Quantum mechanics: how to use Everett theory practically

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Abstract. Among theories of the physical world, quantum mechanics remains a topic of lively discussions on its so-called interpretation. For some it remains an open question to understand how deterministic equations of this theory, as established long ago, may combine with a fundamental uncertainty. We consider the process of spontaneous emission by an atom interacting with infinite number of degrees of freedom of the electromagnetic field. There is uncertainty in the evolution of the photo-emission process which was characterized as Markovian by using the equations of quantum mechanics when the decay of the atom is due to the coupling with the vacuum field. The Markovian property leads us naturally to describe spontaneous emission by using the classical Kolmogorov equation for the probability evolution of a parameter defining the state of the atom. We explain that Everett's many-worlds interpretation weld together our description, and appears therefore as a consequence of the equations of quantum mechanics, that solves in this case the riddle of deterministic equations describing random events.

1 Short history of the statistical interpretation of quantum mechanics

Quantum mechanics started with the huge step(s) forward made by Planck in 1900 when he explained [1] the spectral distribution of the black-body radiation. He did that in two steps. His first derivation was purely thermodynamical without modeling the interaction of atoms and light. At the urging of Boltzmann, Planck managed to derive the black-body spectrum from a more detailed physical model [2]. This model was based upon the idea that the energy of an oscillator instead of changing continuously, as in classical mechanics, changes by random jumps of amplitude $h\nu$, h being a new physical constant (called Planck' constant now) and ν frequency of this oscillator. It was already clear that quantum mechanics, a least for simple situations, mixes classical concepts (the harmonic oscillator) and new physics, the one depending on Planck's constant, a very small quantity when measured with units of macroscopic physics (in MKSA units its value is $6.62\ 10^{-34}$ J.s) . This constant has the physical dimension of a classical action.

The foremost example of randomness in quantum systems is the emission of photons by atoms in an excited state. A stunning example is in the paper by Dehmelt and collaborators [3] where one can see in figure 2 the time records of quantum jumps of a three level atom showing an on/off signal of fluorescence of a single ion. This records shows clearly that the jumps are quasi instantaneous and of finite amplitude as implied somehow by the wording "quantum jump". Moreover the analysis of this time records shows that the jumps occur randomly (see also [4]). At first sight this looks hard to reconcile with the fact that the wave function of an atom interacting with the electromagnetic (EM) field is the solution of well defined equations giving a fully predictive and smooth evolution. However a closer inspection shows that this is not unexpected at all, and was practically stated by Dirac, then 25 years old. In 1927, Dirac published a paper explaining how to derive Einstein's coefficients from the quantum dynamical equations for the coupled electromagnetic field (the vacuum field) and electrons in atoms [5] (the interested reader may read a simpler presentation by Fermi [6] five years later). Somehow this provided a fully rational basis for the calculation of a fundamental quantity of quantum mechanics. In particular Dirac's derivation allows to understand why the return to the ground state by emission of a photon is a Markov process with a rate of occurence $1/\gamma$ derived from the "deterministic" quantum dynamics. Because of the coupling between the electrons of the atom and the EM field, the initial condition (excited atom)+EM field is a priori not an exact eigenstate of the full system atom + radiation. Following the general principles of quantum mechanics, the system has another state with the same energy, namely the emitted photon and the atom returned to the ground state. Therefore the mixing between the two states of equal energy is possible and once the interaction between the EM field and the atom is turned on, the amplitude of the "other state" (emitted photon + atomic ground state) grows from its zero initial value. This squared amplitude grows at the beginning proportional to time, as γt. The coefficient γ is interpreted by Dirac as a rate of transition from one state to the other, later γ was named the bandwidth of the excited state and $1/\gamma$ its lifetime. Moreover Dirac notices also a very im-

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portant property, namely that the duration of the quantum jump τ_{qj} is very short, of order of the period of the emitted light [1], then we have

$$\tau_{qj} << \frac{1}{\gamma}. \tag{1}$$

This relation leads us to claim that spontaneous emission must be described using methods of non equilibrium statistical physics, namely within the frame of a "stochastically determined process" [2], as named by Kolmogorov in chapter I of Ref. [7], instead of trying to remain in a purely deterministic framework. Below we outline the basic ideas behind the Kolmogorov equation for stochastic process which evolves with continuous changes of a set of parameters including jumps. We explain how to use it concretely to describe a single atom pumped at resonance between an excited and its ground state. This part was covered already in a publication [8] to which we refer an interested reader. Additionally this publication, written with Jean Ginibre, deals also with the fairly complex 3-level case of Dehmelt experiments.

In the case of a two-level atom considered below, the intervals between successive emission of photons are independent variables because after each emission the phase of the atomic state is different from its (random) value before the jump. We emphasize that a convincing interpretation of quantum mechanics compatible both with the idea of reduction of the wave packet and the constraint of unitarity of the evolution, or of conservation of the probability, is Everett's theory [9]. In the case of the emission of a photon by an atom in an excited state, if one follows Everett's theory, at each quantum jump, a photon is emitted in the universe of the observer, and another new universe is created without emitted photon, so that the subsequent history is consistent with both disconnected universes, the one with emitted photon and the other universe without the emitted photon. We explain below how the statistics introduced by quantum mechanics amounts to make averages over all universes corresponding to different quantum jumps. Because we are discussing something related to physics and not philosophy, there are consequences of this line of reasoning in the physical and mathematical picture of processes. This relies on definite equations for probability distributions, as illustrated below.

Note that Everett's profound idea makes everything consistent, at the price of introducing a direction of time, that is not questionable. This direction of time plays the same role as the one explaining the arrow of time of thermodynamics: it represents the practical impossibility to reverse the history of a peculiar system.

[1]This is what we understand based on the sentence below Eq.(23) in Ref.[5], where Dirac states that the coefficient of the secular term he computes reaches its asymptotic value at times much longer than the period of the emitted wave. We note that Fermi did not take up this viewpoint

[2]Kolmogorov never mentioned the Markovian property of stochastic processes which are eligible to such description, although he possibly knew Markov's work in 1930, when he wrote this paper. nevertheless his definition of stochastically determined process covers those fulfilling the Markov' s property based on the set of conditional probabilities, namely $p(x_n|x_{n-1}, ...x_1) = p(x_n|x_{n-1})$

We also notice that among the so-called interpretations of quantum mechanics, Everett's theory is sometime rejected because it is seen as going against everyday experience of a single history of the universe and of every thinking individual in it. This raises an interesting issue which has been there forever: does human mind behave according to the laws of physics or is there something special about it? There is no evidence that our brain does not behave according to the laws of physics. For instance this behavior seems to be consistent with the conservation of energy and the increase of entropy to name two important laws of physics[3].

In section 2 we present the dynamical equation for the probability of a parameter set describing a system which evolves evolving through random discontinuous and also continuous phases, named below "Kolmogorov equation". This equation is then applied to the case of a single two-level atom pumped by an intense coherent field, as detailed in Sec. 3 where we deduce the statistical properties of the fluorescence radiated by the atom. In Sec. 4 we present the Kolmogorov equation for an atom illuminated by a thermal field, namely an incoherent field which destroys the correlation between the amplitudes of the two-states (commonly named " atomic coherences"). We comment finally in Sec.5 on why we believe that the statistical underpinning used in our theory fully agrees with Everett's theory of multiple universes.

2 The Kolmogorov equation for Markov stochastic process

Kolmogorov theory describes the changes in the probability distribution of a random quantity under the effect of quick jumps of the relevant variable together with a smooth dynamics (see Ref.[7] p.106-107). The best known example of such a process is the shift of position of a diffusing heavy particle due to collisions with a background of lighter atoms or molecules moving around randomly, the particle being also subject to an external force like buoyancy or due to an electric field. This yields the familiar equation of Brownian diffusion of a particle in an external field. Many examples of this situation exist in the physical world.

We plan to describe the evolution of the probability distribution of a system (in our case the quantum state of a single atom or ion) under the effect of two processes. One is deterministic and changes smoothly the values of the parameters of the system, the other one triggers quick random jumps of these parameter values. Let $\Theta(t)$ be this set of time dependent parameters with the time derivative, a function of Θ written as

$$\partial_t \Theta = \upsilon(\Theta).$$

where ∂_t is here and elsewhere for the derivative with respect to time.

[3]Once one admits that Everett's interpretation includes everything in each different universe (not only photons but also human beings etc..) and one has to admit too that thinking and feeling people have multiple lives, each one in a specific universe, even though each one believes he lives a unique life, without bifurcation, at least in the physical sense.

For this deterministic dynamics the conservation of probability yields the equation of evolution of the probability distribution $p(\Theta, t)$:

$$\partial_t p(\Theta, t) + \partial_\Theta(v(\Theta)p(\Theta, t)) = 0, \tag{2}$$

where ∂_Θ for the derivative with respect to Θ.

Kolmogorov equation adds to this equation a right-hand side representing instantaneous transitions (or jumps) occurring at random instants of time. This effect of the transitions is represented by a positive valued function $\Gamma(\Theta'|\Theta)$ which is the probability density for the system being in a given state Θ at time t, to jump towards another state Θ' in the time interval $(t, t+dt)$ much shorter than the time scale of the deterministic dynamics, but much longer than the duration of the jump (named τ_{qj} in the introduction).

The Kolmogorov equation describing both the deterministic dynamics and the jump process takes the form

$$\begin{aligned}\partial_t p(\Theta, t) + \partial_\Theta(v(\Theta)p(\Theta, t)) = \int \mathrm{d}\Theta_1 \Gamma(\Theta\,|\,\Theta_1)p(\Theta_1, t))\\ -p(\Theta, t)) \int \mathrm{d}\Theta' \Gamma(\Theta'\,|\,\Theta)\end{aligned}. \tag{3}$$

On the right-hand side the first (positive) term (or gain term) describes the increase of probability of the Θ-state due to jumps from other states to Θ. The second term represents the loss of probability because of jumps from Θ to any other state Θ'. We assume that the jump probability $\Gamma(\Theta'\,|\,\Theta)$ is smooth enough near $\Theta = \Theta'$ for not having to care about jumps from Θ to almost identical states (as it may happen for instance because of grazing two-body collisions in Boltzmann kinetic theory with a $1/r$ pair potential). The very existence of the probability transition $\Gamma(.)$ implies that we are considering a Markov process where the transition rate depends on the present state of the system only.

By integration over Θ one finds that the L^1-norm $\int \mathrm{d}\Theta \; p(\Theta, t)$ is constant (if it converges, as we assume it). The physical results we shall derive from the present Kolmogorov approach will rest on explicit calculations of time dependent correlations of the fluctuations of the system around its steady state, given by a time independent solution of Kolmogorov equation. Let $F(\Theta, t)$ and $G(\Theta, t)$ be two functions of the state of the system at time t. We are interested in the computation of the time dependent correlation $< F(\Theta, t)G(\Theta, t') >$ where the average is done on the steady state $P_{st}(\Theta)$, the time independent solution of equation (3). When $t = t'$ the correlation $< F(\Theta, t)G(\Theta, t) >$ is time independent and is simply

$$< F(\Theta, t)G(\Theta, t) >= \int \mathrm{d}\Theta P_{st}(\Theta)F(\Theta)G(\Theta).$$

The two-time correlation can be found as follows. Let $P(\Theta, t\,|\,\Theta_0)$ be the solution of Kolmogorov equation (3) with variables (Θ, t) and initial condition $P(\Theta, 0) = \delta(\Theta_0 - \Theta)$, $\delta(.)$ being Dirac delta function. The time dependent correlation writes

$$\begin{aligned}< F(\Theta, 0)G(\Theta, t) >= \int \mathrm{d}\Theta_0 P_{st}(\Theta_0)\\ \int \mathrm{d}\Theta P(\Theta, t\,|\,\Theta_0)F(\Theta_0)G(\Theta)\end{aligned}. \tag{4}$$

Our approach of single atom fluorescence by using a Kolmogorov equation is not the one used in the literature, although the quantum treatment of the spontaneous emission provides us the argument leading to the derivation of our Kolmogorov equations below in Eqs.(9) for a coherent pump field and (51) for a thermal pump field. We describe the atom+ EM radiated field system using a probabilistic point of view, which amounts to define a probability distribution for the density matrix of the system atom+emitted field, although standard treatments works with a single matrix evolving from a given initial state. In standard treatments the decay of the excited state is often represented by the addition of a non hermitian damping term to a Schrödinger-like evolution equation for the state of the atom [10]-[11]-[12]. Such a non hermitian (or damping) term yields a formal decay of the total probability that has to be balanced by adding other terms that are hard to justify, particularly when the linear damping term becomes large. This could explain why our results agree with the standard approach for small damping, but disagree for large damping.

Below we present an explicit Kolmogorov equation for a model physical problem, the spontaneous emission of a two-level atom, either isolated without any pump field, or illuminated by a coherent quasi-resonant laser field , and also by a thermal field.

3 A model physical problem: fluorescence of a single two-level atom

The spontaneous emission of a photon by an atom in an excited state was considered by Einstein and by Dirac. This introduced a quantum process with a fundamental randomness showing up in the random character of the emission. Thanks to the progress of experimental atomic physics one can observe it in a slightly more complex situation because two pump fields (and three atomic levels a,b,c) are involved in the emission process of a single ion [4]. A strong pump field, resonant with the transition $a \to b$, induces an intense fluorescence easily visible because the pump is intense, moreover a very weak field (a classical lamp) induces transitions (with very small probability) from the state $|a>$ to a state $|c>$ having a very long lifetime. The transition $|a> \to |c>$ can be seen on the fluorescence of the level b as a live stream (see Fig.7.12 of [4]) which displays instantaneous transitions. This result is interesting from the point of view of the statistical interpretation of quantum mechanics, because it paves the way to a description of spontaneous emission by using the Kolmogorov theory presented here for a two-level atom.

Let us first introduce the rate of transition γ for an atom at rest interacting with its surroundings very large volume. The environment is supposed to be the vacuum EM field which is in its ground state initially, namely what is called the vacuum state which contains an infinity of modes, with zero photon in each mode. This infinity of degrees of freedom explains why one can derive an irreversible dynamics from a formally reversible one. In this respect the situation is close to the one of a dilute gas where even though two body collisions are perfectly reversible, the resulting dynamics described by Boltzmann kinetic equation is not reversible at all because this system also contains infinitely many degrees of freedom. The infinity of degrees of freedom comes from the very large number of particles in the gas in the latter case, whereas in the case of spontaneous emission, the situation is the one of a single excited atom interacting with the infinite number of degrees of freedom of the EM field. In both cases it is impossible to return the time because of the spreading of energy among the infinitely many modes in the EM field in the case of the atomic spontaneous emission, and similarly for the dilute gas where the entropy stored in the velocity distribution of all particles diffuses irreversibly among all of them because of the H-theorem.

As written in the introduction, we know since Dirac that a quantum mechanical treatment of the interaction between the atom and the vacuum, including some approximations[4] shows that if the atom is initially in the excited state, it will emit a photon later, so that the population of the excited state decreases with time as

$$\rho_e(t) = \exp(-\gamma t) \tag{5}$$

where γ is the coefficient A introduced by Einstein for the spontaneous emission. Below we shall consider the emission of the atom in various situations within our approach. We show in particular. that when the initial state is a superposition of the excited and ground state, the transition time is longer than γ. Let us first present the Kolmogorov equation for the fluorescence of an atom illuminated by an intense monomode laser of frequency ω_L in quasi-resonance with the atomic transition.

3.1 Statistical theory of fluorescence of a two-level atom

Let us consider a two-level atom whose wave-function is of the form,

$$\Psi_{at}(t) = \cos\theta(t)\,|\,g > +ie^{-i\omega_L t+i\varphi}\sin\theta(t)\,|\,e>, \tag{6}$$

where φ is constant between two successive jumps, but changes abruptly at each jump as explained in [8]. The two atomic states are the ground state $|\,g>$ and the excited state $|\,e>$. They are such that $< g\,|\,g >=< e\,|\,e >= 1$, whereas $< g\,|\,e >=< e\,|\,g >= 0$. The factors $\cos\theta(t)$ and $\sin\theta(t)$ are there to take into account the normalization condition $\overline{\Psi}_{at}(t)\Psi_{at}(t) = 1$ $\overline{\Psi}$ being the Hermitian conjugate of Ψ. The Kolmogorov equation (3) dealing explicitly with the probability distribution $p(\theta,\varphi,t)$ for the atomic state, here indexed by the two variables θ and φ, becomes

$$\partial_t p + \frac{\Omega}{2}\partial_\theta p = \gamma\,\delta(\theta)\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi'\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta'\;p(\theta',\phi',t)\sin^2\theta' -\gamma\,p(\theta,\phi,t)\,\sin^2\theta. \tag{7}$$

The function $v(\Theta)$ in (7) is here the time derivative of $\theta(t)$, which is equal to $\Omega/2$ (at exact resonance) between two jumps, Ω being the Rabi frequency (see below). Note that possible fluctuations of Ω can be included by the addition of a diffusion term in the left hand side of Eq. (7), of the form

$$\frac{1}{2}\int_0^\infty \mathrm{d}t < \Omega(0)\Omega(t) > \;\partial_{\theta^2}p \tag{8}$$

but here this diffusion term will be assumed to have negligible effects.

The randomness of the phase ϕ is a consequence of the randomness of the time of jump. After each jump, the oscillations of the wave functions start again but at a random time at the time scale of period of the laser field, which changes very rapidly with respect to the amplitudes a_0 and a_1. After each jump, the phase values in successive time intervals are supposed to be independent variables, uniformly distributed in the interval $(0, 2\pi)$. It follows that the inter-emission intervals are statistically independent. By integrating both sides of this equation with respect to ϕ one get the Kolmogorov equation for the probability $p(\theta,t) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\; p(\theta,\phi,t)$ of the single variable θ,

$$\frac{\partial p}{\partial t} + \frac{\Omega}{2}\frac{\partial p}{\partial \theta} = \gamma\,\delta(\sin\theta)\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta'\;p(\theta',t)\sin^2\theta' -\gamma\,p(\theta,t)\,\sin^2\theta, \tag{9}$$

In the right-hand side of equation (3), the probability $\Gamma(\theta'\,|\,\theta)$ for the atom to make a quantum jump from the state θ towards the state θ' is proportional to $\delta(\sin\theta')$ (where $\delta(.)$ is the Dirac distribution) because any jump lands on $\theta' = 0$ in the interval $[-\pi/2, \pi/2]$, and this probability is proportional to $\gamma\sin^2\theta$ because it comes from the state $|\,e>$ with the squared amplitude $\sin^2\theta$, and γ is the emission rate of the atom in the excited state, that gives

$$\Gamma(\theta'\,|\,\theta) = \gamma\;\sin^2\theta\;\delta(\sin\theta'). \tag{10}$$

The calculation of any averaged physical quantity requires the knowledge of both the stationary probability distribution $p_{st}(\theta)$ and the conditional probability $p(\theta,t\,|\,\theta_0)$. The stationary distribution, derived in Sec. 3.4 of Ref.[8], is a π-periodic function which is discontinuous for $\theta = k\pi$, and smooth elsewhere as shown in Fig.1. The height of the discontinuity is given by the expression

$$p_{st}(0_+) - p_{st}(0_-) = \frac{1}{\int_0^\pi \mathrm{d}\theta\; e^{-\frac{\tilde{\gamma}}{4}(2\theta-\sin(2\theta))}} \tag{11}$$

which depends on the ratio $\tilde{\gamma} = \frac{2\gamma}{\Omega}$.

The calculation of the conditional probability $p(\theta,t\,|\,\theta_0)$ is detailed in Sec.3.3.1 of our paper

[4] the rotating wave, plus Born and Markov approximations, also named Wigner-Weiskopf approximation.

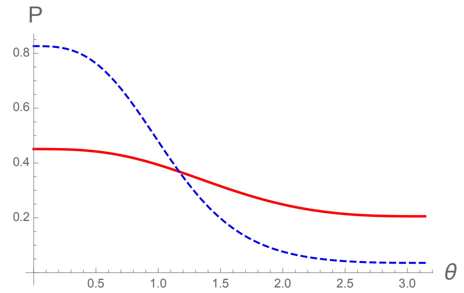

Fig. 1. Stationary probability distribution $p_{st}(\theta)$, solution of equation (9) with $\partial_t p = 0$. It is a π-periodic distribution. The solid line is for $2\gamma/\Omega = \tilde{\gamma} = 0.5$, dashed line for $\tilde{\gamma} = 2$.

[8] where t stands for $\tilde{t} = \Omega t/2$. It requires to solve the following Fredholm integral equation for $b(\tilde{t}) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' \; p(\theta', \tilde{t}) \sin^2(\theta')$,

$$b(\tilde{t}) = m(\tilde{t}) + \int_0^{\tilde{t}} \mathrm{d}t' \; b(t') l(t - t'). \tag{12}$$

where

$$m(\tilde{t}) = \int_{\pi} \mathrm{d}\theta \, p(\theta, 0) f(\theta + \tilde{t}) \alpha(\theta + \tilde{t}, \tilde{t}), \tag{13}$$

with

$$f(\theta) = \tilde{\gamma} \sin^2 \theta \quad \alpha(\theta, \tilde{t}) = e^{-\int_0^{\tilde{t}} \mathrm{d}t' \; f(\theta - t')}$$
$$l(\tilde{t}) = f(\tilde{t}) \, \alpha(\tilde{t}), \quad \alpha(\tilde{t}) = \alpha(\tilde{t}, \tilde{t}). \tag{14}$$

After solving Eq.(12) (by iterations) we get the conditional probability

$$p(\theta, \tilde{t} \,|\, \theta_0, 0) = \delta(\theta - \tilde{t} - \theta_0) \; \alpha(\theta_0, \tilde{t}) + \int_0^{\tilde{t}} \mathrm{d}t' \alpha(\theta, t') \, g(\theta - t', \tilde{t} - t'), \tag{15}$$

where $\delta(.)$ is the Dirac distribution and $g(\theta, \tilde{t}) = \tilde{\gamma} \, \delta(\sin\theta) \, b(\tilde{t})$. The knowledge of $p_{st}(\theta)$ and of the conditional probability (15) allows to derive correlation functions, by using the relation Eq.(4). An exemple of such procedure is presented in Sec. 3.3.3 below.

Note that in the present treatment of photo-emission by a single two-level atom, the probability distribution of the atomic state depends on the single continuous variable θ (and the phase φ which is constant in the time inter-emission interval), that amounts to put a probability on the atomic density matrix elements. We emphasize that this description is a way to take into account all possible trajectories emanating from the emission of a single photon in the universe of the observer, with a new value of the number of photons which are radiated in any direction at each quantum jump (and a new value of the phase of the atomic state). Such procedure allows to deal correctly with the infinite number of possible trajectories, since Boltzmann's genius lies precisely in transforming the classical statistical theory based on unknown initial conditions into statistics for an ensemble of indeterminate trajectories help to the ergodic hypothesis.

In the present case one can say, following Everett, that the probability distribution $p(\theta, t)$ allows to make averages over the states of the atom in different universes, each being labeled by a value of θ at a given time t. In this point of view, physical phenomena like the observation of a quantum state decay measured by emission of a photon, is relative to the measurement apparatus which takes place in the universe associated to the observer. At every emission of photon a new history begins, with a new phase φ, and new dynamical amplitudes represented by the right-hand side of Eqs. (24) and (25) below. In what follows the words "emitted photons" are used for the *observable photons*, namely those emitted in the universe of the observer. In summary the creation of new universes at each step defines a Markov process, which can be described by a Kolmogorov statistical picture, and cannot be considered as a deterministic process depending in a simple way on averaged quantities like population values.

3.2 Rabi oscillations

Let us go back to basic notion and explain why we set $v(\Theta) = \Omega/2$ to describe the deterministic part of the dynamics, in between two successive jumps. Let us consider an atom illuminated by coherent resonant pump field, $\omega_L = \omega_0$ where $\omega_0/(2\pi)$ is the frequency difference between the two states, the energy of the ground state being set to zero. In the time interval where there is no emission, let us write the wave-function of the atom, yet written in (6), in the form

$$\Psi_{at}(t) = a_0(t) \,|\, g > + a_1(t) \,|\, e > , \tag{16}$$

up to a global phase, not written and arbitrary, in factor of the right hand side (r.h.s.) parenthesis. The complex amplitudes $a_0(t)$ and $a_1(t)$ are functions of time. Without pump field those two amplitudes behave according to the equations of motion, $i\hbar\dot{a}_0 = 0$, and $i\hbar\dot{a}_1 = \hbar\omega_0 \, a_1$, where dots are for time derivatives and $\hbar = \frac{h}{2\pi}$. This simple writing is obtained by setting to zero the energy of the ground state, whence the lack of any right-hand side in the equation for a_0. The energy of the excited state is $\hbar\omega_0$ and the amplitude of the excited state oscillates as $e^{-i\omega_0 t}$

In presence of a small amplitude resonant pump field (with angular frequency ω_0) the above equations get a non diagonal term coupling the two amplitudes a_1 and a_0 while keeping the norm $\overline{\Psi_{at}}(t)\Psi_{at}(t) = (|a_0|^2 + |a_1|^2) = 1$. This is done with the new equations:

$$i\dot{a}_0 = -\Omega \cos(\omega_0 t - \varphi) \, a_1 \tag{17}$$

and

$$i\dot{a}_1 = \omega_0 \, a_1 - \Omega \cos(\omega_0 t - \varphi) \, a_0. \tag{18}$$

Thanks to the same sign of the real coupling terms in equations for $\dot{a}_0$ and $\dot{a}_1$ the norm is constant. In those equations $\Omega = -dE/\hbar$ is an angular frequency called the Rabi frequency, a real number proportional to the amplitude of the pump field and to the dipole moment ($d =< g|q\,\mathbf{r}\,|e >$, q being the electron charge) of the two-level atom. There does not seem to exist a general solution of the above set of equations, but a solution can be readily found in the (realistic) limit Ω small (the dimensionless condition is $\Omega/\omega_0 << 1$).

This set of equations is solved in the limit Ω small by taking as leading order solution $a_0(t) = A_0(t)$ and $a_1(t) = A_1(t)e^{-i\omega_0 t+i\varphi}$ where A_0 and A_1 depend slowly on time, namely with typical time scales much larger than $1/\omega_0$. Putting this into the equations and neglecting the terms depending on t like $e^{2i\omega t}$, one finds an equation for the slow dynamics of the amplitudes A_0 and A_1, with the solution

$$A_0(t) = \cos\theta(t) \tag{19}$$

and

$$A_1(t) = i\sin\theta(t) \tag{20}$$

with

$$\dot{\theta} = \Omega/2. \tag{21}$$

Without decay by spontaneous emission we find that the resonant pump field induces periodic oscillations of the atomic state which is in a linear superposition of the two states (excited state and ground state) during this deterministic stage. During this so-called Rabi nutation stage, the atom exchanges energy with the pump field, a process interpreted as stimulated emission and absorption, which does not radiate apart from the laser direction. If the atom is initially in the ground state, the population of the excited state evolves as $\sin^2 \Omega t/2$, and the imaginary part of the atomic dipole as $d\sin\omega t$.

The Rabi oscillations do not yield the full picture of what happens physically when an atom is resonantly pumped, because when this atom lies in the excited state it may jump back spontaneously and randomly to its ground state by emitting a photon in any direction, at the resonant frequency too. This is the phenomenon of fluorescence that we shall study within the frame of Eq.(9). Because the fluorescence of a single atom can be recorded, one can compare the predictions of a theory of this fluorescence with the experimental data. This gives a rather unique access to the consequences of some basic principles of quantum mechanics.

3.3 Statistics of photo-emission

Let us look at a simple problem, namely the probability distribution of the time intervals between two photons emitted by the atom that is resonantly pumped. This interesting question has first an interesting answer and implies also in a non trivial way the statistical principles of quantum mechanics. The random process in the present case is the point process composed of the emission times of observable photons, by the atom when it is in the excited state (in the observer universe). We shall look at its statistical property by using Eq.(9).

Considering the atomic wave function in Eq.(6), the populations of the two levels, or probabilities for the atom to be in the excited or in the ground state at time t, are respectively,

$$\rho_1(t) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, p(\theta,t)\sin^2\theta. \tag{22}$$

and

$$\rho_0(t) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, p(\theta,t)\cos^2\theta. \tag{23}$$

Their sum is one, as it should be, if $p(\theta,t)$ is normalized to one. From (9) one can derive an equation for the time derivative of $\rho_1(t)$ and $\rho_0(t)$ by multiplying (9) by $\sin^2\theta$ and by $\cos^2\theta$ respectively and integrating the result over θ. It gives,

$$\dot{\rho}_1 = -\frac{\Omega}{2}\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \sin^2\theta \frac{\partial p}{\partial\theta} - \gamma\left(\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, p(\theta,t)\sin^4\theta\right) \tag{24}$$

and

$$\dot{\rho}_0 = -\frac{\Omega}{2}\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \cos^2\theta \frac{\partial p}{\partial\theta} + \gamma\left(\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, p(\theta,t)\sin^4\theta.\right). \tag{25}$$

In the r.h.s of the rate equations (24-25), the first term, proportional to the Rabi frequency Ω, describes the effect of the Rabi oscillations, whereas the second term, proportional to γ, displays the effect of the quantum jumps responsible for the *observable* photo-emission. Because $p(\theta,t)$ includes both the fluctuations due to the quantum jumps and the streaming term, the right hand side of Eqs.(24)-(25) represents the new history beginning at each step. After integration by parts they become,

$$\dot{\rho}_1(t) = -\dot{\rho}_0(t) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, p(\theta,t)\, (\frac{\Omega}{2}\sin 2\theta - \gamma\sin^4\theta). \tag{26}$$

Note that the set of equations (24-25), or (26), is not closed. It *cannot* be mapped into equations for $\rho_1(t)$ and $\rho_0(t)$ only because their right-hand sides depend on higher momenta of the probability distribution $p(\theta,t)$, momenta that cannot be derived from the knowledge of $\rho_1(t)$ and $\rho_0(t)$. The unclosed form of (24)-(25) is a rather common situation. To name a few cases, the BBGKY hierarchy of non-equilibrium statistical physics makes an infinite set of coupled equations for the distribution functions of systems of interacting (classical) particles [13] where the evolution of the one-body distribution depends explicitly on the two-body distribution, that depends itself on the three-body distribution, etc. In the theory of fully developed turbulence, for instance, the average value of the velocity depends on the average value of the two-point correlation of the velocity fluctuations, depending itself on the three-points correlations, etc. Fortunately, one can solve the Kolmogorov equation (9) via an implicit integral equation [8], then there is generally no need to manipulate an infinite hierarchy of equations as in those examples.

To illustrate how one can use Kolmogorov equation, we derive the time dependent probability of photo-emission by a single atom, first without any pump field, then in presence of a resonant laser.

3.3.1 *Emission without pump*

We consider first an isolated atom initially in pure state $\Psi_{at}(0)$ given by Eq.(6) with $\theta(0) = \theta_0$. The solution of Eq.(9) with $\Omega = 0$ (no pump) and $p(\theta,0) = \delta(\theta-\theta_0)$, is

$$p(\theta,t) = (1-q(t))\,\delta(\theta) + q(t)\,\delta(\theta-\theta_0). \tag{27}$$

with

$$q(t) = e^{-(\gamma\sin^2\theta_0)t} \tag{28}$$

The evolution of the probability that the atom is in excited state at time t, is given by (22) and the emission of a photon occurs randomly in time with a rate,

$$\dot{\rho_1} = -\gamma \sin^2 \theta(t)\rho_1(t). \quad (29)$$

Once the atom "jumps" to its ground state, it cannot emit another photon, then the emission of a photon, if recorded, is a way to measure the state of the atom. The solution of Eq.(29) leads to the population of the excited state

$$\rho_1(t) = \sin^2 \theta_0 e^{-(\gamma \sin^2 \theta_0)t} \quad (30)$$

when taking into account the initial condition, and the photo-emission rate is,

$$\dot{\rho_1}(t) = -\gamma \sin^4 \theta_0 e^{-(\gamma \sin^2 \theta_0)t}. \quad (31)$$

The probability of photo-emission in the interval $(0, \infty)$ is the integral of $\dot{\rho_1}$

$$\int_0^\infty \gamma \sin^4 \theta_0 \, e^{-(\gamma \sin^2 \theta_0)t} dt = \sin^2 \theta_0, \quad (32)$$

which means that the final state of the coupled system atom+emitted photon field is

$$\Psi(\infty) = \sin \theta_0 \,|\, g, 1 > + e^{i\phi'} \cos \theta_0 \,|\, g, 0 > \quad (33)$$

where the indices $(1, 0)$ correspond to one and zero photon state respectively. The relation (32) means that if we consider N atoms initially prepared in a given pure state with $\theta(0) = \theta_0$, namely with total energy $N \sin^2 \theta_0 \; \hbar\omega$, we get, at infinite time, N atoms in the ground state and $N \sin^2 \theta_0$ photons of individual energy $\hbar\omega$. In the final state only a fraction of those N atoms, $N \sin^2 \theta_0$, did jump from the excited state to the ground state with the emission of a photon, the other part, $N \cos^2 \theta_0$, simply stayed in the ground state.

3.3.2 Emission of a coherently pumped atom

In the case of an atom submitted to an intense coherent pump field, the atom will emit photons at random times forming a point process. Here we assume that the process is Markovian, but more generally any process with time-dependent history, is completely characterized by its conditional intensity function $\lambda(t\,|\,\mathcal{H}_t)$, the density of points at time t, where $\mathcal{H}_t$ is the history of the emission activity up to time t, and the time interval probability distribution is given by the relation, $\ell(\tau) = \lambda(\tau\,|\,\mathcal{H}_\tau)e^{-\int_0^\tau \mathrm{d}t\lambda(t\,|\,\mathcal{H}_t)}$. In the present case the conditional intensity of the point process, which is the probability of emission of a photon at time t, only depends on the value of θ at this time, therefore one has simply

$$\ell(\tau) = \lambda(\tau)e^{-\int_0^\tau \lambda(t)dt} \quad (34)$$

From Eq.(24) we deduce the intensity of the point process at time t, conditionally to the knowledge of $\theta(t)$,

$$\lambda(t) = \gamma \sin^4 \theta(t). \quad (35)$$

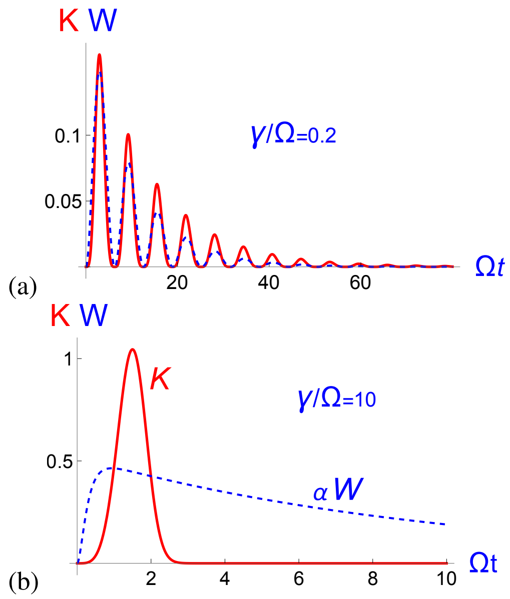

Fig. 2. Inter-emission time distribution $\ell(t)$ in two opposite cases, (a) for weak (b) for strong dissipative rate (with the respect to the Rabi frequency). The solid red curves pointed by the letter K are for our Kolmogorov statistical theory, equation (36). The blue dashed curves labelled W display the inter-emission time distribution deduced in [10] for same values of γ/Ω which are equal to 0.2 in (a) and 10 in (b). In curve (b) the dashed curve displays $\alpha W(t)$ (α is a numerical coefficient allowing to compare the behavior of the two curves on the same figure).

This result which is not so obvious, is of prime importance to derive the statistical properties of the radiated field. In this relation the exponent 4 comes from two conditions, one that the atom is in the excited state, and the other one that it emits an observable photon, as deduced from Eq.(31) describing an emission without any pump field. Assuming a photon is emitted times at time $t = 0$, the atom undergoes Rabi oscillations in between two successive emission times, that gives $\theta(t) = \Omega t/2$. Therefore the inter-emission time distribution for an atom driven by a resonant pump is given by the expression

$$\ell(\tau) = \gamma \sin^4(\frac{\Omega}{2}\tau) \;\; e^{-\gamma \int_0^\tau \mathrm{d}t \sin^4(\frac{\Omega}{2}t)}, \quad (36)$$

which gives $\int_0^\infty \ell(\tau)d\tau = 1$, as expected[5]. Let us notice that the expression (36) for $\ell(\tau)$ can be deduced directly from the Kolmogorov Eq.(9) by using a proof similar to the one detailed in Sec. 3.3.3 just below. The result is shown in Figs.2 in the two opposite limits of large and small values of the ratio Ω/γ, and compared to so-called "the delay function" derived in [10] (which has not the expected form for a Markovian process and is not normalized to unity). For the case of weak damping, or strong input field, $\gamma \ll \Omega$, the two methods agree approximately, see Fig.2-a. But they differ noticeably in the opposite case shown in Fig.2-b. In the latter case (γ bigger than the Rabi frequency) the Kolmogorov derivation gives a mean delay between successive photons of order $\tau_K = (\Omega^4\gamma)^{-1/5}$,

[5] let us note that Eq.(36) was mistaken in our previous paper [8] , where the conditional intensity of the photo-emission process was written as $\sin^2(\frac{\Omega}{2}\tau)$ instead of $\sin^4(\frac{\Omega}{2}\tau)$.

which decreases slowly as the damping rate γ increases which seems reasonable. However the authors of Ref.[10] find a delay time $\tau_Q = \gamma/\Omega^2$, a time scale much longer than the inverse of γ, as illustrated by Fig.2-(b). We hope this discrepancy will be elucidated by an experiment.

3.3.3 *Correlation fonction of the emitted field and spectrum*

Let us now use these results to give the expression of the correlation function and spectrum of the emitted field. Using Eq. (35) for the intensity of the point process at time t, conditionally to the knowledge of $\theta(t)$, we deduce that the amplitude of the emitted field is quadratic with respect to the amplitude of the excited state [6]. The correlation function of the radiated field by the atom is

$$C(\tau) =< \sin^2\theta_0 \, \sin^2\theta(\tau)\, e^{2i(\phi_0-\phi(\tau)} > e^{-i\omega_L\tau} \tag{37}$$

where the quantity inside the brackets must to be weighted by the probability $p(\theta,\phi,t)$ deduced from the Kolmogorov Eq.(7) with i.c. θ_0,ϕ_0. As written above, the set of phases $\phi_m = 1,2...$ (m indexing the inter-emission intervals) form a set of independent random variables uniformly distributed between $[0,\pi]$. It follows that the correlation $C(\tau)$ vanishes if a photon is emitted in the time interval $[0,\tau]$. Then the average weighted by $p(\theta,\phi,t)$ in Eq.(37) becomes an average of another quantity weighted by the probability $p(\theta,t)$ which obeys the Kolmogorov equation (9). This expression can be written as

$$C(\tau) =< \sin^2\theta_0 \;\sin^2\theta(\tau)\; e^{-\gamma\int_0^\tau \mathrm{d}t\,\sin^4\theta(t)\,|\mathcal{H}_t} > e^{-i\omega_L\tau}, \tag{38}$$

where the exponential $e^{-\gamma\int_0^\tau \mathrm{d}t\,\sin^4\theta(t)|\mathcal{H}_t}$ implements the constraint that zero photon is emitted in the time interval $[0,\tau]$, and the label $\mathcal{H}_t$ means, as in Sec. 3.3.2, that at each time step $[t-dt,t]$ belonging to this interval, the conditional intensity $\gamma\sin^4(t)$ for a photon to be emitted has to deduced from the Kolmogorov equation (9) with a priori an unknown the initial condition $\theta(t-dt)$. We prove just below that the two terms in the r.h.s. of Eq.(9) cancel each other at first order in dt, so that the solution of the Kolmogorov equation without the r.h.s, $\theta(t) = \theta_0 + \Omega t/2$, is the one to be put in Eq.(38) which becomes

$$C(\tau) = e^{-i\omega_L\tau}\int_0^\pi \mathrm{d}\theta_0 p_{st}(\theta_0)\sin^2\theta_0\;\sin^2(\theta_0+\Omega t/2) \times e^{-\gamma\int_0^\tau \mathrm{d}t\,\sin^4(\theta_0+\Omega t/2)} \tag{39}$$

3.3.4 *Proof*

Let us set

$$C(\tau) = e^{-i\omega_L\tau}\int_0^\pi \mathrm{d}\theta_0 p_{st}(\theta_0)\, C_0(\tau) \tag{40}$$

[6] In our paper [8] we claimed that the emitted field should be linear with respect to amplitude $\sin(\theta)\exp i\varphi$, but the linearity is odds with the energy conservation constraint through out the emission of a photon, as stated in Sec.3.3.1. Here we take up the calculation of the correlation function.

and write the integral in Eq.(38) as a Riemann sum by discretizing the interval $[0,\tau]$ into n subintervals, $\tau = n\,dt$, that gives

$$\int_0^\tau \mathrm{d}t\;\sin^4\theta(t) = dt\sum_{j=1}^{n}\sin^4\theta(j\,dt)$$

which is valid in the limit $dt\to 0$. The conditional average $C_0(\tau)$ becomes

$$C_0(\tau) = \sin^2\theta_0\int\cdots\int_0^\pi \mathrm{d}\theta_1...\mathrm{d}\theta_n\sin^2\theta_n \times p(\theta_1,,,,,\theta_n\;|\theta_0)\prod_{j=1}^{n}F(\theta_j), \tag{41}$$

where $\theta_j = \theta(j\,dt)$ and

$$F(\theta_j) = e^{-\gamma\,dt\,\sin^4\theta_j}. \tag{42}$$

As written above, the Kolmogorov Eq.(9) drives the Markovian dynamics of the variable θ representing the atomic state, therefore we have the relation

$$p(\theta_1,,,,,\theta_n\;|\,\theta_0) = \prod_{j=1}^{n}p(\theta_j\;\;|\,\theta_{j-1})$$

where the transition probability $p(\theta_j\,|\,\theta_{j-1})$ can be deduced by expanding the solution given by Eq.(15) for small dt. Setting $\tilde{t} = \Omega\,t/2$, and $\tilde{\gamma} = 2\gamma/\Omega$, hence $\gamma t = \tilde{\gamma}\tilde{t}$, we get

$$C_0(\tilde{\tau}) = \prod_{j=1}^{n-1}\int_0^\pi \mathrm{d}\theta F(\theta)p(\theta,d\tilde{t}\,|\,\theta_{0j}) \times\int_0^\pi \mathrm{d}\theta\sin^2\theta\,F(\theta)p(\theta,d\tilde{t}\,|\,\theta_{n-1}), \tag{43}$$

where θ_{0j} is the initial value of θ in the j^{th} interval.Now we have to calculate the integral

$$c_j = \int_0^\pi \mathrm{d}\theta F(\theta)p(\theta,d\tilde{t}\;|\theta_{0j})$$

by using the solution (15) of the Kolmogorov equation which becomes

$$p(\theta,d\tilde{t}\;|\theta_{0j}) = \delta(\theta-(\theta_{0j}+jd\tilde{t}))\alpha(\theta_{0j}+d\tilde{t},\,d\tilde{t}) +\gamma'\,d\tilde{t}\,\alpha(\theta,0)\,\delta(\theta)\,b(d\tilde{t}), \tag{44}$$

where the functions $b(t)$ and $\alpha(x,t)$ are given in Eqs. (12) and (14). At first order with respect to the small increment $d\tilde{t}$, it gives

$$c_j = F(\theta_{0j}+d\tilde{t})e^{-\gamma' d\tilde{t}\,\sin^2\theta_{0j}} +F(0)\gamma' d\tilde{t}\sin^2(\theta_{0j}+d\tilde{t})e^{-\gamma' d\tilde{t}\,\sin^2(\theta_{0j}+d\tilde{t})}. \tag{45}$$

Using the definition of $F(.)$ in Eq.(42), we have $F(\theta_{0j}+d\tilde{t}) = e^{-\gamma' d\tilde{t}\,\sin^4(\theta_{0j}+d\tilde{t})}$ and $F(0) = 1$, that gives at first order in $d\tilde{t}$, $c_j = 1-\gamma' d\tilde{t}\,\sin^4(\theta_{0j}+d\tilde{t})$, or

$$c_j = e^{-\gamma' d\tilde{t}\,\sin^4(\theta_{0j}+d\tilde{t})}\quad j = 1,...,(n-1)$$
$$c_n = \sin^2(\theta_{0(n-1)}+d\tilde{t})e^{-\gamma' d\tilde{t}\,\sin^4(\theta_{0(n-1)}+d\tilde{t})}\quad j = n. \tag{46}$$

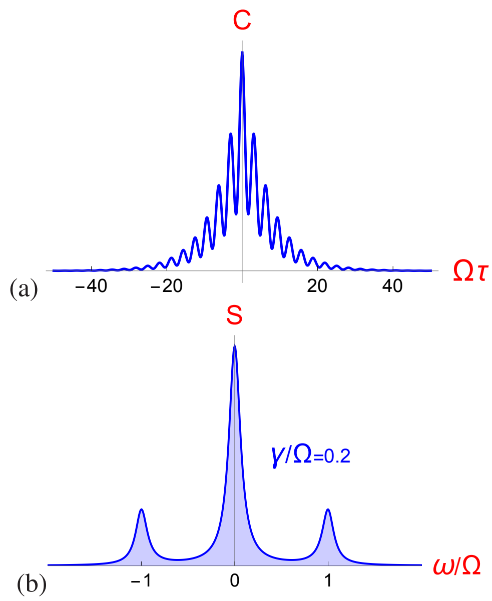

Fig. 3. (a) Scaled correlation functions versus $\Omega\tau$ without the rapid phase, defined by Eq.(48). (b) Fourier transform of $c(\Omega\tau)$, or spectrum versus the shifted frequency $\omega - \omega_L$, for small damping, $\gamma/\Omega = 0.2$.

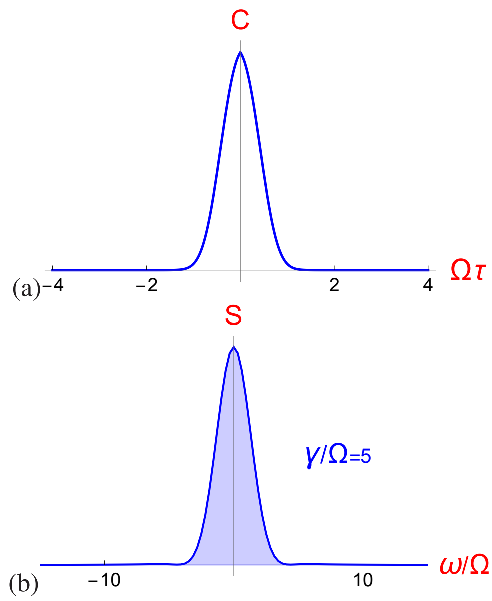

Fig. 4. (a) Scaled correlation functions of the radiated field, without the rapid phase, $c(\Omega\tau)$. (b) Fourier transform of $c(\Omega\tau)$, or spectrum versus the shifted frequency $\omega - \omega_L$, for large damping, $\gamma/\Omega = 5$.

This result shows that in the case of F functions, the product in Eq.(43) can be deduced by putting at each step $\theta_j = \theta_{j-1} + d\tilde{t}$, or returning to the physical variable $t = \Omega\tilde{t}/2$, Eq.(46) shows that one may write

$$\theta(t) = \theta_0 + \omega\, t/2 \tag{47}$$

in the expression (38) of the correlation function. The solution in Eq.(47) obeys the Kolmogorov equation (9) but with the r.h.s. equal to zero. Actually we note that each of the two r.h.s. terms of the Kolmogorov equation gives a contribution (to c_j) of order dt, but their difference is of order dt^3, so that the r.h.s. of Eq.(9) provide no contribution to c_j and finally to $C(\tau)$. In summary we found that everything happens as if the r.h.s. of the Kolmogorov equation (9) plays no role in this calculationt Finally, we find that The solution (47) depicts the deterministic stage of the dynamics , that is not surprising because the correlation function $C(\tau)$ vanishes if there is one (or more) emitted photon in the time interval $[0, \tau]$ in the universe of the observer, which means that the atom undergoes Rabi oscillations during this time in this universe.

The correlation function is drawn in Figs.(3)-(a) and (4)-(a) for small damping and large damping respectively. The curves display the correlation of the radiated field as defined by Eq.(38) but omitting the fast phase factore$^{-i\omega_L\tau}$, and scaled to the value at $\tau = 0$. In other words the curves plotted in these figures are given by the expression

$$c(\tau) =< \sin^2\theta_0\, \sin^2\theta(\tau)\, e^{2i(\phi_0-\phi(\tau))} > / < sin^4\theta > \tag{48}$$

The spectrum of the EM field radiated by the pumped atom,

$$S(\omega) = 2\,Re[\int_0^\infty \mathrm{d}\tau\, C(\tau)]\, e^{i\omega\tau} \tag{49}$$

is centered on the laser frequency ω_L. The curves in Figs.3-(b) and 4-(b) display the Fourier transform of $c(\tau)$, which is equal to the shifted spectrum $S(\omega - \omega_L)$. For small damping (or large pumping, $\Omega \gg \gamma$), the spectrum displays the well known Mollow's triplet [14] with a central peak and two smaller side peaks at frequencies $\omega_L \pm \Omega$. It has been observed in many different systems: ions, single molecules, quantum dots, and recently in a cloud of cold atoms [15]. Let us note that the spectrum derived from the Kolmogorov equation agrees with the one deduced by usual quantum theory (Bloch equations and dressed atom formalism), since the two sidebands in Fig. 3-b have an amplitude about three times smaller than the central peak, a ratio predicted in the limit of large ratio Ω/γ. For large damping, the correlation function is narrower in time (compare the abscissa of Fig.4-a with the one of Fig.3-a), and the spectrum displays a single peak centered on the laser frequency, as illustrated in Fig.4.

4 Kolmogorov equation in case of a thermal light illumination

As explained at the beginning of this article the early developments in the theory of interaction of light with atoms were devoted to the interaction with a black-body radiation so that it is not without interest to look at this case within the formalism of the Kolmogorov equation explained above. The idea for doing that is inspired by a return to the meaning of the r. h. s. of the Kolmogorov equation and its formal writing. First the effect of a thermal radiation is to trigger transitions from ground state to excited state by absorption of one photon and vice versa by stimulated emission. These transitions occur with a rate γ' that is proportional to Einstein coefficient B and to the spectral density $\mathcal{I}(\omega_0)$ of the thermal field at frequency ω_0,

$$\gamma' = \mathcal{I}(\omega_0)B. \tag{50}$$

The jumps between the ground and excited state by absorption and stimulated emission are quasi instantaneous, as in the case of the spontaneous emission of a photon by an atom in the excited state. Therefore this process is also described by the same kind of kinetic term as the spontaneous emission, but for obvious changes because the emission terms in the r.h.s. of Eq.(9) are now proportional to $\gamma + \gamma'$, and the absorption terms, proportional to γ', have to be modified so that the role of the two states, excited and ground state, is exchanged. The final result is that, in presence of this thermal radiation, equation (9) is changed into the following one

$$\begin{aligned}\frac{\partial p}{\partial t} = (\gamma + \gamma')\delta(\sin\theta) \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' p(\theta', t) \sin^2\theta' \\ -(\gamma + \gamma')p(\theta, t)\sin^2\theta \\ +\gamma'\delta(\cos\theta) \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' p(\theta', t) \cos^2\theta' \\ -\gamma' p(\theta, t)\cos^2\theta\end{aligned} \tag{51}$$

One notices that the absorption terms (the last ones proportional to γ') in this equation are derived from the r.h.s. of equation (9) by exchanging the role of the ground state and the excited state. This is obtained by taking the following kernel to describe the absorption

$$\Gamma'(\theta' \,|\, \theta) = \gamma' \;\cos^2\theta\, \delta(\cos\theta'). \tag{52}$$

which is similar to Eq.(10) but γ' replaces γ and $\cos\theta$ replaces $\sin\theta$.

The dynamical equations for the diagonal elements of the density matrix (or populations) defined in Eqs.(22)-(23), can be derived by multiplying both sides of Eq.(51) respectively by $\sin^2\theta$ and $\cos^2\theta$, and integrating this over θ, as performed above to get Eq. (24). We get

$$\dot{\rho}_1(t) = \gamma'(\rho_0 - \rho_1) - \gamma \left(\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' \; p(\theta', t) \sin^4\theta' \right), \tag{53}$$

and similarly , after multiplication by $\cos^2\theta$ and integration over θ, we get

$$\dot{\rho}_0(t) = \gamma'(\rho_1 - \rho_0) + \gamma \left(\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' \; p(\theta', t) \sin^4\theta' \right). \tag{54}$$

which displays the norm conservation.

If a steady state exists, the probability $p_{eq}(\theta)$ should be of the form

$$p_{eq} = \bar{q}\delta(\theta) + (1 - \bar{q})\delta(\pi/2 - \theta). \tag{55}$$

which satisfies Eq.(51) with $\frac{\partial p}{\partial t} = 0$. In this expression $0 < \bar{q} < 1$. Moreover using the relation (22) for the population of the excited state, we get $\bar{\rho_1} = (1 - \bar{q})$ and $\bar{\rho_0} = \bar{q}$. The equilibrium occurs if the ratio of the populations is given by the Boltzman factor

$$\frac{\bar{\rho_1}}{\bar{\rho_0}} = \exp\left(-\frac{\hbar\omega_0}{k_B T} \right). \tag{56}$$

where k_B is Boltzmann constant, T the absolute temperature of the black-body radiation (which makes up the thermal field) and $\hbar\omega_0$ the difference of energy of the excited state and of the ground state. On the other hand this ratio depends on the coefficients γ and γ'. This can deduced by looking at at the dynamical equation (53) with l.h.s. equal to zero. It gives $\gamma'(\bar{\rho_0} - \bar{\rho_1}) = \gamma(1 - \bar{q})$, so that the equilibrium occurs if one has,

$$\frac{1 - \bar{q}}{\bar{q}} = \frac{\gamma'}{\gamma + \gamma'} = \exp\left(-\frac{\hbar\omega_0}{k_B T} \right). \tag{57}$$

This defines the peculiar value of the spectral density $\bar{\mathcal{I}}$ which permits the equilibrium state to be reached, because the rate γ' is proportional to this component of the power spectrum, see Eq.(50), although γ is given by Dirac formula in function of the wave functions of the two states of the atom. The steady state occurs for a pump field spectral density depending on the temperature of the black body radiation as

$$\bar{\mathcal{I}} = \frac{\gamma/B}{\exp\left(\frac{\hbar\omega_0}{k_B T}\right) - 1}. \tag{58}$$

Notice that the off-diagonal part of the density matrix, given by

$$\rho_{01} = e^{-i\omega_0 t + \varphi} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' \; p(\theta) \cos\theta \sin\theta,$$

vanishes in the case of a thermal pump field, as expected, whereas it is nonzero in the case of coherent pumping.

4.1 Solution of the Kolmogorov equation (51)

The dynamical equation (51) for the probability $p(\theta, t)$ is easier to solve than Eq.(9) since there is a single differential term $\partial p/\partial t$. Let us set as above

$$b(t) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' \; p(\theta', t) \sin^2(\theta'). \tag{59}$$

and let us define the two terms appearing in the r.h.s. of Eq.(51),

$$f(\theta) = \gamma \sin^2\theta + \gamma' \tag{60}$$

$$g(\theta, t) = b(t) \left((\gamma + \gamma')\delta(\sin\theta) - \gamma'\delta(\cos\theta) \right) + \gamma'\delta(\cos\theta). \tag{61}$$

Equation (51) is now on the form

$$\frac{\partial p}{\partial t}(\theta, t) = -p(\theta, t) f(\theta) + g(\theta, t). \tag{62}$$

The solution is

$$p(\theta, t) = p(\theta, 0)e^{-f(\theta)t} + \int_0^t \mathrm{d}t' \; g(\theta, t')e^{-f(\theta)(t - t')}. \tag{63}$$

We shall take, as above, a pure state as initial condition $p(\theta, 0) = \delta(\theta - \theta_0)$, namely a superposition of ground and excited state. The integral term in the r.h.s. of Eq.(63) contains $b(t)$ which obeys an implicit integral equation deduced by integrating over θ the quantity $p(\theta, t)\, \sin^2\theta$ with p given by Eqs. (63) and (60)-(61). We get the integral equation

$$\begin{aligned}b(t) = \sin^2\theta_0 e^{-f(\theta_0)t} - \gamma' \int_0^t \mathrm{d}t' \; b(t')e^{-(\gamma+\gamma')(t - t')} \\ + \frac{\gamma'}{\gamma + \gamma'}(1 - e^{-(\gamma+\gamma')t})\end{aligned} \tag{64}$$

4.2 correlation function

let us assume that the system atom+ thermal pump field+ radiated field is in steady state, which occurs if the incident intensity fulfills relation (58). In this case the probability for the atomic variable θ doesn't depend on time, and is given by Eq.(55). The correlation function of the emitted field is still given by Eqs.(38) and (41) which include the phase dependence (a random variable, which changes after each quantum jump). Following the derivation detailed in Sec. 3.3.4 and using the same notations, Eq. (45) becomes

$$c_j = \int_\pi \mathrm{d}\theta(\bar{q}\delta(\theta) + (1-\bar{q})\delta(\pi/2-\theta)\,)\,e^{-\gamma\,dt\,\sin^4\theta} \quad (65)$$

for $j = 1, .., (n-1)$, and $c_n = \int_\pi \mathrm{d}\theta(\bar{q}\delta(\theta) + (1-\bar{q})\delta(\pi/2 - \theta)\,)\,e^{-\gamma\,dt\,\sin^4\theta}\sin^4\theta$. It gives

$$c_j = \bar{q} + (1-\bar{q})e^{-\gamma\,dt} \quad j = 1...n-1$$
$$c_n = (1-\bar{q})e^{-\gamma\,dt}. \quad (66)$$

where t is the physical time since $\gamma t = \tilde{\gamma}\tilde{t}$ as noted above. After putting the product of these c_j in Eq.(41), we obtain

$$C_0(\tau) = \sin^2\theta_0\,e^{-\gamma(1-\bar{q})\tau}\,(1-\bar{q}). \quad (67)$$

Finally the correlation function of the field radiated by the atom (in the universe of the observer), defined by Eq.(38), becomes

$$C(\tau) = e^{-i\omega_0\tau}(1-\bar{q})^2\,e^{-\gamma(1-\bar{q})\tau} \quad (68)$$

when the condition (57) is fulfilled, namely for

$$1-\bar{q} = \gamma'/(2\gamma'+\gamma),$$

so that Eq.(68) is given by the expression

$$C(\tau) = e^{-i\omega_0\tau}\left(\frac{\gamma'}{2\gamma'+\gamma}\right)^2 e^{-\frac{\gamma\gamma'}{2\gamma'+\gamma}\tau} \quad (69)$$

at thermal equilibrium.

The spectral density, which is the Fourier transform of this expression, is a Lorentzian function, the width of which depending on temperature according to Eq. (57).

5 Statistical picture of the emission of photons and Everett's theory

In the nineteen fifties Everett presented [9] a theory of quantum physics that is sometimes considered as philosophical speculations without connection with real physics. In this section we explain how Everett's ideas are needed to understand the statistical effects observed in fluorescence. One fundamental idea of Everett when applied to the problem of emission of photons by a single atom is that, after the emission the universe splits in two: in the universe of the observer the photon has been emitted and new universe is created in which no photon is emitted, the latter one being compatible with relativity theory (expanding with light velocity). A 3D schema illustrating a possible set of trajectories coming from successive emission times t_i, is drawn in Fig. 5, see captions, with the aim of illustrating that the various universes do not overlap.

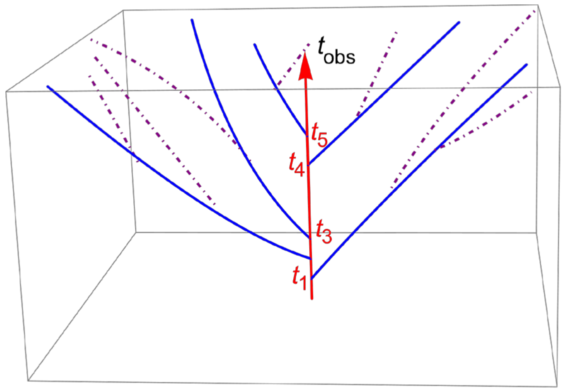

Fig. 5. Schema of the possible trajectories of the atom emitting photons at times t_i, $\{i, 1, 5\}$ in the universe of the observer. The vertical red line with arrow is the trajectory seen by the observer, where the atom makes Rabi oscillations between t_i and t_{i+1}. The solid blue lines stemming from each t_i, illustrate the successive splitting of the observer trajectory (universe) in two parts. On the blue trajectory (virtual for the observer) no photon is emitted at t_i, but Rabi oscillations go along, until a photon is emitted in this universe. This occurs at the crossing points of blue curves with purple dotted-dashed curves. At these crossing points the "virtual blue trajectory" splits into two parts, one (blue) with an emitted photon and another one (purple) with no photon emitted.
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By different universes one implies two related things. First the histories of the two universes are a priori different after the emission event. This does not imply a big difference of course between the two universes because their initial condition at the instant of the emission are almost the same, but for the absence of presence of a single photon. Mathematically the two universes are separated because their density matrices do not overlap: in one of the universe the photon number for the emitted photon is one and it is zero in the other. Therefore one can define in each universe a density matrix which will evolve in the future without any relationship with the density matrix of the other universe. Concretely the emission of photons occurs on a very short time, of the order of the period of the atomic motion, which is also the period of the EM waves emitted by the atom in its excited state. Therefore there is a continuous emission of photons and so a continuous creation of new Everett's universes. In the case of fluorescence what happens in all universes can be described only statistically, the statistics being carried over all universes existing at a given time. This defines a kind of super statistics because probability distributions are defined themselves over an object with a statistical meaning, namely the density matrix for the quantum state in the universe under consideration. In the case of a pumped two-level atom, this density matrix depends on the angle θ and the phase φ, so that the probability distribution is a probability depending on these two variables only.

Contrary to other theories of fluorescence of a single atom, such a statistical theory has a built-in statistical structure which is, we believe, necessary to describe the randomness of the emission process. Such a randomness is intrinsic to the emission process, and it represents bifurcation from one Everett universe to two, every time a photon is emitted. Let us notice that from an experimental point of view, it would be impossible to make averages over all possible universes, because of the lack of overlap of the density matrices attached to the different universes. Therefore one is practically restricted to make time averages in the universe where the observer lies, that poses the question of ergodicity: are these two averages identical? The case of pumped atom could be interesting to investigate from the point of view of ergodic theory.

6 Summary and conclusions

The purpose of this paper was to show first how the view of quantum mechanics as a statistical theory grew from the very beginning of this theory and how things got clarified by Everett's bold idea of multi-universes. We felt also that it was not sufficient to discuss those questions abstractly, as points of metaphysics (here understood in its original meaning by Aristotle, just after physics, although the word "metaphysics" is not by Aristotle) but as a point of physics. This was demonstrated on a model problem with a non trivial "solution", namely a model where the statistical analysis needs to be done very carefully even though its mathematics is actually fairly simple. This model has also the interest to be connected with the problems raised first by the founding fathers focused on the interaction of matter and light. We thought it was instructive to show how the general concepts of quantum mechanics as a statistical theory work "concretely" in a given case. By "concretely" we mean in a probabilistic mathematical framework using probability distributions and their evolution equation. The hope is that this discussion of a specific model brings more light on this difficult subject than a more abstract discussion.

To take a wider view of the problem, it is of interest to recall that *classical* mechanics took a long time to become a theory not requiring a specific philosophical approach. For instance in the middle of the eighteen century a fierce debate took place between the deterministic view of Newtonian mechanics, where the initial conditions determine the future, and the Maupertuis view where the evolution is dictated by the minimization of an action integral with boundary conditions at the two ends of the time interval. Nowadays we know that the two pictures of classical mechanics are equivalent, but "philosophically" there is an obvious difference between the two "interpretations" of classical mechanics (via differential equation or via a minimization principle with ends fixed). So it is not that surprising, seen on the long term evolution of Science, to observe that after a century of development still some difficulties and misunderstanding remain in the interpretation of quantum mechanics. This work tried to contribute, not only to the exposition of the topic but also to put forward the idea that the treatment of quantum mechanics as a statistical theory is not such a trivial matter and that this should be done carefully, somehow by using at least implicitly the general principles of statistical theory, which could be seen perhaps as the "metaphysical side" of this physics.

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