight line

$$\ln\frac{r}{r_0} = d\,\theta\,, \tag{34}$$

with angular coefficient $\tan^{-1} d$. The *fractal dimension* or *self-similarity dimension* is given by the slope d. Self-similarity is immediately recognized by observing that the the power $(r/r_0)^n$ implies $\theta \to n\,\theta$.

The *golden spiral* [50] is obtained for $d = d_g \equiv 2\ln\phi/\pi$, with the golden ratio $\phi = (1+\sqrt{5})/2$. In Eq. (33), $r_g(\theta) = r_0\, e^{d_g\,\theta}$ and $r_g(\theta = n\pi/2)/r_0 = e^{d_g\,(n\pi/2)} = (\phi)^n$ (see [17, 57] for the connection with Fibonacci progression).

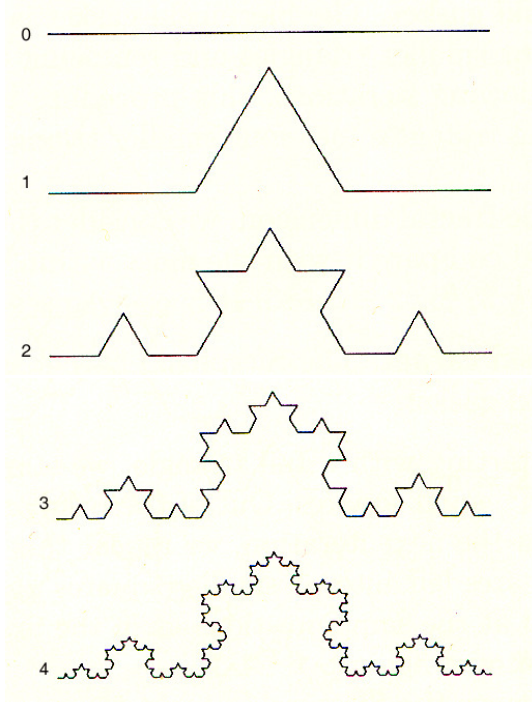

Figure 1. First five stages of the Koch curve

The parametric equations in the complex plane $z = \xi + i\,\eta = r_0\, e^{d\,\theta}\, e^{i\,\theta}$ are

$$\xi \;=\; r(\theta)\,\cos\theta = r_0\, e^{d\,\theta}\,\cos\theta\,, \tag{35}$$

$$\eta \;=\; r(\theta)\,\sin\theta = r_0\, e^{d\,\theta}\,\sin\theta\,. \tag{36}$$

Mathematical completeness requires that both components $q = e^{\pm d\,\theta}$ of the hyperbolic basis $\{e^{-d\,\theta}, e^{+d\,\theta}\}$ must be considered. The same for the (circular) basis $\{e^{-i\theta}, e^{+i\theta}\}$. Thus two linearly independent choices $z_1 = r_0\, e^{-d\,\theta}\, e^{-i\,\theta}$ and $z_2 = r_0\, e^{+d\,\theta}\, e^{+i\,\theta}$ determine points on the z-plane and z_1 and z_2 are solutions of

$$m\,\ddot{z}_1 + \gamma\,\dot{z}_1 + \kappa\, z_1 \;=\; 0\,, \tag{37}$$

$$m\,\ddot{z}_2 - \gamma\,\dot{z}_2 + \kappa\, z_2 \;=\; 0\,, \tag{38}$$

where m, γ and κ are positive real constants, "dot" denotes derivative with respect to t and, up to an arbitrary additive constant, taken to be zero for simplicity, the parametrization $\theta = \theta(t)$ is assumed

$$\theta(t) = \frac{\gamma}{2\,m\,d}\,t = \frac{\Gamma}{d}\,t\,. \tag{39}$$

Here $\Gamma \equiv \gamma/2m$. Let $\Omega^2 \equiv (1/m)(\kappa - \gamma^2/4m) = \Gamma^2/d^2$, and $\kappa > \gamma^2/4m$. We may then write $z_1(t) = r_0\; e^{-i\,\Omega\,t}\, e^{-\Gamma\,t}$, $z_2(t) = r_0\; e^{+i\,\Omega\,t}\, e^{+\Gamma\,t}$. $|\,d\theta/dt\,| = |\,\Gamma/d\,|$ gives the angular velocity by interpreting t as the time parameter.

Summarizing, z_1 and z_2, solutions of the damped and amplified harmonic oscillator, (37) and (38), describe two logarithmic spirals of opposite chirality.

The use of the two components of the basis $\{e^{-d\,\theta}, e^{+d\,\theta}\}$ is also physically justified by considering that only then the canonical formalism can be used [58]. In fact, the oscillators (37) and (38) form in that case a *closed* system. Separately considered, each one of them is a non-hamiltonian *open* system and the canonical formalism cannot be applied to their analysis.

At a microscopic level, we use the QFT canonical quantization of the couple of the two oscillators and introduce the operators a and b, respectively (the momentum subscript k is omitted for notational simplicity).

Their CCR are $[a, a^\dagger] = 1 = [b, b^\dagger]$, $[a, b] = 0 = [a, b^\dagger]$. It is convenient to define the operators $A = (1/\sqrt{2})(a + b)$ and $B = (1/\sqrt{2})(a - b)$, with usual CCR, together with the vacuum state by them annihilated:$|0\rangle$, $A|0\rangle = 0$ and $B|0\rangle = 0$, with $|0\rangle \equiv |0\rangle \otimes |0\rangle$, (see details in [57–59]).

In the Appendix D, the algebraic structure related to the two degrees of freedom a and b is briefly discussed and the generator of the time evolution of the vacuum state $|0(t)\rangle$ is given. The system Hamiltonian $H = H_0 + H_I$ [58] is

$$H = \sum_k [\hbar\Omega(A_k^\dagger A_k - B_k^\dagger B_k) + i\hbar\Gamma(A_k^\dagger B_k^\dagger - A_k B_k)] \,. \quad (40)$$

We see that H is made by the generators of the $su(1,1)$ algebra: $[K_+, K_-] = -2K_3$, $[K_3, K_\pm] = \pm K_\pm$, where, omitting again for simplicity the k subscript, $K_+ = A^\dagger B^\dagger$, $K_- = K_+^\dagger = AB$, $K_3 = (1/2)(A^\dagger A + B^\dagger B + 1)$, . The Casimir operator is $C^2 = (1/4)(A^\dagger A - B^\dagger B)^2$. Note that $[\,H_0, H_I\,] = 0$, which guaranties that the positiveness condition imposed on eigenvalues of H_0 is preserved under time evolution induced by H_I, $\mathcal{U}(t) = e^{-i(t/\hbar)H_I}$ (see Appendix D).

The time evolution $A_k(t)$ and $B_k(t)$ of the operators A_k and B_k is given by

$$A_k \to A_k(t) \;=\; A_k \cosh(\Gamma_k t) - B_k^\dagger \sinh(\Gamma_k t) \quad (41)$$

$$B_k \to B_k(t) \;=\; B_k \cosh(\Gamma_k t) - A_k^\dagger \sinh(\Gamma_k t), \quad (42)$$

and h.c.. These are the canonical (i.e. preserving CCR) Bogoliubov transformations. The explicit expression of $|0(t)\rangle$ is

$$|0(t)\rangle = \prod_k \frac{1}{\cosh(\Gamma_k t)} \exp\left(\tanh(\Gamma_k t) A_k^\dagger B_k^\dagger\right)|0\rangle \,. \quad (43)$$

It is annihilated by $A(t)$ and $B(t)$, and is normalized at each t, $\langle 0(t)|0(t)\rangle = 1$. Its decay (dissipativity) is expressed by

$$\lim_{t\to\infty}\langle 0|0(t)\rangle \propto \lim_{t\to\infty} \exp\,(-\sum_k \Gamma_k t) = 0, \quad (44)$$

with $\sum_k \Gamma_k$ assumed to be finite and positive. $|0(t)\rangle$ is a squeezed, q-deformed coherent state of $SU(1,1)$ [36, 60–63] (cf. Appendix D).

The B modes may be interpreted to describe the heat bath of the A-oscillators [58], and/or the hole of the As (and vice-versa), since destruction of A modes is equivalent to creation of B modes (and vice-versa). Remarkably,

$$\langle 0|0(t)\rangle \to 0 \;\text{ as }\; V \to \infty \quad \forall\, t\,, \quad (45)$$

$$\langle 0(t)|0(t')\rangle \to 0 \;\text{ as } V \to \infty \quad \forall\, t \text{ and } t',\; t' \neq t \quad (46)$$

for $\int d^3\kappa\, \Gamma_\kappa$ finite and positive, which shows that the representation $\{|0(t)\rangle\}$ is unitarily inequivalent to other representations $\{|0(t')\rangle,\ \forall\, t' \neq t\}$ in the $V \to \infty$ limit. The emerging picture is thus that time evolution occurs over trajectories in the manifold of uir. They are classical chaotic trajectories [64–66] and the manifold of uir is known to be a symplectic Kählerian manifold [36].

The dissipative character of the dynamics [17, 58] also suggests that entropy S plays a relevant role. The Hamiltonian H is actually the free energy $\mathcal{F} = U - TS$, with the temperature $T = \hbar\Gamma$ (the Boltzmann constant k_B is put equal to one). The entropy is then $S = 2K_2$, $K_2 \equiv -(i/2)(K_+ - K_-)$, heat contribution is $2\hbar\Gamma K_2$ ($dQ = (1/\beta)dS$) , and $T \propto \hbar\Omega/2$ = the zero point energy [6, 67]. $|0(t)\rangle$ is therefore a finite temperature state in the Thermo Field Dynamics formalism [4, 6, 58].

Incidentally, I note that the oscillators (37) and (38) form a deterministic systems à la 't Hooft. By imposing the constraint $K_2|0\rangle = 0$ on physical states the quantization formalism is obtained [67–69].

The number of A_k condensed in $|0(t)\rangle$ at t is

$$\mathcal{N}_{A_k}(t) = \langle 0(t)|A_k^\dagger A_k|0(t)\rangle$$

$$= \langle 0(t)|B_k(t)B_k^\dagger(t)|0(t)\rangle = \sinh^2 \Gamma_k t. \quad (47)$$

The Bose-Einstein distribution for $\mathcal{N}_{A_k}$ is obtained by minimizing the free energy [58]. Notice that the number of A-mode is measured by its double B-mode. In computing $\mathcal{N}_{A_k}(t)$ the only non-vanishing contributions come indeed from the $B_k(t)$-modes (acting as a "lens" to look in the physical vacuum $|0(t)\rangle$).

One also finds that A and B are entangled modes [17, 65, 70]. The *phase-mediated* correlation between A_k and B_k is measured by the linear correlation coefficient $J(N_A, N_B)$ (see Appendix D) [65, 70], with N_A, N_B the number operators. One finds $J(N_A, N_B) = 1$, i.e. maximal entanglement.

By inverting the Bogoliubov transformations (41) and (42), one may provide an example of dynamical map: the "Heisenberg field operators" A_k and B_k are expressed at the level of "physical field operators" $A_k(t)$ and $B_k(t)$ and physical vacuum $|0(t)\rangle$ at time t.

It is the coherent structure of the vacuum condensate that generates the strong (A, B)-pair correlation. A and B modes share their common phase in the $|0(t)\rangle$ vacuum without exchanging any messenger signal between them. Therefore, there is no contradiction with relativity principle (and causality). There are no "spooky force at a distance". On the other hand, the macroscopic manifestations of the condensate provide realistic observables, as realistic are, e.g., observables for a magnet, a crystal, etc. (cf. Appendix D).

Extension of the discussion to iteratively constructed fractals (deterministic and random ones) may be also done. For example, the equations in the z-plane can be written also for the case of the Koch fractal (Fig. 1) by using $q = e^{-d\theta}$, $u_1/u_0 = 1$ [16–18]. The self-similarity equation $q\alpha = 1$ is written in polar coordinates as $u_1 = u_0\alpha e^{d\theta}$, obtaining then, as done for the logarithmic spiral, the fractal

Hamiltonian, fractal free energy and the SU(1, 1) generalized coherent state.

Summing up, fractal *forms* are *macroscopic manifestations* of the microscopic coherent dynamics and they evolve (*metamorphoses*) through classical chaotic trajectories in the space of uir of the CCR.

8 Conclusions

In this paper, I have reviewed the mechanism of SBS in QFT and the coherent NG condensation in the system ground state.

The coherence of the NG condensate allows the taming of quantum fluctuations, so that the system behaves as a macroscopic quantum systems, and the order parameter M is a classical field (i.e. independent of quantum fluctuations and excitations of low momenta NG quanta (the low energy theorems, section 6)).

The formation of observable ordered patterns (*forms*), consequent to SBS, describes the process of *morphogenesis*. The space-time evolution of the order parameter describes the process of *metamorphosis*.

The dynamical rearrangement of symmetry $G \to G'$, with G and G' the symmetry groups of the interacting field equations and free field equations, respectively, has been studied in a variety of models with physical applications [3–6], with $SU(n)$, $SO(n)$, chiral $SU(2) \times SU(2)$, $SU(3) \times SU(3)$ symmetry groups, phase, chiral phase and scale invariance, with scalar isotriplets, in the $SU(3)$ $T - t$ Jahn-Teller systems [30].

It is a general result in SBS that G' is the (Inönü-Wigner) group contraction of G [10, 27–31, 71, 72]. Translations of the NG fields, generating the coherent boson condensation, thus the formation of ordered patterns (*forms*), are induced by G'.

The unitary inequivalence among the representations of CCR implies that transitions among them (phase transitions) are characterized by the appearance of singularities. Thus they are characterized by *criticality* and by the formation of topologically non-trivial extended objects, such as kinks, vortices, etc. The boson condensation becomes non-homogeneous, space-time dependent, the order parameter behaves as a 'form factor' with topological singularities. Its time-evolution describes classical chaotic trajectories in the 'space' of the phases [36, 64, 65].

Remarkably, q-deformed coherent states turn out [15–17] to be isomorph to fractal self-similar patterns (under convenient conditions, isomorphism to quantum electrodynamics can also be shown [18]).

Physical states and observables have an intrinsic localization character. One then misses infrared contributions of the order of $1/V$, with the volume $V \to \infty$. The missing of these contributions is responsible of the difference between the observed symmetry (G') and the one (G) of the basic interaction equations, namely of the metamorphosis process $G \to G'$.

Besides the case of "internal" symmetries, explicitly discussed for the ferromagnet, "external" symmetries, such as space-time continuous translational symmetries, may also be broken, as in the formation of crystals (space periodic structures) (section 4) and dissipative systems, where time translational and time-reversal symmetries (cf. Eq. (44)) are broken. The transformation K_2 belonging to $SU(1, 1)$ leaves invariant the Hamiltonian (40), but not the vacuum $|0\rangle$ ((45) and (46)) and time-translation symmetry is broken, together with time-reversal symmetry.

External (Poincaré) symmetries are broken in the presence of curved background [73], in inflationary cosmology [74–76], in the study of the cosmic microwave background (CMB) [77], loop-antiloop symmetry [78]. Extension to general covariant theory is an open question [79, 80].

In order to trigger SBS, the system must be coupled to the environment, although through a very weak coupling ϵ ($\epsilon \to 0$) (sections 3 and 4). The canonical formalism applies to closed systems. The whole {system-environment} must be then considered, which amount to the doubling of the degrees of freedom, since the environment, in the energy fluxes balance, behaves just like the time-reversed image of the system. It is at this point that temperature enters into the discussion. In fact, the vacuum $|0(t)\rangle$, which turns out to be an entangled state, is a finite temperature state in the Thermo Field Dynamics formalism [4, 6, 58]. Time evolution is irreversible due to time-reversal breakdown and the *the arrow of time* enters in the scheme*. Stability of the system is guaranteed by the minimization of free energy at each t (Appendix D).

Many of the results reviewed in this paper, and their extension to gauge field theory, are widely confirmed in experiments. They may be extended to living matter and brain studies [81–91] and to neural networks [92, 93]. They have been also extended to some open questions in the transition from syntax to semantics and the generation of meanings in linguistics [94]. The challenge is whether the coherent dynamics formalism might be applied to networks and behaviours in social studies (see [95, 96]).

Summing up, the lesson that comes from realizing that long-range correlations are dynamically generated and are responsible of the formation of ordered structures shows that any distinction or antinomy between structure and function is dissolved. The result is the transition from the "atomistic" vision of the world, to its integrated, "dynamical" vision, where not only the individual components, but also their coherent collective behaviour are considered, their "playing together" in a large orchestra...

Finally, I might close, as already done elsewhere [8, 9, 57], by observing that metamorphoses are the *manifestation* of the basic dynamical flow of evolving forms, *of the being unfolding in the kaleidoscopic variety of the existence* [9], as suggested by the passage by Darwin "*[...] in this view of life, with its several powers, having been originally breathed into a few form or into one; [...] from so simple a beginning endless forms most beautiful and most wonderful have been, and are being, evolved.* [97].

*As observed in ref. [9], *...perhaps it is not a case that in stories and legends undoing a metamorphosis (break a spell, restore a symmetry) requires a miraculous action...only the kiss of the Princess may reverse the arrow of time and let the frog go back to be the most beautiful Prince.*

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Conflicts of Interest: "The author declares no conflict of interest."

APPENDICES

Appendix A. The Goldstone theorem

Let $S^{(i)} = \sum_l S_\psi^{(i)}(x_l), i = 1, 2, 3$, denote the total generators. The $SU(2)$ algebra (9) then gives

$$[S^{(i)}, S^{(j)}] = i\epsilon_{ijk}S^{(k)}. \quad \text{(A.1)}$$

The transformations of $S_\psi^{(i)}(x)$ is, for infinitesimal θ_i,

$$S_\psi^{(i)}(x) \to S_\psi^{(i)'}(x) = S_\psi^{(i)}(x) - \theta_j\epsilon_{ijk}S_\psi^{(k)}(x). \quad \text{(A.2)}$$

The generating functional is [13]

$$W[J, j, n] = \frac{1}{N}\int [d\psi][d\psi^\dagger] \ \exp \ i\int dt\{\mathcal{L}[\psi(x)] + J^\dagger(x)\psi(x) + \psi^\dagger(x)J(x) + j^\dagger(x)S_\psi^{(-)}(x) + S_\psi^{(+)}(x)j(x) + S_\psi^{(3)}(x)n(x) - i\epsilon S_\psi^{(3)}(x)\} \ ., \quad \text{(A.3)}$$

with anticommuting fields $\psi(x)$, and sources J; j are commuting c-numbers. The normalization factor N is

$$N = \int [d\psi][d\psi^\dagger] \ \exp \ i\int dt\{\mathcal{L}[\psi(x)] - i\epsilon S_\psi^{(3)}(x)\}, \quad \text{(A.4)}$$

$S_\psi^{(\pm)}(x) \equiv S_\psi^{(1)}(x) \pm iS_\psi^{(2)}(x)$. The ϵ-term selects the condition of symmetry breakdown. The limit $\epsilon \to 0$ is taken at the end of the computation. For computation details see [13]. The Ward-Takahashi identities we need for our discussion are

$$\langle S_\psi^{(1)}(x)S_\psi^{(1)}(y)\rangle_\epsilon = \langle S_\psi^{(2)}(x)S_\psi^{(2)}(y)\rangle_\epsilon \ , \quad \text{(A.5)}$$

$$\epsilon\int d^4x\langle S_\psi^{(2)}(x)S_\psi^{(1)}(y)\rangle_\epsilon = 0 \ , \quad \text{(A.6)}$$

$$\langle S_\psi^{(1)}(y)\rangle_\epsilon = \langle S_\psi^{(2)}(y)\rangle_\epsilon = 0. \quad \text{(A.7)}$$

$$\langle S_\psi^{(3)}(y)\rangle_\epsilon = \epsilon\int d^4x\langle S_\psi^{(1)}(x)S_\psi^{(1)}(y)\rangle_\epsilon \ , \quad \text{(A.8)}$$

$$\langle S_\psi^{(3)}(y)\rangle_\epsilon = \epsilon\int d^4x\langle S_\psi^{(2)}(x)S_\psi^{(2)}(y)\rangle_\epsilon \ , \quad \text{(A.9)}$$

$$\langle S_\psi^{(i)}(x)S_\psi^{(i)}(y)\rangle_\epsilon = i\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\rho_{(i)}(p) \times \times \left(\frac{1}{p_0 - \omega_p + i\epsilon a_i} - \frac{1}{p_0 + \omega_p - i\epsilon a_i}\right) + c.c., \quad i = 1, 2. \quad \text{(A.10)}$$

where $p(x - y) = -\mathbf{p}\cdot(\mathbf{x} - \mathbf{y}) + ip_0(t_x - t_y)$, $a_i = \eta/\epsilon$; ω_p is the energy of the quasiparticle (in condensed matter physics, physical fields are also called quasiparticles). SBS thus occurs for $\omega_p = 0$ at $p = 0$, and the corresponding singularity in the Green's function signals the existence of the NG particle, the magnon. In Eqs. (A.8) and (A.9) the integration over the infinite volume selects indeed the zero-momentum in the two-point function. Since $S_\psi^{(i)}$ are hermitian, the spectral density $\rho_i(p)$ is non-negative. From Eq. (A.5) we get $\rho_1(p) = \rho_2(p) \equiv \rho$ and $a_1 = a_2 \equiv a$; states with more than one quasiparticle give the continuum contributions (*c.c.*). The result (10) is thus obtained and the Goldstone theorem is proved.

Our derivation is model independent. When the Lagrangian is explicitely given, the Bethe-Salpeter equation may be adopted for explicit computations [3, 14].

M is the local spin density in the third direction. NM is then the total spin in such a direction, with N the number of elementary spin components (lattice points). It is $\langle 0|\mathbf{S}^2|0\rangle = NM(NM + 1)$. From (A.10), for $t_k < t_l$, we find [13] $\langle 0|S^iS^i|0\rangle = \rho N$, $i = 1, 2$, for $t_k \to t_l$ (similar result for $t_l < t_k$). Then $\langle 0|\mathbf{S}^2|0\rangle = 2\rho N + (NM)^2$, $\rho = M/2$ and $a = 1$.

Intuitively, correlations among the system elementary constituents can be established provided that their de Broglie wave-length $\lambda = h/p$ extends beyond the inter-component distance d, $\lambda > d$, with $d = (V/N)^{1/3}$ at given density and temperature, where V is the volume and N the number of components in that volume, $h/(3mk_BT)^{1/2} > (V/N)^{1/3}$, where $p^2/(2m) = (3/2)k_BT$ has been used in the approximation of weakly interacting (quasi-free) components. In such a case, one expects that quantum effects may manifest (see e.g. [98]). Provided that the various tails of the λs join together in a constructive (i.e. *coherent*) interference pattern, collective assemblies (ordered patterns) are then formed by long-range correlated individual components (the singularities of the two-point functions at $p \to 0$, i.e. for $\lambda \to \infty$, in Eqs. (A.8) and (A.9)). We have thus *macroscopic quantum systems*, such as crystals, ferromagnets, etc.

Appendix B. Derivation of Eqs. (15)-(18)

Let Z be the electron wave function renormalization, and $\rho = M/2$. $A(\phi, \phi^\dagger, \chi, \chi^\dagger)$ in Eqs. (13) and (14) is given by

$$A(\phi, \phi^\dagger, \chi, \chi^\dagger) = \int d^4x[\rho^{-1/2}\chi(x)K(\overrightarrow{\partial})S_\psi^{(-)}(x) + \rho^{-1/2}S_\psi^{(+)}(x)K(-\overleftarrow{\partial})\chi^\dagger(x) + Z^{-1/2}\phi^\dagger(x)\Lambda(-\overrightarrow{\partial})\psi(x) + Z^{-1/2}\psi^\dagger(x)\Lambda(-\overleftarrow{\partial})\phi(x)] \ . \quad \text{(B.1)}$$

Denote the transformed fields by ϕ_θ, χ_θ, with $\phi_\theta(x) = \phi(x)$, $\chi_\theta(x) = \chi(x)$, at $\theta = 0$. They must satisfy the quasi-particle equations, leave invariant the S-matrix $\mathcal{S}$, and induce the transformation (A.2) of $S^{(i)}(\phi, \phi^\dagger, \chi, \chi^\dagger)$, i.e.

$$\Lambda(\overrightarrow{\partial})\phi_\theta(x) = 0, \quad \Sigma(\overrightarrow{\partial})\chi_\theta^\dagger(x) = 0, \quad \text{(B.2)}$$

$$\frac{\partial}{\partial\theta_l}\mathcal{S}(\phi_\theta, \phi_\theta^\dagger, \chi_\theta, \chi_\theta^\dagger) = 0, \quad \text{(B.3)}$$

$$\frac{\partial}{\partial\theta_l}S^i(x, \phi_\theta, \phi_\theta^\dagger, \chi_\theta, \chi_\theta^\dagger) = -\epsilon_{ilk}S^k(x, \phi_\theta, \phi_\theta^\dagger, \chi_\theta, \chi_\theta^\dagger) \quad \text{(B.4)}$$

respectively. Use now ϕ_θ, χ_θ and their h.c. in (B.1). Eqs. (B.3) and (B.4) give

$$\frac{\partial}{\partial\theta_1}\chi_\theta(x) = i\left(\frac{M}{2}\right)^{1/2}, \quad \frac{\partial}{\partial\theta_1}\phi_\theta(x) = 0, \tag{B.5}$$

$$\frac{\partial}{\partial\theta_2}\chi_\theta(x) = -\left(\frac{M}{2}\right)^{1/2}, \quad \frac{\partial}{\partial\theta_2}\phi_\theta(x) = 0, \tag{B.6}$$

$$\frac{\partial}{\partial\theta_3}\chi_\theta(x) = -i\chi_\theta(x), \quad \frac{\partial}{\partial\theta_3}\phi_\theta(x) = i\lambda_3\phi_\theta(x)\,, \tag{B.7}$$

and their h.c.. The transformations (15)-(18) are obtained by solving these equations.

Appendix C. Finite volume and infrared effects

The magnon field $\chi(x)$ is written as

$$\chi(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\chi_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}-i\omega_p t}, \tag{C.1}$$

with $\omega_p = 0$ at $p = 0$ (cf. Eq. (10)). The commutation relations are $[\chi(x), \chi^\dagger(y)]_{t_x-t_y} = \delta(\mathbf{x}-\mathbf{y})$ and the other commutators are zero. In Eq. (23) $\chi_\eta(x)$ is given by

$$\chi_\eta(x) = \frac{1}{2}\eta\int_{-\infty}^{+\infty} dt e^{-\eta|t|}\chi(x) = \frac{1}{2(2\pi)^{1/2}}\,\eta\int d^3k\delta_\eta(k)\chi_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{C.2}$$

with $\delta_\eta(p) \to \delta(p)$ for $\eta \to 0$. By using (23), Eq. (14) gives

$$S^{(i)}(y, \phi, \phi^\dagger, \chi, \chi^\dagger) = \langle S^{(i)}_\psi(y) : \exp[-iA(\phi, \phi^\dagger, \chi_t + \chi_\eta, \chi^\dagger_t + \chi^\dagger_\eta)] :\rangle =$$

$$s^{(i)}_t(y) - i\rho^{-1/2}\chi_\eta\int d^4x\langle S^{(i)}_\psi(y)K(\overrightarrow{\partial})S^{(-)}_\psi(x) : e^{-iA_t} :\rangle$$

$$-i\rho^{-1/2}\chi^\dagger_\eta\int d^4x\langle S^{(i)}_\psi(y)S^{(+)}_\psi(x)K(-\overleftarrow{\partial}) : e^{-iA_t} :\rangle \tag{C.3}$$

with $s^{(i)}_t(y) = \langle S^{(i)}_\psi(y) : e^{-iA_t} :\rangle$. A_t contains only hard momenta. The spin density operators are then [13]:

$$S^{(1)}(y) = s^{(1)}_t(y) + (1/2M)^{1/2}\,(\chi_\eta + \chi^\dagger_\eta)s^{(3)}_t(y) \tag{C.4}$$

$$S^{(2)}(y) = s^{(2)}_t(y) - i\,(1/2M)^{1/2}\,(\chi_\eta - \chi^\dagger_\eta)s^{(3)}_t(y) \tag{C.5}$$

$$S^{(3)}(y) = s^{(3)}_t(y) + (1/2M)^{1/2}\times \left[i(\chi_\eta - \chi^\dagger_\eta)s^{(2)}_t(y) - (\chi_\eta + \chi^\dagger_\eta)s^{(1)}_t(y)\right]. \tag{C.6}$$

Matrix elements of $S^{(i)}(y)$ between physical states $|i\rangle$ and $|j\rangle$ are

$$\langle i|S^{(i)}(y)|j\rangle = \langle i|s^{(i)}_t(y)|j\rangle\,, \tag{C.7}$$

which shows that the localized nature of physical states implies that infrared contributions are missing ($\eta \to 0$). For $i = 3$, Eq. (C.7) is

$$\langle i|S^{(3)}(y)|j\rangle = \langle i|s^{(3)}_t(y)|j\rangle = M\,, \tag{C.8}$$

i.e. $s^{(3)}_t(y) = M +: s^{(3)}_t(y) :$. We split also $f(x)$ in hard and soft parts, $f(x) = f_t(x) + f_\eta(x)$. $f_\eta(x) \to 0$ as $\eta \to 0$. We thus obtain Eqs. (24)-(26) and

$$[S^{(1)}_f, S^{(2)}_f] = iM\int d^3x|f(x)|^2 \tag{C.9}$$
$$+i\,(1/2)\,[f^*_\eta(x) + f_\eta(x) + f^*_{\bar\eta}(x) + f_{\bar\eta}(x)] : s^{(3)}_t :$$
$$-(1/2M)^{1/2}\left[i(\chi_{\bar\eta} - \chi^\dagger_{\bar\eta})s^{(2)}_t - i(\chi_\eta + \chi^\dagger_\eta)s^{(1)}_t\right],$$

where two cutoffs η and $\bar\eta$ are involved. Let $\bar\eta \ll \eta$. Since $|f_{\bar\eta}(x)|^2 \ll |f_\eta(x)|^2$, $f_{\bar\eta}$ and $f^*_{\bar\eta}$ can be ignored. Eqs. (27)-(29) are obtained.

Appendix D. Doubling the degrees of freedom and entanglement

The doubling of the degrees of freedom is obtained by the doubling maps $H_M \to H_M \otimes H_M$ and $\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ of state space and algebra $\mathcal{A}$, allowing to consider the two oscillators together by considering the q-deformed Hopf algebra, with coproduct $\Delta A = A \otimes q + q^{-1} \otimes A$. This is non-commutative under exchange $1 \leftrightarrow 2$, thus distinguishing between the system and its environment. The q-deformation parameter for bosons or fermions is $q = e^\theta$ or $q = e^{i\theta}$, respectively [44, 60, 62]. Non-commutative deformed Hopf algebra also enters in the discussion of non-commutative geometry [63]. $\ln q = \zeta$ acts as a "label" for the uir. Time dependence of q, e.g. for $\zeta = \Gamma t$, occurs in the study of the vacuum $|0(t)\rangle$ in the dissipative system case. Transitions between uir, namely from *form* to *form*, may be thus obtained by tuning the value of the q-parameter, and with it other characterizing parameters, such as phase and translation parameters, in a 'deterministic iterated function process' (also called 'multiple reproduction copy machine process') [50].

The time evolution operator, which is also the two mode squeezing operator [58], is

$$\mathcal{U}(t) = \exp\left(-\frac{\Gamma t}{2}\left((a^2 - a^{\dagger 2}) - (b^2 - b^{\dagger 2})\right)\right), \tag{D.1}$$

or

$$\mathcal{U}(t) = \prod_k \exp\left(\Gamma_k t(A^\dagger_k B^\dagger_k - A_k B_k)\right), \tag{D.2}$$

where the k subscript has been restored. The Hamiltonian H_I in (40) is recognized in the exponent of $\mathcal{U}(t)$ [58].

At each t, the system may be thought to be at the equilibrium and the minimization of the free energy F, $dF = 0$, gives [6, 58] $dU = \sum_k \hbar\omega_k\dot{\mathcal{N}}_k(t)dt = k_B T dS$. Here $\dot{\mathcal{N}}_k$ denotes the time derivative of the NG modes density $\mathcal{N}_k$. Thus, provided that dF is kept zero, some of the internal (kinetic) energy U may be 'invested' in ordering (with corresponding diminution of entropy S), or vice-versa, energy stored in the ordering correlations may be released (with increase in dS) to increase internal energy U.

The entanglement between the A_k and B_k modes is quantitatively measured by the linear correlation coefficient $J(N_A, N_B)$ [65, 70]:

$$J(N_A, N_B) = \frac{cov(N_A, N_B)}{(\langle(\Delta N_A)^2\rangle)^{1/2}\,(\langle(\Delta N_B)^2\rangle)^{1/2}}, \tag{D.3}$$

defined of course for $\langle(\Delta N_A)^2\rangle \neq 0$ and $\langle(\Delta N_B)^2\rangle \neq 0$. The subscripts k are omitted for simplicity. The variance is $\langle(\Delta N)^2\rangle \equiv \langle(N - \langle N\rangle)^2\rangle = \langle N^2\rangle - \langle N\rangle^2$, and the covariance is denoted by $cov(N_A, N_B) \equiv \langle N_A N_B\rangle - \langle N_A\rangle\langle N_B\rangle$. The expectation value in $|0(t)\rangle$ is denoted by the symbol $\langle **\rangle$. $\langle N_A N_B\rangle = \langle N_A\rangle\langle N_B\rangle$ for non-correlated modes and the covariance is zero. One finds $J(N_A, N_B) = 1$ for the $SU(1,1)$ coherent state $|0(t)\rangle$ given in Eq. (43). This state can be writtens also as

$$|0(\theta)\rangle = \left(\prod_k \frac{1}{\cosh\theta_k(t)}\right)\times$$

$$\left[|0\rangle\otimes|0\rangle + \sum_{\mathbf{k}} \tanh\theta_k(t)\,(|A_{\mathbf{k}}\rangle\otimes|B_{-\mathbf{k}}\rangle) + \ldots\right], \quad \text{(D.4)}$$

where $\theta_k(t) \equiv \Gamma_k t$ and $\theta \equiv \theta(t)$. The entanglement of the two modes is explicit since it cannot be factorized into the product of two single-mode states. If one introduces the dependence of the operators also on some internal degree of freedom, e.g. on spin $\mu = 1$, the pairs $\{A, B\}$ condensed in the vacuum must carry zero momentum and zero spin and states of the Bell type, i.e. $|\psi_\pm\rangle \propto \left(|A_{\mu,\mathbf{k}}\rangle\,|B_{-\mu,-\mathbf{k}}\rangle \pm |A_{-\mu,-\mathbf{k}}\rangle|B_{\mu,\mathbf{k}}\rangle\right)$, for any $\mathbf{k}$, are recognized to be included in the summation term, thus allowing the usual QM analysis in terms of Bell's theorem and disequalities. In the QFT state there are, however, also the higher power terms denoted by the dots in the in (D.4) and the $\{A, B\}$-pair correlation is due to the coherent condensate of $|0(\theta)\rangle$.

The minimization of free energy $\partial F_a/\partial\theta_k = 0$, for all k leads to [58]:

$$\beta\omega_k = -\ln\tanh^2\theta_k \quad \text{(D.5)}$$

and then to the Bose-Einstein distribution function

$$N_{A_{\mu,k}}(\theta) = \sinh^2\theta_k = \frac{1}{e^{\beta\omega_k} - 1}\,, \quad \text{(D.6)}$$

where $\beta^{-1} = k_B T$, k_B is the Boltzmann constant, T the temperature and we used for simplicity $\theta_k \equiv \theta_k(t)$. The density matrix operator is $\hat{\rho}_{\mathbf{k},\mu} = f_k^{A^\dagger_{\mu,\mathbf{k}}A_{\mu,\mathbf{k}}}$ (similar expression for the B-modes), where $f_k \equiv e^{-\beta\omega_k} = \tanh^2\theta_k$, which makes explicit the sharing of the same phase by the $\{A, B\}$-pairs, for all $\mathbf{k}$ (cf. Eqs. (43) and (D.4)).

As shown by Eq. (47), measurements (say, made by Alice) on the A-modes are actually determined by measurements (say, made by Bob) on the B-modes and they are given by expectation values on the coherent vacuum state. Thus the pair correlations have a probability distribution nature, controlled by the Bose-Einstein distribution function. Bob's measurement determines only a probability distribution for the Alice's measurements, in agreement with the usual entanglement analysis.

However, the influences of Bob's measurements on the ones by Alice at space-like distances are not due to the action of spooky forces, but to the "phase correlations" built in the coherent QFT vacuum condensate. Bob's and Alice's observations are made within the coherent background of the vacuum $|0(\theta)\rangle$ shared by them. There are no forces mediated by messanger particles, and therefore, even if correlations extend over space-like distances, there is no violation of locality (causality) and relativity principles.

On the other hand, the coherent QFT vacuum also provides realistic observables ("element of reality", such as those in condensed matter physics).

Problems with non-locality thus do not arise in the QFT infinite volume limit where phase mediated correlations are possible (the phase velocity in establishing the coherent phase correlations is not bounded by light velocity). In the QM approximation, correlations are mediated by a messenger and non-locality is in conflict with relativity principles (and causality). This is connected with the basic difference between QM and QFT mathematical structures: in QFT there exist infinitely many uir of the CAR, with different, coherent condensate contents, which do not exist in QM (the von Neumann theorem).

A further measure of the pair correlation in $|0(\theta)\rangle$ is provided by the entropy. This can be seen as follows. $|0(\theta)\rangle$ may be also written as [3, 4, 6, 58]:

$$|0(\theta)\rangle = \exp\left(-\frac{1}{2}S_A\right)|\,\mathcal{I}\rangle \;= \exp\left(-\frac{1}{2}S_B\right)|\,\mathcal{I}\rangle\,, \quad \text{(D.7)}$$

where S_A is the entropy operator for the mode A:

$$\mathcal{S}_A \equiv -\sum_{\mathbf{k},\mu}\left\{A^\dagger_{\mu,\mathbf{k}}A_{\mu,\mathbf{k}}\ln\sinh^2\theta_k - A_{\mu,\mathbf{k}}A^\dagger_{\mu,\mathbf{k}}\ln\cosh^2\theta_k\right\}. \quad \text{(D.8)}$$

and $|\,\mathcal{I}\rangle \equiv \exp\left(\sum_{\mathbf{k}} A^\dagger_{\mu,\mathbf{k}}B^\dagger_{-\mu,-\mathbf{k}}\right)|0\rangle$. A similar expression is obtained for $\mathcal{S}_B$ by replacing $A_{\mu,\mathbf{k}}$ and $A^\dagger_{\mu,\mathbf{k}}$ with $B_{-\mu,-\mathbf{k}}$ and $B^\dagger_{-\mu,-\mathbf{k}}$, respectively. We write $\mathcal{S}$ for either $\mathcal{S}_A$ or $\mathcal{S}_B$. Use of Eq. (D.7) leads to [3, 4, 6, 58]

$$|0(\theta)\rangle = \sum_{n=0}^{+\infty}\sqrt{W_n(\theta)}\,(|n\rangle\otimes|n\rangle). \quad \text{(D.9)}$$

$W_n(\theta)$ is a decreasing monotonic function of n, gives the probability of pair correlation of the two sets of $\{n\}$ modes A and B, and is given by:

$$W_n(\theta) = \prod_{k,\mu}\frac{\sinh^{2n_{\mu,k}}\theta_k}{\cosh^{2(n_{\mu,k}+1)}\theta_k}, \quad 0 < W_n < 1, \quad \text{(D.10)}$$

and $\sum_{n=0}^{+\infty} W_n = 1$. One then finds

$$\langle 0(\theta)|\mathcal{S}|0(\theta)\rangle = -\sum_{n=0}^{+\infty} W_n(\theta)\ln W_n(\theta)\,, \quad \text{(D.11)}$$

which shows that pair correlation is measured by the entropy $\mathcal{S}$.

Summing up, in QFT the coherent vacuum background is responsible of phase-mediated long-range correlations, there are no spooky forces over space-like distances and no violations of special relativity bounds.

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