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From PT quantum mechanics to PT holography

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Abstract. $\mathcal{PT}$ quantum mechanics studies the physics of non-Hermitian Hamiltonians having an anti-linear involution denoted by $\mathcal{PT}$ as a symmetry. We extend the concept of $\mathcal{PT}$ non-Hermitian quantum theory to gauge-gravity duality. Non-Hermiticity is introduced via boundary conditions in asymptotically AdS spacetimes. We identify how key properites such as the $\mathcal{PT}$ phase transition are implemented in the gauge-gravity duality. Then we move on to study time dependent systems (holographic quantum quenches) and discuss how Hermitian and non-Hermitian time evolution arise in $\mathcal{PT}$ quantum mechanics and how it is implemented in the gauge-gravity duality. A key feature is that the non-Hermitian time evolution violates the null energy condition (NEC) and speculate about possible applications.

1 Introduction

Quantum mechanics is based on the axiom that observable quantities are represented by Hermitian operators. It comes then as a surprise that some seemingly non-Hermitian Hamiltonians can have real energy spectra: [1–4] found that the underlying reason for this is that such Hamiltonians have an anti-linear Z_2 symmetry. This is commonly denoted as $\mathcal{PT}$. The reason is that in many cases it can be thought of as the product of parity $\mathcal{P}$ and time-reversal $\mathcal{T}$. The study of such Hamiltonian has raised much interest and is reviewed in the recent book [5]. A basic property of a $\mathcal{PT}$ symmetric Hamiltonian is the existence of two regimes separated by a $\mathcal{PT}$ critical point. In the so-called $\mathcal{PT}$ symmetric regime the eigenvectors of the Hamiltonian are simultaneous eigenvectors of $\mathcal{PT}$ which implies that the energies are real. In the $\mathcal{PT}$ broken regime the eigenvectors of the Hamiltonian come in doublets under the $\mathcal{PT}$ symmetry and the energy eigenvalues are complex conjugate to each other.

In the $\mathcal{PT}$ symmetric regime the Hamiltonian is actually pseudo-Hermitian, i.e. there exists a Hermitian similarity transformation η such that $\eta H \eta^{-1} = h$ with $h^\dagger = h$ being Hermitian [6–9].

A key observation for the implementation of $\mathcal{PT}$ in holography (the gauge-gravity) duality is that the similarity transformation η can be thought of switching on certain non-Hermitian operators by analytically continuing in the couplings of these operators. The simplest possible example is a complex operator O. One can add this to a given (Hermitian) Hamiltonian via a coupling $\lambda O + \bar{\lambda} O^\dagger$ with the coupling λ being just a complex number. If we write the coupling as $|\lambda| e^{i\alpha}$ we can think of the non-Hermitian theory to arise from analytically continuing in the angle $\alpha \to i\beta$. Now we have two real couplings coupling λ, $\bar{\lambda}$ and the

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resulting theory is not-Hermitian! But as long as these couplings obey the constraint $\lambda\bar{\lambda} > 0$ we can think of it as arising from the analytic continuation in the angle α of the Hermitian theory. This indeed describes the $\mathcal{PT}$ symmetric regime in which eigenvalues are real. The $\mathcal{PT}$ critical point is the point where $\lambda\bar{\lambda} = 0$. If however $\lambda\bar{\lambda} < 0$ this regime can not be reached by analytic continuation in the angle α and eigenvalues will come in complex pairs.

It has to be emphasized that despite the the fact that the $\mathcal{PT}$ symmetric regime is mathematically equivalent to quantum mechanics with a Hermitian Hamiltonian the physical interpretation is very different. A standard Hermitian Hamiltonian describes a closed isolated quantum system whereas a $\mathcal{PT}$ symmetric Hamiltonian describes an open quantum system in which the non-Hermitian couplings describe a dynamical equilibrium with an exact balance between inflow and outflow of the quantity measured by the operator O.

Let us suppose now that we can manipulate the in- and outflow in a time dependent manner while maintaining exact balance at all times. In this situation the non-Hermitian Hamiltonian at any time t would be pseudo-Hermitian at all times

$$H(t) = \eta(t)^{-1} h(t) \eta(t) \,. \tag{1}$$

The question is if such a time dependent system is still equivalent to a Hermitian system and answer is "it depends".

We can insist that the time evolution is determined by a Schrödinger equation based on the Hamiltonian (1)

$$i\partial_t = H|\Psi\rangle \,. \tag{2}$$

This time evolution does not respect unitarity. It is easy to see that the product of two vectors in the Hilbert space is not conserved using (2)

$$\partial_t(\langle\Psi|\Phi\rangle_\eta) = 2\langle\Psi|(\eta^{-1}\partial_t\eta)|\Phi\rangle_\eta \,. \tag{3}$$

A cure to this problem is as follows. One defines the modified time dependent Schrödinger equation

$$i(\partial_t + \eta^{-1}\dot{\eta})|\Psi\rangle = H|\Psi\rangle \,. \tag{4}$$

It is easy to check that indeed now time evolution is unitary.

A natural interpretation is that formally we have introduced a time component of a gauge field for the gauged Schrödinger equation

$$iD_t|\Psi\rangle = i(\partial_t - iA_t)|\Psi\rangle = H|\Psi\rangle \,. \tag{5}$$

Formally treating the time dependent similarity transformation as a gauge transformation one sees that the Hermitian system described by h, $|\psi\rangle$, $A_t = 0$ is gauge equivalent to $H = \eta^{-1}h\eta$, $|\Psi\rangle = \eta^{-1}|\psi\rangle$ and $A_t = i\eta^{-1}\partial_t\eta$. A more detailed discussion of the time dependent pseudo-Hermitian systems can be found in the recent reviews [10] and [11] and a more general situation allowing also space dependence has recent been investigated in [12].

It is worth emphasizing that this is not a true gauge transformation. A gauge transformation does not change in any way the physical system under consideration. It merely describes it in a different language. In contrast here we do have to carefully engineer a quantum system with balanced in- and outflow at each point in time and in addition switch on the coupling corresponding to $A_t = i\eta^{-1}\partial_t\eta$ in order to achieve a time evolution that is equivalent to the one of a closed quantum system with Hermitian Hamiltonian!

2 PT Holography

Gravity with asymptotically anti-de Sitter boundary conditions allows a dual interpretation in terms of strongly coupled quantum systems. Useful introductions to applications fo this to strongly coupled quantum many body physics are [13, 14]. We will now review the holographic dual to a non-Hermitian quantum field theory [15, 16]. The key is that in the holographic duality the asymptotic values of the fields encode the couplings of the dual field theory. The model of holographic field theory is defined by the action

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R + \frac{6}{L^2} - \overline{D_\mu\phi}D^\mu\phi - m^2\phi\overline{\phi} - \frac{V}{2}\phi^2\overline{\phi}^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{6}$$

The important point is that it contains a complex scalar field ϕ dual to a complex scalar operator O. According to the holographic dictionary the boundary values of the scalar field encode the couplings to the operator O. The non-Hermitian holographic field theory will then be defined by choosing appropriate boundary conditions.

We use coordinates such that the boundary is located at $u = 0$, and the boundary coordinates (v, x_1, x_2) are collectively denoted as x

$$ds^2 = -f dv^2 - \frac{2}{u^2}e^g du dv + S^2(dx_1^2 + dx_2^2) \tag{7}$$

where (f, g, S) are functions of (u, v) whose near boundary expansion recover the AdS geometry

$$\lim_{u\to 0}(u^2 f) = 1\,, \quad \lim_{u\to 0} g = 0\,, \quad \lim_{u\to 0} S = 1\,. \tag{8}$$

With the choice $m^2L^2 = -2$ for the mass of the scalar field its asymptotic expansion is

$$\phi(u, x) = u\phi_1(x) + O(u^2)\,, \tag{9}$$

$$\bar{\phi}(u, x) = u\bar{\phi}_1(x) + O(u^2)\,. \tag{10}$$

The leading terms ϕ_1, $\bar{\phi}_1$ acts as the couplings of the scalar operator O, $\bar{O}$ of conformal dimension $\Delta = 2$. The non-Hermitian theory is obtained with the boundary conditions

$$\phi_1(x) = (1 - \xi(x))\varphi_1(x)\,, \tag{11}$$

$$\overline{\phi}_1(x) = (1 + \xi(x))\varphi_1(x)\,. \tag{12}$$

We can also define the analytically continued (hyperbolic) angle

$$e^{\beta(x)} = \sqrt{\frac{1 + \xi(x)}{1 - \xi(x)}}\,. \tag{13}$$

For $\xi(x) = 0$ we recover the standard Hermitian boundary conditions.

So far we have only considered the boundary conditions on the scalar fields. As we emphasized in the introduction, such a procedure will result in a non-unitary time evolution as soon as the parameter ξ or equivalently β becomes time dependent. A unitary time evolution can be obtained by also introducing a gauge field. In our case and generalizing to full spacetime dependence this gauge field would be given by

$$A_\mu = i\partial_\mu\beta = \frac{i\partial_\mu\xi}{1 - \xi^2}\,. \tag{14}$$

We will use such a gauge field to show explicitly that the time evolution in that case is equivalent to a usual time evolution with Hermitian boundary conditions obeying $\phi_1^* = \bar{\phi}_1$ where the star denotes complex conjugation.

2.1 Time independent case ξ = const.

We will consider first solutions at zero temperature. In the dual gravitational theory these are solutions which are regular in the interior and do not contain a black hole. These solutions correspond to domain wall geometries interpolating between two AdS regions of different cosmological constant. This is so as long as $|\xi| < 1$. At the critical points $|\xi| = 1$ the scalar field equation actually decouples from the metric and one finds a simple AdS-4 geometry as solution. For $|\xi| > 1$ on the other hand one find two solutions with complex conjugate metrics for the same boundary conditions. This is expected since this $|\xi| = 1$ are precisely the critical points separating the $\mathcal{PT}$ symmetric from the broken regime. Gravity backgrounds containing a black hole are dual to the field theory at finite temperature (the Hawking temperature of the black hole). In this case real solutions exist even for $|\xi| > 1$. It turns out however that they are all unstable since a quasinormal mode has moved into the upper half-plane. Despite much numerical effort no stable ground state could be found in this regime [15]. This is a qualitatively new behavior that has not appeared before in non-Hermtian quantum systems or quantum field theories. It would be interesting to see if it persists in a weakly coupled $\mathcal{PT}$ theory at finite temperature.

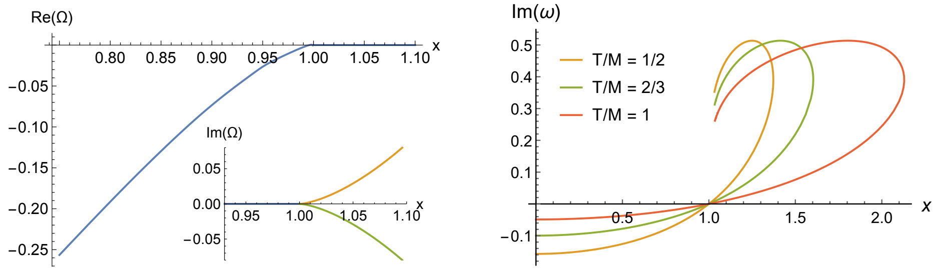

Figure 1. *Left panel:* The free energy (Ω) of the dual field theory as function of the non-Hermiticity parameter ξ (denoted by "x" in the plot). For $|\xi| > 1$ there are two complex conjugate solutions as expected for the $\mathcal{PT}$ broken regime. *Right panel:* Lowest quasinormal mode for solutions containing a black hole (finite temperature solutions). While real solution can be found even for $|\xi| > 1$ the are all unstable since there is a quasi-normal mode in the upper half plane.

2.2 Time independent case $\xi(v)$.

Now we focus on the time evolution of the system for some interesting examples when the non-Hermiticiy parameter is a function of time [16]. Initial and final states correspond to equilibrium finite temperature solutions. To study the response of the system when ξ interpolates between two different values we use the profile function

$$\xi(v) = \xi_i + \frac{\xi_f - \xi_i}{2}\left[1 + \tanh\left(\frac{v - v_m}{\tau}\right)\right]. \tag{15}$$

Here τ is the timescale over which the non-Hermitian coupling varies. Following standard usage call this a holographic quantum quench. We start at the Hermitian point $\xi = 0$ and end either in the $\mathcal{PT}$-symmetric phase, i.e. $|\xi| < 1$, or at the exceptional point. In these type of quenches no additional gauge field is switched on. Therefore it is expected to be dual to a non-unitary time evolution. Two signature of this in the gravity theory are as follows. First the expectation value of the charge operator J^0 dual to the gauge field is purely imaginary

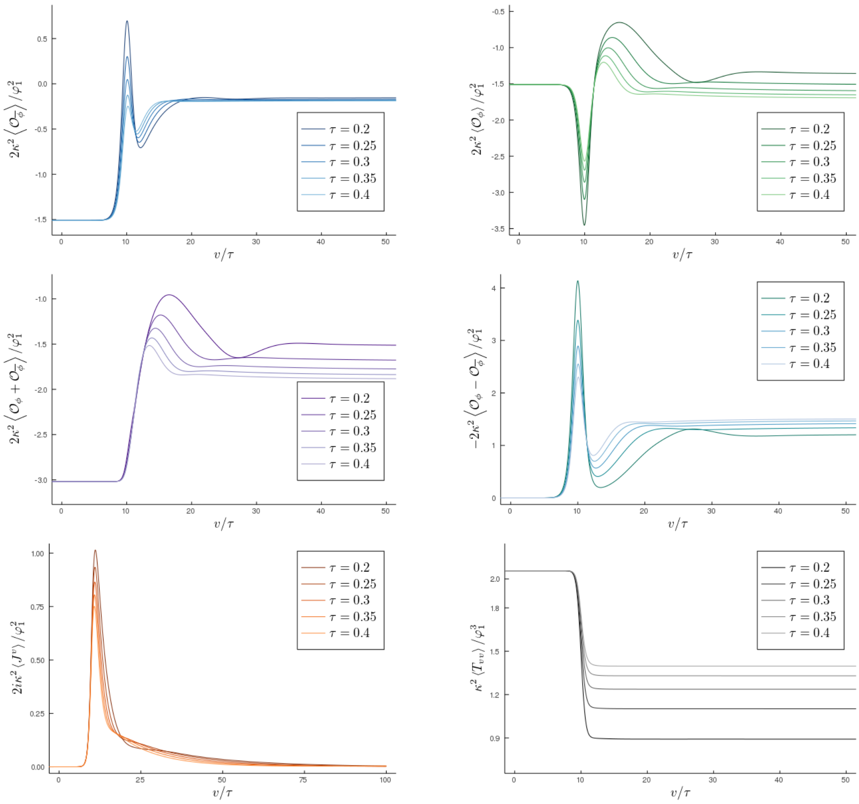

Figure 2. Expectation values of the holographic operators for a quench with profile as in eq. (15) for several values of τ which interpolates between the Hermitian point $\xi_i = 0$ and a final value $\xi_f = 0.8$. We set $v_m = 10\tau$.

during the time evolution. The second one is that the energy (the expectation value of the T_{vv} component of the energy-momentum tensor) diminishes. This also means that the horizon of the dual geometry shrinks, a situation that does not arise is sufficiently "reasonable" types of matter fall into a black hole. Indeed the so called null energy condition is violated. This condition states that $T_{\mu\nu}d^\mu d^\nu > 0$ for any null vector d^μ. As long as this condition holds for matter falling into a black hole the horizon expands. In the non-Hermitian quantum quench however this is violated as can be clearly seen in figure 3. Another interesting type of holographic quantum quench is defined by the profile function

$$\xi(v) = \xi_i + \frac{\xi_m - \xi_i}{2}\left[1 + \tanh\left(\frac{v - v_m}{\tau}\right)\right] + \frac{\xi_f - \xi_m}{2}\left[1 + \tanh\left(\frac{v - \tilde{v}_m}{\tilde{\tau}}\right)\right]. \tag{16}$$

where ξ_m is some intermediate value. This is interesting because it is possible to chose this intermediate value to lie in the unstable regime $|\xi| > 1$. From the static analysis it is known that there is an instability. Indeed in figure 4 this is clearly seen as the onset of exponential growth. One can stay in this unstable regime for sime finite time and upon entering into the $\mathcal{PT}$ symmetric regime the system settles down again to a stationary state at late times after the quench has ended.

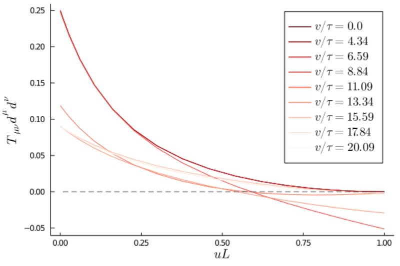

Figure 3. The NEC $T_{\mu\nu}d^\mu d^\nu \geq 0$ for the particular simulation of figure 2 with $\tau = 0.4$ at different stages v/τ of the evolution. It is clearly violated, especially at the horizon $uL = 1$.

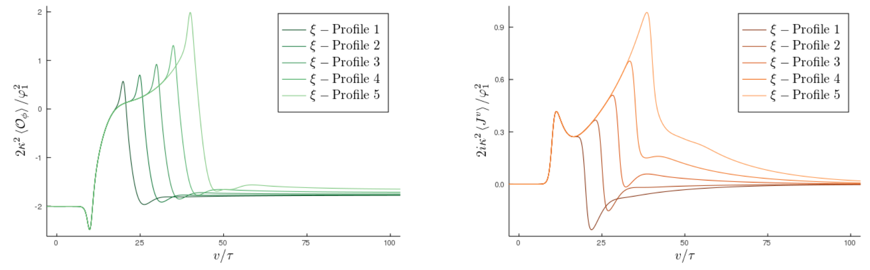

Figure 4. Expectation values of the scalar operator and the charge operator for a quench with profile 16 for $\tilde{v}_m/\tilde{\tau} = \{20, 25, 30, 35, 40\}$. The initial and final states are located at $\xi = 0.8$, besides we set $\xi_m = 1.1$ and $\tau = \tilde{\tau} = 0.25$. An instability starts to develop, but it is suppressed as we re-enter the $\mathcal{PT}$-symmetric region.

2.3 Time independent case $\xi(v)$: Hermitian evolution

We have also studied the time evolution corresponding in which one also introduces a gauge field. More precisely the gauge field in AdS is sourced by

$$A_0|_{u=0} = \frac{i\partial_v \xi}{1-\xi}, \tag{17}$$

corresponding to the time evolution according to (4). In this case the time evolution should be exactly equivalent to a Hermitian time evolution. This is indeed the case (see figure 5).

3 Conclusions

We have shown that the concept of $\mathcal{PT}$ symmetry can be implemented in strongly coupled holographic field theories. Several interesting observations can be made that give hints about what to look for in weakly coupled $\mathcal{PT}$ symmetry field theory. Finally we also would like to remark that during the conference we have been made aware of some problem arising in the renormaliation of weakly coupled $\mathcal{PT}$ quantum field theory [17, 18]. In contrast we found not obvious problems in carrying out the holographic renormalization. This is certainly a point that merits further investigation.

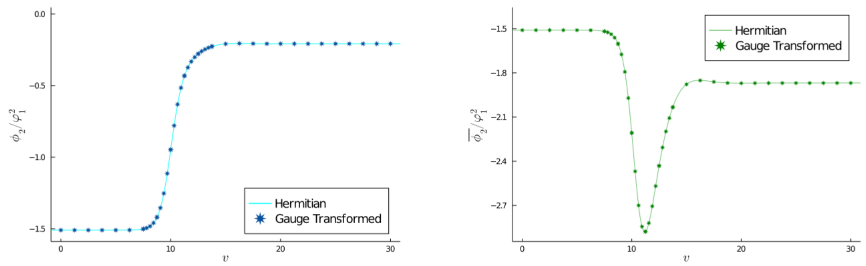

Figure 5. Vevs of the scalar operators for a purely Hermitian quench and the corresponding non-Hermitian one with the gauge field (17). Both give exactly the same results.

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