Computation of aberration coefficients for plane-symmetric reflective optical systems using Lie algebraic methods

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Abstract. The Lie algebraic method offers a systematic way to find aberration coefficients of any order for plane-symmetric reflective optical systems. The coefficients derived from the Lie method are in closed form and solely depend on the geometry of the optical system. We investigate and verify the results for a single reflector. The concatenation of multiple mirrors follows from the mathematical framework.

1 Introduction

The calculation of Seidel coefficients of rotationally symmetric optical systems is well known in the optical design literature [1, 2]. Seidel sums provide a clear insight into the contribution of each interface of the system to the third order transverse ray aberrations at the image plane and provide analytic expressions for the aberration coefficients. We aim to derive similar closed form expression for aberration coefficients of plane-symmetric optical systems. To construct the optical mapping for plane-symmetric reflective systems, we employ Lie algebraic methods [3–5]. This mapping systematically produces the analytic aberration expansion up to the desired order in terms of initial position and direction.

2 Analytic Ray-Tracing Equations for Reflecting Interfaces

In classical Hamiltonian optics, the propagation of light rays in an optical medium with refractive index $n(\boldsymbol{q}, z)$ is determined by the Hamiltonian equations, where the Hamiltonian H is given by [6]:

$$H(\boldsymbol{q}, \boldsymbol{p}, z) = -\sqrt{n(\boldsymbol{q}, z)^2 - |\boldsymbol{p}|^2}. \tag{1}$$

Here, $\boldsymbol{q}$ are the space and $\boldsymbol{p}$ are the momentum coordinates of the ray. Since at each z-plane, $z =$ const., a ray is fully determined by $(\boldsymbol{q}, \boldsymbol{p})$, they represent the phase space as in classical Hamiltonian mechanics, with the only restriction that $|\boldsymbol{p}| \leq n(\boldsymbol{q}, z)$.

In order to follow the ray path from object to image plane, we need to be able to reflect the ray at a plane-symmetric mirror, to rotate the coordinate system such that the optical axis ray (OAR) remains aligned with the z-axis and to propagate the reflected ray further with respect to the rotated system. For instance, reflection at an interface $z = \zeta(\boldsymbol{q})$ can be described by its vectorial formulation as:

$$\boldsymbol{p}' = \boldsymbol{p} - 2\frac{\nabla\zeta(\bar{\boldsymbol{q}})}{1 + |\nabla\zeta(\bar{\boldsymbol{q}})|^2}(\boldsymbol{p} \cdot \nabla\zeta(\bar{\boldsymbol{q}}) - p_z), \tag{2}$$

where $\boldsymbol{p}'$ are the momentum coordinates after reflection and $\bar{\boldsymbol{q}}$ are the position coordinates of the point of impact between the ray and the interface. Position coordinates before and after reflection are projected onto the z-plane at the point of impact of the OAR and the coordinate system rotates in order to keep the z-axis aligned with the OAR before and after its reflection, respectively, see Figure 1.

The optical axis ray, with coordinates $\boldsymbol{q} = \boldsymbol{0} = \boldsymbol{p}$, will have reflected coordinates in the rotated system $\boldsymbol{q}^R = \boldsymbol{0} = \boldsymbol{p}^R$ and all the considered rays will lie in a neighbourhood of it. As such, the mapping $\mathcal{M}$ representing reflection with rotation of the coordinate system can be expanded around the OAR.

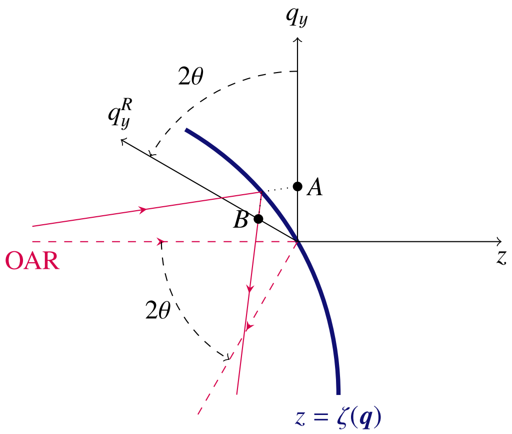

Figure 1. The reflection operator maps the point A of the incoming ray to the point B of the outgoing ray in the rotated coordinate system. The OAR (dashed) is mapped from the origin to itself. The incidence angle of the OAR on the reflector is equal to θ.

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3 Lie Algebraic Tools

The Poisson bracket is an operator $[\cdot,\cdot]$, which maps any pair of functions f, g in $(\boldsymbol{q}, \boldsymbol{p})$ to a single function of $(\boldsymbol{q}, \boldsymbol{p})$, denoted by $[f, g]$:

$$[f, g] := \frac{\partial f}{\partial \boldsymbol{q}} \cdot \frac{\partial g}{\partial \boldsymbol{p}} - \frac{\partial f}{\partial \boldsymbol{p}} \cdot \frac{\partial g}{\partial \boldsymbol{q}}.$$

The Poisson bracket turns the space of polynomials on phase space into a Lie algebra [3–5]. We define the linear Lie operator $[f, \cdot\,]$ associated with f acting on g as follows:

$$[f, \cdot\,]g = [f, g].$$

If the argument of $[f, \cdot\,]$ is a vector-valued function $\boldsymbol{g}$, then it acts component-wise on $\boldsymbol{g}$. The main Lie algebraic tool we use is the Lie transformation. The Lie transformation $\exp([f, \cdot\,])$ associated with f and generated by $[f, \cdot\,]$ is defined as:

$$\exp([f, \cdot\,]) = \sum_{k=0}^{\infty} \frac{[f, \cdot\,]^k}{k!}. \tag{3}$$

The powers in (3) follow the recursive definition

$$[f, \cdot\,]^0 = I, \quad [f, \cdot\,]^k = [f, [f, \cdot\,]^{k-1}], \quad k = 1, 2, \ldots.$$

It can be proven that the propagation and the reflection with rotation mappings are representable as infinite concatenations of Lie transformations [3, 7] of the form

$$\mathcal{M} = \exp([g_2, \cdot\,]) \exp([g_3, \cdot\,]) \exp([g_4, \cdot\,]) \cdots. \tag{4}$$

Here, the generators g_2, g_3, etc. are homogeneous polynomials on phase space of degree $2, 3$, etc. For instance, the second-order generator h_2 for free propagation in a medium of constant refractive index n over a distance d along the optical axis reads:

$$h_2(\boldsymbol{p}) = -\frac{d}{2n} |\boldsymbol{p}|^2. \tag{5}$$

The polynomial g_2 associated with the combined action of reflection and rotation to the new coordinate system reads:

$$g_2(\boldsymbol{q}, \boldsymbol{p}) = \frac{\cos(\theta)}{\rho} q_x^2 + \frac{1}{R \cos(\theta)} q_y^2, \tag{6}$$

where ρ and R are the sagittal and tangential radii of curvature, respectively, and θ is the incidence angle of the OAR on the mirror, see Figure 1.

4 The Fundamental Element and Optical Systems

By representing each mapping in the form (4) and concatenating propagation from an intermediate object plane, reflection by a tilted interface with rotation to the outgoing reference system and, lastly, propagation to an intermediate image plane, we can describe a so-called fundamental optical element by its mapping of the form (4). If we are interested in aberration terms up to third order, then the final mapping $\mathcal{M}_{\mathrm{el}}$ corresponding to one single optical element will be

$$\mathcal{M}_{\mathrm{el}} = \exp([\tilde{g}_2, \cdot\,]) \exp([\tilde{g}_3, \cdot\,]) \exp([\tilde{g}_4, \cdot\,]), \tag{7}$$

where all the information about the optical element is contained in the coefficients of $\tilde{g}_2, \tilde{g}_3, \tilde{g}_4$, that are thus dependent on the geometry of the reflector.

By concatenating multiple optical elements, using the mathematical tools described in [3–5], it is possible to derive a mapping $\mathcal{M}_{\mathrm{tot}}$ that describes the complete optical system up to third order aberrations, which has the same form as in Eq. (7). We therefore have

$$\mathcal{M}_{\mathrm{tot}} = \exp([t_2, \cdot\,]) \exp([t_3, \cdot\,]) \exp([t_4, \cdot\,]), \tag{8}$$

where the coefficients of the generators t_i are related to the geometry of the complete system.

5 Results

We have been able to recover the results recently presented in [8], where the authors determine the fourth order surface expansion coefficients of a reflector. The requirement for such reflector is that all field-independent aberrations, i.e., $\boldsymbol{p} = \boldsymbol{0}$ for incoming rays, are zero up to third order. By constructing the optical mapping of the form (7) and investigating its action on the initial phase space coordinates, we reproduced exactly the results presented in [8, Table 3].

We proceeded similarly for the case where we required that $\boldsymbol{q} = \boldsymbol{0}$ at the object plane for incoming rays and zero third order aberrations. The spherical ellipsoid is known to be a perfect imager for an object and image point lying at its respective foci [9]. The surface coefficients we calculated from the requirement $\boldsymbol{q} = \boldsymbol{0}$ agree exactly with the surface expansion of the corresponding spherical ellipsoid.

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