Tsallis statistics and QCD thermodynamics*

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Abstract. We summarize recent progress on the applications of Tsallis statistics to high energy and heavy ion physics. We also address the possible connections of this statistics with a fractal structure of hadrons.

1 Introduction

Before the quark structure of the hadrons was widely accepted, another approach resulted in an interesting description of many observed phenomena. Supporting this approach is the self-consistent model (Bootstrap Model) proposed by Frautschi and others [1], which gave an exponential mass spectrum for the hadrons that was able to describe the high mass sector of the observed hadrons.

Hagedorn proposed a Self-Consistent Thermodynamics for the fireball formed at high energy collisions, where fireball was defined as a thermodynamical system formed by fireballs [2]. With this definition, it was possible to describe the thermodynamical aspects of the fireballs, as well as the exponential mass spectrum. Another fundamental result of the theory was the limiting temperature, called Hagedorn's temperature, representing the maximum temperature at which the fireball could be heated. According to the Hagedorn's theory, increasing the energy given to the system would result in the production of a larger number of particles, keeping the temperature below its asymptotic value.

The success of the Self-Consistent Thermodynamics motivated the development of the Hadron Resonance Gas models, where different exponential mass spectra were used to investigate the thermodynamical characteristics of the fireballs [3–6]. With the advance of the theory based on the quark structure, the Hagedorn temperature was reinterpreted as a phase-transition temperature, corresponding to the interface between the confined and deconfined regimes of the hadronic matter. The deconfined regime is known as Quark-Gluon Plasma, and has being identified and studied experimentally since 2005.

The Self-Consistent Thermodynamics predicts exponential distributions for the energy and momentum of the particles produced in the collisions between high energy hadrons. However, with the increasing of the collision energy promoted by the advance of the colliders, it was observed that this was not observed experimentally, that is, $(\frac{d^2N}{dp_\perp dy} \not\approx \; e^{-p_\perp/T})$.

*Talk given by E. Megías at the QCD@Work: International Workshop on QCD, 27-30 June 2022, Lecce, Italy.
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After 2000, it was clear that Hagedorn theory could not describe the full range of the multiparticle production, and the distributions should be described by the Tsallis distributions. When using Tsallis statistics, then power-law distributions are obtained and the agreement between theory and experiment is perfect in many orders of magnitude [7, 8].

2 Tsallis statistics

The Tsallis statistics is a generalization of Boltzmann-Gibbs (BG) statistics for systems with correlations that can emerge from non Markovian evolution, long-range forces or complex internal structure. It is based on a new definition for the Entropy, given by

$$S_q \equiv -k_B \sum_i p_i^q \ln_q p_i \,, \qquad \ln_q(p) \equiv \frac{p^{1-q}-1}{1-q} \quad \left[\underset{q\to 1}{\longrightarrow} \log(p) \right] . \tag{1}$$

A consequence is that the entropy of the system is non-additive, and for two independent systems A and B [9]

$$S_{A+B} = S_A + S_B + k_B^{-1}(1-q) S_A S_B \,, \tag{2}$$

where the entropic index q measures the degree of non-extensivity. Tsallis generalized statistics leads to q-exponential and q-logarithm functions,

$$e_q^{(\pm)}(x) = [1 \pm (q-1)x]^{\pm 1/(q-1)} \,. \tag{3}$$

It was proved that the thermodynamical relations are kept intact if one maximize the Tsallis entropy, and the whole thermodynamics can be derived from the new entropic formula, as for instance, the occupation number, given by

$$n_q^{(\pm)}(x) = \frac{1}{\left(e_q^{(\pm)}(\beta(\varepsilon-\mu)) - \xi \right)^{1\pm(q-1)}} \qquad \left[\underset{q\to 1}{\longrightarrow} \frac{1}{e^{\beta(\varepsilon-\mu)} - \xi} \right] , \tag{4}$$

where $\xi = \pm 1$ for bosons and fermions, respectively, and the density of hadron species in an ideal gas of massive particles, given by

$$\rho_i \equiv \frac{\langle N_i \rangle}{V} = \frac{g_i}{2\pi^2} \left[\int_0^{p_{i\star}} dp\, p^2 n_q^{(-)}(x_i) + \int_{p_{i\star}}^{\infty} dp\, p^2 n_q^{(+)}(x_i) \right] , \tag{5}$$

where

$$x_i \equiv \beta \left(\varepsilon_i - \sum_a \mu_a q_{ai} \right) , \quad \varepsilon_i \equiv \sqrt{p^2 + m_i^2} \,, \quad \sum_q \mu_a q_{ai} \equiv \mu_u u_i + \mu_d d_i + \mu_s s_i \,, \tag{6}$$

and $p_{i\star} > 0$ if $\sum_a \mu_a q_{ai} > m_i$.

3 Non-Extensive Self-Consistent Thermodynamics

The Self-Consistent Thermodynamic Theory was generalized by the inclusion of Tsallis statistics. It was shown that the critical temperature is obtained in this generalization, but with a new spectrum mass formula that increases as a power-law instead of the exponential increase from the Hagedorn theory, that is,

$$\rho(m) = \rho_o \cdot [1 + (q-1)m/T_H]^{1/(q-1)} \,.$$

This new spectrum mass formula describes the observed data better than the Hagedorn's spectrum mass formula, describing the full range of observed masses from the pion mass up to approximately 2.5 GeV. Moreover, it predicts a constant value for the parameter q after the limiting temperature is reached.

The energy and momentum distributions also change, resulting in q-exponential distribution instead of the exponential ones, that is,

$$\frac{d^2N}{dp_\perp dy} = gV\frac{p_\perp m_\perp}{(2\pi)^2} e_q^{(+)}\left(-\frac{m_\perp}{T}\right). \tag{7}$$

4 Thermofractals

Emergence of the non-extensive behavior has been attributed to different causes: 1) long-range interactions, correlations and memory effects; 2) temperature fluctuations; 3) finite size of the system [10]. We will study a natural derivation of non-extensive statistics in terms of thermofractals.

Thermofractals are systems in thermodynamical equilibrium presenting the following properties [11]:

1. Total energy is given by
$$U = F + E, \tag{8}$$
where F is the kinetic energy, and E is the internal energy of N constituent subsystems, so that $E = \sum_{i=1}^N \varepsilon_i^{(1)}$.

2. Constituent subsystems are thermofractals, so that the distribution $P_{\text{TF}}(E)$ is self-similar or self-affine, i.e. at level n of the subsystem hierarchy, $P_{\text{TF}(n)}(E)$ is equal to the distribution in the other levels:
$$P_{\text{TF}\,(n)}(\varepsilon) \propto P_{\text{TF}\,(n+m)}(\varepsilon). \tag{9}$$

3. At level n the phase space is so narrow that one can consider $P_{\text{TF}}(E_n)dE_n = \rho dE_n$.

In thermofractals the phase space must include momentum degrees of freedom of free particles as well as internal degrees of freedom. According to property 2 of self-similar thermofractals

$$P_{\text{TF}(0)}(U)dU = A'F^{\frac{3N}{2}-1}\exp\left(-\frac{\alpha F}{kT}\right)dF\left[P_{\text{TF}(1)}(\varepsilon)\right]^{\nu}d\varepsilon, \tag{10}$$

with $\alpha = 1+\frac{\varepsilon}{kT}$ and $\frac{\varepsilon}{kT} = \frac{E}{F}$, and ν is an exponent to be determined. By imposing self-similarity

$$P_{\text{TF}(0)}(U) \propto P_{\text{TF}(1)}(\varepsilon) \tag{11}$$

one finds

$$P_{\text{TF}}(\varepsilon) = A\left[1+\frac{\varepsilon}{kT}\right]^{-\frac{3N}{2}\frac{1}{1-\nu}} \quad \Longrightarrow \quad P_{\text{TF}(n)}(\varepsilon) = A_{(n)}\,e_q\left[-\frac{\varepsilon}{k\tau}\right]. \tag{12}$$

Therefore, the distribution of thermofractals then obeys Tsallis statistics with $q-1 = \frac{2}{3N}(1-\nu)$.

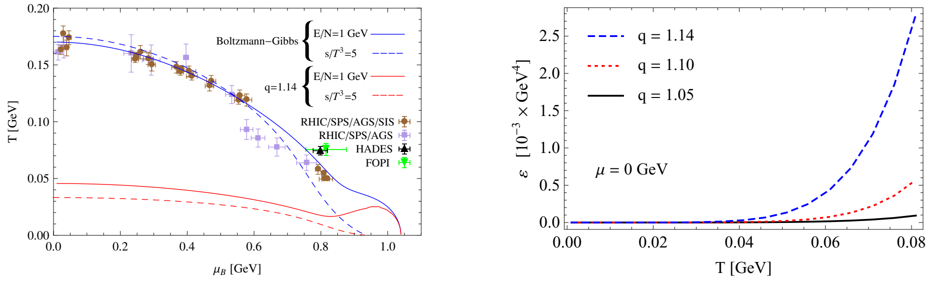

Figure 1. Chemical freeze-out line $T = T(\mu_B)$ (left panel) and energy density as a function of temperature (right panel). We plot the result by using BG statistics, and Tsallis statistics.

From the generalization of Hagedorn's thermodynamics, it is possible to derive the thermodynamical functions that will show a non-extensive character [12]. Indeed,

$$\log Z_q(V,T,\mu) = -\xi V \int \frac{d^3p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \log_q^{(-r)} \left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)} \right), \tag{13}$$

which results in the Boltzmann partition function in the limit $q \to 1$. The non-extensive thermodynamics, then, can describe all the observables of the thermodynamically equilibrated system formed at high energy collisions. On this respect, the thermodynamics of QCD in the confined phase can be studied within the Hadron Resonance Gas approach, for which physical observables can be expressed in terms of hadronic states [3]. In particular, the partition function is given by

$$\log Z_q(V,T,\{\mu\}) = \sum_{i \in \text{hadrons}} \log Z_q(V,T,\mu_i), \tag{14}$$

and from that one can compute all the QCD thermodynamics quantities:

- Energy density: $\longrightarrow$ $\varepsilon \equiv \frac{\langle E \rangle}{V} = -\frac{1}{V}\frac{\partial}{\partial \beta} \log Z_q\Big|_{\mu_B} + \frac{1}{V}\frac{\mu_B}{\beta}\frac{\partial}{\partial \mu_B} \log Z_q\Big|_{\beta}$.
- Pressure: $\longrightarrow$ $P = \frac{1}{\beta}\frac{\partial}{\partial V} \log Z_q$.
- Entropy: $\longrightarrow$ $S = -\beta^2 \frac{\partial}{\partial \beta}\left(\frac{\log Z_q}{\beta}\right)\Big|_{\mu_B}$.
- Baryon density: $\longrightarrow$ $\rho_B \equiv \frac{\langle B \rangle}{V} = \frac{1}{3V}\left(\langle N_{\text{quarks}} \rangle - \langle N_{\text{antiquarks}} \rangle\right)$.

The non-extensive hadronic matter presents interesting aspects. The equation of states becomes harder as the parameter q increases, therefore the hadronic matter with $q = 1.14$, as it results from the thermofractal QCD, leads to a higher pressure. Applying this reasoning to a possible deconfined phase in Neutron Stars, one obtains large masses for the star for the same radius, giving a possible explanation for the massive objects observed. Some results on the QCD thermodynamics are displayed in Fig. 1.

5 Scales in YM theory

At this point, it is interesting to investigate the relations between the thermofractal structure and the QCD. The two main features of fractals are the scaling invariance and the complex internal structure. These two aspects are also present in the description of the effective partonic structure of hadronic matter.

The effective parton has a complex structure arising from the self-energy interactions, which becomes apparent when one describes the high-order contributions. The inclusion of the self-energy interaction leads, indeed, to the infinities in the QCD theory, or in general, in any Yang-Mills theory.

It is possible to cut-off the infinity contributions because of another property of the quantum fields, that is represented mathematically by the renormalization group equation given by [13, 14]

$$\left[M\frac{\partial}{\partial M}+\beta_{\bar{g}}\frac{\partial}{\partial \bar{g}}+\gamma\right]\Gamma=0\,.$$

The equation above establishes the scaling invariance of the field theory, and fixing a maximum value for the momentum is equivalent to renormalising the parameters, like mass and charge, of the partons described by the theory.

With the two characteristics above, a fractal theory of the quantum fields was proposed in Ref. [15]. The fractal structure leads to a recursive equation that allows to obtain the running-coupling as

$$\bar{g}(\varepsilon)=\prod_{i=1}^{\tilde{N}}G\left[1-(q-1)\frac{\varepsilon_i}{\lambda}\right]^{\frac{1}{q-1}}\,. \tag{15}$$

With this coupling, and comparing with the predictions of the QCD, one can obtain the value for the parameter q in terms of the number of colours and the number of flavours, given by

$$\frac{1}{q-1}=\left[\frac{11}{3}N_f-\frac{4}{3}N_c\right]\,.$$

For $N_f=3$ and $N_c=6$, it results $q=8/7=1.14$, in excellent agreement with the experimental analysis.

6 Conclusions

We have reviewed the non-extensive statistics in the form of Tsallis statistics of a quantum gas at finite T and μ. It is shown that the Tsallis statistics allows for a generalization of the self-consistent thermodynamics first proposed by Hagedorn. The phase-transition temperature is still obtained, and a new spectrum mass formula for hadrons can describe the observed states better than the original Hagedorn's formula.

It is discussed that the emergence of the generalized statistics in HEP can be due to the formation of fractal structures of the thermodynamics distributions, which can be associated with a fractal structure in the momentum space. These structures, called thermofractals, would lead to the correlations needed for the emergence of Tsallis statistics.

Finally, recent work is discussed where thermofractal structures can appear in any field described on the basis of a Yang-Mills theory. The structure would be due to the scaling properties that allow the renormalization of the theory. This result allows the calculation of the entropic index, q, in terms of the fundamental parameters of the field theory, as the numbers of colours and flavours in QCD. In this case, the value contained, $q=8/7$, is in very good agreement with the experimental value.

Finally, let us point out that Tsallis statistics has many applications, including high energy collisions [8, 16, 17], hadron models [18, 19], hadron mass spectrum [7], hadron structure [20], neutron stars [21], lattice QCD [22], Bose-Einstein condensation [23, 24], and non-extensive statistical mechanics [12, 25, 26]. The study of all these features deserves further investigation.

Acknowledgement

E.M. would like to thank the Instituto de Física and the Instituto de Astronomia e Geofísica of the Universidade de São Paulo, Brazil, for hospitality during the process of writing this work. A.D. is supported by the Project INCT-FNA (Instituto Nacional de Ciência e Tecnologia - Física Nuclear Aplicada) Proc. No. 464898/2014-5, by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil), grant 304244/ 2018-0, by Project INCT- FNA Proc. No. 464 898/2014-5, and by FAPESP, Brazil grant 2016/17612-7. The work of E.M. is supported by the project PID2020-114767GB-I00 financed by MCIN/AEI/10.13039/501100011033, by the FEDER/Junta de Andalucía-Consejería de Economía y Conocimiento 2014-2020 Operational Programme under Grant A-FQM-178-UGR18, by Junta de Andalucía under Grant FQM-225, and by the Consejeríıa de Conocimiento, Investigación y Universidad of the Junta de Andalucía and European Regional Development Fund (ERDF) under Grant SOMM17/6105/UGR. The research of E.M. is also supported by the Ramón y Cajal Program of the Spanish MICIN under Grant RYC-2016-20678.

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