

# Hadronic light-by-light contribution to the anomalous magnetic moment of the muon: The role of scalar resonances in a holographic model of QCD

Luigi Cappiello<sup>1,2,\*</sup>

<sup>1</sup>Dipartimento di Fisica “Ettore Pancini”, Università di Napoli ‘Federico II’, Via Cintia, 80126 Napoli, Italy

<sup>2</sup>INFN-Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio 6, 80126 Napoli, Italy

**Abstract.** We review the evaluation of scalar mesons contribution to the hadronic light-by-light piece of the muon anomalous magnetic moment, using a holographic model of QCD. We evaluate the contributions of the lightest, sub-GeV scalars  $\sigma(500)$ ,  $a_0(980)$  and  $f_0(980)$  and their associated towers of excited states. Our results point at a negative contribution, overwhelmingly dominated by the  $\sigma(500)$  meson, that we estimate at  $a_\mu^{\text{HLbL,S}} = -9(2) \cdot 10^{-11}$ , in very good agreement with recent determinations from dispersive analyses.

## 1 Introduction

The Fermilab measurement of the muon anomalous magnetic moment [1], based on the first data run, has confirmed the previous BNL E871 result [2], with a similar precision. Compared to the most recent theory determination [3] (based on Refs. [4–23]) the discrepancy with the theory expectation is now beyond  $4\sigma$ . In few years, Fermilab precision will be improved by a factor 4, thereby reaching the projected 0.14ppm.

On the theory side, hadronic contributions are currently the focus of attention. Rather surprisingly, the most serious issue concerns now the hadronic vacuum polarization contribution (HVP), with an evident discrepancy between increasingly precise lattice simulations (see, e.g., Ref. [24]) and data-driven analyses [12, 13] and low-energy phenomenology [25–28].

The hadronic light-by-light (HLbL) shows instead a reasonable level of precision, with an overall agreement between data-driven dispersive analysis [29, 30] and lattice simulations [22, 31], still having room for more precise determinations in the near future.

## 2 Holographic QCD

In a parallel theoretical approach, some hadronic resonances contributions to HLbL have been evaluated using five-dimensional Lagrangian-based models using techniques inspired by the AdS/CFT correspondence [32–34]. When compactified to four dimensions, these models have a natural interpretation as hadronic models of QCD with an infinite number of states for the different meson channels, as it is expected in the large- $N_c$  limit of QCD [35, 36]. These

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\*e-mail: [luigi.cappiello@na.infn.it](mailto:luigi.cappiello@na.infn.it)

HQCD models [37–42] are provided with the following relevant features: (i) at high energies, they are conformal invariant and match the QCD asymptotic behaviour of correlators, reproducing leading short distance (SD) constraints of perturbative QCD; (ii) spontaneous chiral symmetry breaking can be implemented, so that they contain a pion multiplet, together with all the consequences of chiral dynamics built in; (iii) chiral anomaly is consistently implemented at all energy scales; (iv) in a minimal settings, HQCD models contain very few free parameters (e.g. the size of the extradimension, the 5D gauge coupling), which can be fitted to physical quantities, making the HQCD model rather predictive (or actually over constrained); (v) correlators of QCD quark currents, relevant to the evaluations of hadronic contributions to  $a_\mu$  can be obtained almost analytically [43–45].

Thus, HQCD models appear especially suited to address conceptual issues in simplified but consistent settings, in particular those related to the duality between hadronic contributions at high energies and QCD short-distance constraints. A clear example, shown in Refs. [46, 47], was the nontrivial combination of the Goldstone and axial-vector contributions, dictated by anomaly matching at all energy scales, that naturally resolved a long-standing puzzle [15, 20, 21] concerning the saturation of short-distance constraints with hadronic states in the HLbL. The holographic models also predict a more sizeable axial-meson contribution than previously estimated (see, e.g., Ref. [48]). This conclusion has been shown to hold even in the presence of quark-mass corrections [49]. It is worth mentioning that HQCD models allow simple expressions for the HVP contribution [50], however the ballpark of values is large, showing a strong dependence on the models and the chosen values of their parameters.

### 3 Scalars contributions from HQCD

The generalized consensus is that the scalar contributions are modest, largely dominated by the  $\sigma(500)$  resonance and with the opposite sign with respect to the Goldstone- and axial-vector contributions. As is usual for HQCD models, apart from the contribution of the lightest, sub-GeV scalars, our model allows us to estimate the contribution of the infinite towers of excited scalar states, and thereby get an idea of the uncertainties that can be ascribed to heavier, not experimentally accessible, scalar states.

All HQCD model treat resonances as narrow-width states, and we are forced to do so even for the  $\sigma(500)$  whose width is quite big. This raises the question of whether its contribution to the HLbL, as predicted by the model, is reliable. In order to estimate the  $\sigma(500)$  parameters (e.g., its mass and its decay width into photons), the most rigorous approach uses dispersive methods including coupled channels applied to  $\gamma\gamma \rightarrow \pi\pi$  (see, e.g., Refs. [51–53]) or, more recently, to  $\gamma\gamma \rightarrow \pi\pi$  and  $\gamma\gamma \rightarrow KK$  data [54, 55]. To the best of our knowledge, the set of values closest to our parametrization are the ones provided in Ref. [56] from a fit to  $\gamma\gamma \rightarrow \pi\pi$  data, as we shall discuss in the numerical analysis.

#### 3.1 The 5D action

The action of our HQCD model reads [57]:

$$S[L_M, R_M, X] = \int d^5x \sqrt{g} \mathcal{L}(L_M, R_M, X) + S_{CS}, \quad (1)$$

where

$$\begin{aligned} \mathcal{L}(L_M, R_M, X) = & -\lambda \text{tr} \left[ F_{(L)}^{MN} F_{(L)MN} + F_{(R)}^{MN} F_{(R)MN} \right] + \rho \text{tr} \left[ D^M X^\dagger D_M X - m_X^2 X^\dagger X \right] \\ & + z \delta(z - z_0) V(X) + \zeta_+ \text{tr} \left[ X^\dagger X F_{(R)}^{MN} F_{(R)MN} + X X^\dagger F_{(L)}^{MN} F_{(L)MN} \right] + \zeta_- \text{tr} \left[ X^\dagger F_{(L)}^{MN} X F_{(R)MN} \right] \end{aligned} \quad (2)$$

and

$$S_{CS} = c \int \text{tr}[\omega_5(L) - \omega_5(R)]. \quad (3)$$

The Chern-Simons term makes sure that the model correctly reproduces the effects the axial anomaly. Its explicit expression in terms of the gauge fields reads  $\omega_5(L) = LF_{(L)}^2 + \frac{i}{2}L^3F_{(L)} - \frac{1}{10}L^5$ , where the wedge product of forms is implicitly understood. The same conventions and definitions apply to the right-handed sector.

The five-dimensional background metric is anti-de Sitter,

$$g_{MN}dx^M dx^N = \frac{1}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2), \quad (4)$$

where  $\mu, \nu = (0, 1, 2, 3)$ ,  $M, N = (0, 1, 2, 3, z)$  and  $\eta_{\mu\nu}$  has a mostly negative signature. For convenience, we have normalized the AdS curvature to unity. The fifth dimension is taken to be compact, with  $\epsilon < z < z_0$ .

The first term in Eq. (2) collects the Yang-Mills term for the 5D gauge fields  $L_M$  and  $R_M$ . Together with the Chern-Simons term, this action was used in Ref. [47] to describe the contribution of axials and Goldstone bosons to the HLbL, with  $F_{(L)MN} = \partial_M L_N - \partial_N L_M - i[L_M, L_N]$ ,  $L_M = L_M^a t^a$  and  $\text{tr}(t^a t^b) = \frac{1}{2}\delta_{ab}$ , where  $t^a$  are the eight Gell-Mann matrices extended with  $t^0 = \mathbf{1}_3/\sqrt{6}$ . The second term in Eq. (2) contains the complex scalar field  $X = X^{a\prime}$  transforming as a bifundamental of  $U(3)_L \times U(3)_R$ ,  $X \rightarrow g_L X g_R^\dagger$ , with  $D_M X = \partial_M X - iL_M X + iX R_M$ .

We have also introduced a potential term, localized on the infrared boundary [40]:

$$V(X) = \frac{1}{2}\mu^2 \text{tr}[X^\dagger X] - \eta \text{tr}[(X^\dagger X)^2]. \quad (5)$$

The last two additional terms in Eq. (2), generate the  $S\gamma\gamma$  interaction and are therefore essential to obtain a prediction for the single scalar exchange contributions to the HLbL.

## 4 Numerical analysis and discussion

Here we summarize some of the steps of the numerical analysis done in [57], where the explicit analytic expressions for the various physical quantities obtained in the HQCD model are presented.

The model contains nine parameters: six coefficients of the different terms in the 5D action (2) ( $\lambda, c, \rho, \zeta_\pm, m_X$ ), the size of the fifth dimension  $z_0$  and two parameters from the scalar boundary potential ( $\mu, \eta$ ). Actually, for the scalar contributions to the HLbL, only the combination  $\zeta = \zeta_+ + \frac{1}{2}\zeta_-$  is relevant. The parameters of the boundary potential, can be traded for the quark condensate  $\langle \bar{q}q \rangle$  and  $\gamma$ . The five-dimensional scalar mass  $m_X$  is fixed to the value dictated by the AdS/CFT correspondence,  $m_X^2 = -3$ .

We obtain the values of  $\zeta$  and  $\rho$  by matching the QCD OPE short-distance constraint for the three-point function  $\langle S V V \rangle$  and the decay width of the lowest-lying scalars into two photons. Experimentally, the partial decay widths of the  $f_0(980)$  and  $a_0(980)$  are well known and given by [58]

$$\Gamma_{\gamma\gamma}^{(f_0)} = 0.29_{-0.06}^{+0.11} \text{ keV}; \quad \Gamma_{\gamma\gamma}^{(a_0)} = 0.30(10) \text{ keV}. \quad (6)$$

For the  $\sigma(500)$ , however, rescattering effects are dominant and the determination is less precise. The most recent analyses suggest  $\Gamma_{\gamma\gamma}^{(\sigma)} = (1.3 - 2) \text{ keV}$  [52, 55, 59].

To evaluate the scalar contributions to the HLbL in a realistic way, we need to introduce flavour breaking. Following what was done in [47] for the case of Goldstone and axial-vector

towers, we consider independent copies of the original Lagrangian for each of the different light scalar states, with a threefold set of parameters associated to the three neutral scalar towers, whose lightest states are the  $\sigma(500)$ ,  $a_0(980)$  and  $f_0(980)$ . For simplicity, we will assume a flavour-invariant quark condensate  $\langle \bar{q}q \rangle = (-260 \text{ MeV})^3$ , which is consistent with the phenomenological determinations at 1 GeV. The size of the fifth dimension is inversely proportional to the vector meson mass scale as and the value  $z_0 = (322 \text{ MeV})^{-1}$ , guarantees that the first vector multiplet lies at  $m_\rho = 775 \text{ MeV}$ .

The parameters  $\gamma$ ,  $\rho$  and  $\zeta$ , will be flavour dependent, with each  $\gamma$  fixed by the mass of the lightest scalars. A flavour-independent short-distance constraint can be used to fix the ratio  $\zeta_j/\rho_j \simeq -7.72$ , while another constraint links  $\rho_j$  to  $\gamma_j$  and  $\Gamma_{j\gamma\gamma}$

The model has limitations as a model of scalar mesons. The parameter  $\rho$ , for instance, has been determined in previous papers [38, 40, 60] through the asymptotic behaviour of the two-point correlator

$$\delta^{ab}\Pi^{SS}(q^2) = i \int d^4x e^{iq \cdot x} \langle 0|T \{j^a(x)j^b(0)\}|0 \rangle, \quad (7)$$

where  $j^a(x) = \bar{q}(x)t^a q(x)$ . At large Euclidean momenta, the HQCD model successfully reproduces the QCD OPE  $Q^2 \log Q^2$  behaviour. However, the coefficient would be matched with  $\rho = N_c(8\pi)^{-2}$ , while the constraint used predict  $\rho_j \sim \mathcal{O}(10^3)$ . Such numerical mismatches at  $N_c = 3$  are, however, to be expected in a minimal model. Moreover, such kind of HQCD model the predicted spectrum of excited states wick does not show a Regge behaviour, and the  $g_{SPP}$  couplings of the lightest scalars with Goldstone pairs give results of  $\mathcal{O}(10 \text{ MeV})$ . This is the same range that was found in Ref. [61] using a similar holographic model, and it falls short of the experimental results by 2 orders of magnitude.

Despite these limitations, the strategy to fix the parameters is meant to ensure that the model is phenomenologically sound, at least for a determination of the scalar contribution to the HLbL and it works especially well for the  $a_0(980)$  and  $f_0(980)$ , which are rather narrow resonances. Instead, the  $\sigma(500)$  cannot be considered a narrow state and one could argue that the description of this state in the model is not reliable. The most rigorous formalism to extract its parameters combines coupled Roy-Steiner equations with information from chiral perturbation theory (see, e.g., Refs. [17, 51, 52, 54, 55]). In our formalism, however, using the values for the mass and partial decay width extracted with this procedure would be inconsistent. The closest parametrization to our approach, that we have found in the literature, is the one of Ref. [56], in which a Breit-Wigner model was employed to fit  $\gamma\gamma \rightarrow \pi^0\pi^0$  data. The results of two different fitting strategies gave

$$m_\sigma = 547(45) \text{ MeV}; \quad \Gamma_{\sigma\gamma\gamma} = 0.62(19) \text{ keV}, \quad (8)$$

$$m_\sigma = 471(23) \text{ MeV}; \quad \Gamma_{\sigma\gamma\gamma} = 0.33(07) \text{ keV}, \quad (9)$$

where in the second line constraints from chiral perturbation theory were used. The interpretation of the partial widths above should be taken with care, but for our purposes this is not relevant. The important point is that the parameters above fit the experimental data.

Using the strategy outlined above to determine the model parameters for the  $\sigma(500)$  contribution, we find  $a_\mu^S = -8.11 \cdot 10^{-11}$  and  $a_\mu^S = -8.42 \cdot 10^{-11}$ , respectively, depending on whether one takes Eq. (8) or Eq. (9) as input. These values are not only compatible with each other but are rather similar of the one obtained from a dispersion relation analysis,  $a_\mu^S = -9(1) \cdot 10^{-11}$  [17, 62], a strong indication that our strategy for the  $\sigma(500)$  contribution is reliable.

It is difficult, however, to come up with a reasonable estimate of the uncertainty associated to our number for the  $\sigma(500)$  contribution to the HLbL. The mass and partial decay width in

**Table 1.** Results for the scalar contributions to  $a_\mu^{\text{HLbL}} \times 10^{11}$  for the set of values described in the main text.

	$n = 1$	$n = 2$	<b>Total</b>
$a_\mu^S(\sigma)$	-8.5(2.0)	-0.07(2)	-8.7(2.0)
$a_\mu^S(a_0)$	-0.29(13)	-0.025(10)	-0.32(14)
$a_\mu^S(f_0)$	-0.27(13)	-0.025(9)	-0.29(14)
$a_\mu^S$	-9(2)	-0.12(4)	-9(2)

Eqs. (8) and (9) are strongly correlated. However, in Ref. [56], no information is given of the correlation matrix. Our estimate

$$a_\mu^S(\sigma) = (-8.5 \pm 2.0) \cdot 10^{-11} \quad (10)$$

is therefore orientative, but we believe that it correctly captures the right order of magnitude for the uncertainty. Instead, the contributions of  $a_0(980)$  and  $f_0(980)$  can be computed in a less problematic way, and we find:

$$a_\mu^S(a_0) = -0.29(13) \cdot 10^{-11}; \quad a_\mu^S(f_0) = -0.27(13) \cdot 10^{-11}, \quad (11)$$

in agreement with recent work [62, 63], and compatible with earlier estimates [64, 65] (see, however, the remarks in Ref. [62]).

The model also provides a prediction for the contribution of the whole tower of scalar excitations. The results for the first excitation states together with the total contributions of the three scalar towers are shown in Table 1. They amount to an overall 3% correction to the sub-GeV scalar contribution. As expected, the dynamics beyond a few GeV is of little numerical importance.

Taking all the previous points into account, we estimate the scalar contribution to the HLbL at

$$a_\mu^S = -9(2) \cdot 10^{-11}, \quad (12)$$

with roughly 90% of this number coming from the  $\sigma(500)$ . The error budget is likewise dominated by the  $\sigma(500)$ , whereas scalar states above the GeV scale contribute no more than 15% to the total uncertainty. This number is compatible with the one estimated with the Nambu-Jona-Lasinio model,  $a_\mu^S = -7(2) \cdot 10^{-11}$  [66, 67], with dispersive analyses [62], and also in agreement with the values reported in Refs. [68–70].

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