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Padé resummation of divergent Born series and its motivation by analysis of poles

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Abstract. The Born series is in principle a powerful way to solve electromagnetic scattering problems. Higher-order terms can be computed recurrently until the desired accuracy is obtained. In practice, however, the series solution often diverges, which severely limits its use. We discuss how Padé approximation can be applied to the Born series to tame its divergence. We apply it to the scalar problem of scattering by a cylinder, which has an analytical solution that we use for comparison. Furthermore, we improve our understanding of the divergence problem by analyzing the poles in the analytical solution. This helps build the case for the use of Padé approximation in electromagnetic scattering problems. Additionally, the poles reveal the region of convergence of the Born series for this problem, which agrees with actual calculations of the Born series.

1 Introduction

Forward electromagnetic scattering problems can be solved by expanding the electromagnetic field as a perturbation series: the Born series. Its terms are computed recurrently, starting from the zeroth order solution. So one goes from solving a difficult problem to solving a series of easy problems. The higher-order corrections can be added to the zeroth order field until the desired accuracy is achieved. That is, if the series converges. However, the Born series diverges in many cases of practical interest.

We discuss the approach presented in Ref. [1], where Padé approximation is used to tame the divergence of the Born series and retrieve a useful solution. As an example, we apply it to the problem of scattering by an infinitely long cylinder. We analyze the poles in the exact solution to understand the divergence problem. This allows us to determine for which parameters the Born series will converge.

2 Born-Padé method

We consider scattering problems for which the scalar formulation is rigorous. Given a polarization, either the electric or magnetic field has only one nonzero component. That component $U(\mathbf{r})$ then satisfies the inhomogeneous Helmholtz equation:

$$\left[\nabla^2 + k_0^2\right] U(\mathbf{r}) = -k_0^2 \Delta\varepsilon_r(\mathbf{r}) U(\mathbf{r}), \tag{1}$$

where k_0 is the wave number in vacuum and $\Delta\varepsilon_r(\mathbf{r}) = \varepsilon_r(\mathbf{r}) - 1$ the relative permittivity contrast relative to the background (chosen to be vacuum/air). We can consider $-k_0^2 \Delta\varepsilon_r(\mathbf{r}) U(\mathbf{r})$ as an induced source that acts as a perturbation superposed on the background.

The perturbation solution is obtained by writing $U(\mathbf{r})$ as a power series in a perturbation parameter σ, i.e.,

$$U(\mathbf{r}, \sigma) = \sum_{\ell=0}^{\infty} U_\ell(\mathbf{r}) \sigma^\ell. \tag{2}$$

We also add σ in front of the source term to vary the perturbation strength. The terms are computed recurrently, starting from the zeroth order field, an incident plane wave. Then summing the terms one by one is only possible if the series converges. Alternatively, we can recast the terms into a different representation that does not suffer from divergence. We use Padé approximants [2]:

$$P_N^M(\sigma) = \frac{\sum_{m=0}^{M} A_m \sigma^m}{1 + \sum_{n=1}^{N} B_n \sigma^n}, \tag{3}$$

with $M + N + 1$ unknown Padé coefficients A_m and B_n. By equating the Born and Padé representations, one can derive a linear system of $M + N + 1$ equations which yield A_m and B_n when solved. This procedure is repeated for each point in space. The power of the approximants lies in their ability to represent poles, contrary to power series.

3 Application to an infinitely long cylinder

For a normally incident plane wave scattering off an infinitely long cylinder, the solution is invariant along the cylinder's axis. The problem is effectively 2D and Maxwell's equations decouple into TE and TM polarization. Here, we consider a TE-polarized plane wave, so E_z is the only nonzero component of the electric field and it satisfies eq. (1). Figure 1 shows the geometry of the problem, with the incident plane wave propagating in the $+x$-direction. The problem has an analytical solution in terms

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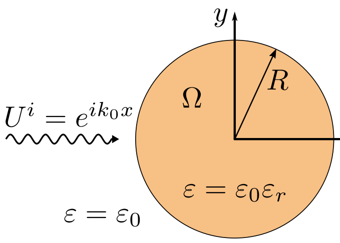
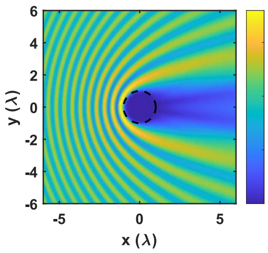


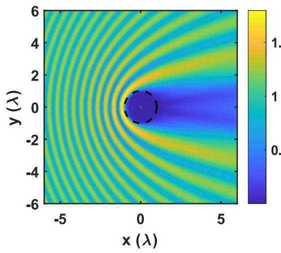
Figure 1: Geometry of the 2D problem of a cylinder. The incident plane wave U_i represents E_z, as we consider TE polarization.

of cylindrical modes, i.e., as an infinite sum of Bessel and Hankel functions [1, 3].

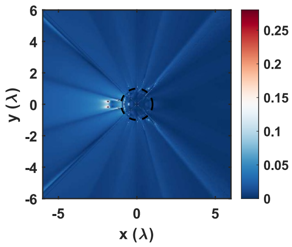
We showcase the Born-Padé method for these values: $\lambda = 400$ nm, radius $R = 400$ nm, and the relative permittivity of silver ($\varepsilon_r = 30.8 + 4.30i$). We compare the most accurate Padé approximant P^{12}_{12} to the analytical solution in fig. 2. We are able to retrieve the solution from a strongly diverging Born series. Namely, the magnitude of the series' terms grows with a factor of 5.67.



(a) Analytical solution.



(b) Padé approximant P^{12}_{12}.



(c) Absolute error.

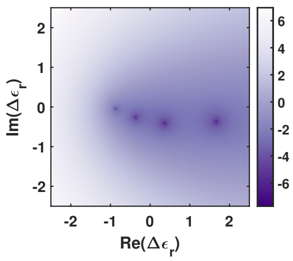
Figure 2: Born-Padé solution for a silver cylinder of radius 400 nm at a wavelength of 400 nm. The plots display the absolute value. The axes are relative to the wavelength.

4 Poles and convergence

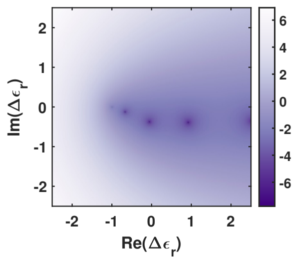
Since the Born series is a Taylor series, it converges inside a disk in the complex plane of σ. The radius of the disk is determined by the pole closest to the point about which the series is expanded. To find the poles we follow the derivation of the analytical solution, by writing the general solution in- and outside the scatterer and requiring their continuity on the cylinder's boundary. This results in a 2x2 linear system of equations for each cylindrical wave mode. The determinant of the system ends up as the denominator in the analytical solution, so the poles are where the determinant vanishes. Hence, for each cylindrical mode m, we set the determinant to 0:

$$H^{(1)\prime}_m(k_0R)J_m(k_0\sqrt{\varepsilon_r}R) - \sqrt{\varepsilon_r}H^{(1)}_m(k_0R)J'_m(k_0\sqrt{\varepsilon_r}R) = 0, \quad (4)$$

where H and J are Hankel and Bessel functions, respectively. Figure 3 shows eq. (4) as function of $\Delta\varepsilon_r = \varepsilon_r - 1$ for $R = 400$ nm and $\lambda = 400$ nm. The dark spots indicate where the determinant approaches 0, and thus where the poles are. The closest pole, out of all modes m, is



(a) Mode $m = 0$.



(b) Mode $m = \pm 1$.

Figure 3: Determinant for a cylinder with radius 400 nm at $\lambda = 400$ nm plotted in the complex plane of $\Delta\varepsilon_r$ (logarithm of absolute value).

$\Delta\varepsilon_r = -0.060 - 0.373i$ in the $m = \pm 1$ modes. Its distance to the origin is $|\Delta\varepsilon_r| = 0.377$, which thus is the radius of convergence. When actually computing the Born series, convergence is indeed seen only for values of $\Delta\varepsilon_r$ inside a disk with that radius.

Why do we look at the convergence in terms of $\Delta\varepsilon_r$ instead of σ? Since σ always appears in the Born series as a product $\sigma\Delta\varepsilon_r$, we can equally well set $\sigma = 1$–which we do anyways because that value corresponds to our problem–and consider the convergence as function of $\Delta\varepsilon_r$.

Furthermore, we see all poles are in the lower half plane and thus have negative imaginary part, corresponding to materials with gain. Also, setting the determinant to 0 is equivalent to solving eq. (1) in the absence of an incident field. We thus interpret the poles as a nonphysical cylinder radiating in the absence of a source. Interestingly, the divergence in physical cases is thus due to poles at nonphysical values. For obliquely incident waves, the poles could have physical values of $\Delta\varepsilon_r$, but that would still be nonphysical due to the infinity of the cylinder [4].

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