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XVth Quark Confinement and the Hadron Spectrum

Non-invertible self-duality defects of Cardy-Rabinovici model and mixed gravitational anomaly

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Abstract. We study new symmetries of the Cardy-Rabinovici model and their dynamical applications. The Cardy-Rabinovici model is a 4d $U(1)$ gauge theory with electric and magnetic matters, which is a good playground for studying the dynamics of the Yang-Mills theory with θ angle. In this model, the electromagnetic $SL(2,\mathbb{Z})$ self-duality is not realized in a naive way. Still, the $SL(2,\mathbb{Z})$ transformations become legitimate duality operations by appropriately gauging the $\mathbb{Z}_N$ 1-form symmetry. We construct new noninvertible symmetries from this duality at self-dual points and determine their non-group-like fusion rules. As an application, we can rule out the trivially gapped phase for some self-dual parameters due to a mixed gravitational anomaly of this new symmetry. We also show how the conjectured phase diagram of the Cardy-Rabinovici model is consistent with this anomaly matching condition.

The $SU(N)$ Yang-Mills theory has a topological θ-term $\frac{i\theta}{8\pi^2}\int \mathrm{tr}\, f \wedge f$ which affects the vacuum structure non-perturbatively. Studying the θ-term dependence is an interesting and challenging topic because it will give an important insight into the nonperturbative aspects of the $SU(N)$ Yang-Mills theory.

One of the most popular understandings of quark confinement is the dual superconductor picture. In this scenario, we assume that the monopole condenses at the vacuum, and this condensation forms the color electric flux tube, leading to the linear quark-antiquark potential. We therefore suppose that the vacuum is the monopole-condensed branch around $\theta = 0$. As θ increases, we expect a dyon condensation since a magnetic monopole acquires an electric charge $\theta/2\pi$, known as the Witten effect. For example, at $\theta = 2\pi$, the dyon with (-1)-electric charge and $(+1)$-magnetic charge should condensate. Thus, in the interval between $\theta \in [0, 2\pi]$, the candidates of ground states are the monopole-condensed branch and dyon-condensed branch, and they will be degenerate two vacua at $\theta = \pi$.

There might be a more nontrivial possibility around $\theta = \pi$, e.g., an exotic dyon condensation, where the oblique confinement realizes [1]. Cardy and Rabinovici proposed a toy model mimicking such a θ dependence, which we call the Cardy-Rabinovici model [2, 3]. This model is a Villain lattice $U(1)$ gauge theory, consisting of $U(1)$ gauge field, charge-N electric matter, and monopole. This model has the monopole-condensed confinement phase, dyon-condensed confinement phase, and oblique confinement phase (see Fig. 1), so it gives an interesting playground to study the physics of the topological θ angle.

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Recently, the notion of symmetry has been extensively generalized (see [4] and references therein). In modern terminology, ordinary symmetry is characterized by a symmetry generator of a co-dimension-1 topological defect with a group-like structure. A lesson from recent works indicates that the essential part is the "topological" condition, which corresponds to the conservation law. Topological defects with higher codimension, known as higher-form symmetries, describe symmetries acting on extended objects and have many applications to gauge theories. There is the other direction of the generalization of symmetry: allowing non-group-like fusion rules of symmetry defects. This type of symmetry is called non-invertible symmetry.

While non-invertible symmetries have been studied mainly in 2d QFTs, there has been excellent progress in higher-dimensional cases very recently. One breakthrough is a systematic construction of non-invertible defects when the theory is self-dual under gauging a discrete symmetry. When this kind of self-duality exists, one can construct a co-dimension-1 topological defect by the half-space gauging of the discrete symmetry (See Sec. 2). This construction is a sort of generalization of the Kramers-Wannier duality defect [5–7]. Moreover, the existence of such duality defects can constrain possible vacuum structures.

Our work [8] is to apply this strategy to the Cardy-Rabinovici model.The main results are as follows.

1. In the Cardy-Rabinovici model, it has been known that an apparent $SL(2,\mathbb{Z})$ "self-duality" generated by S and T transformation. We point out that this "self-duality" is a duality between the Cardy-Rabinovici model and its appropriately $\mathbb{Z}_N^{[1]}$-gauged model.

2. From this duality, at a self-dual parameter, we can construct a non-invertible defect by a half-space $\mathbb{Z}_N^{[1]}$-gauging operation.

3. For some cases, we find a mixed gravitational anomaly, which constrains the dynamics, e.g., rules out the trivially gapped vacuum.

1 Cardy-Rabinovici model

Here we review some basics of the Cardy-Rabinovici model: what the Cardy-Rabinovici model is, why this model is interesting for studying dynamics related to θ-angle, and the conjectured phase diagram of this model.

1.1 Description of the model

Although the Cardy-Rabinovici model is originally formulated as a Villain-type lattice gauge theory, we skip lattice details and describe only the essentials of its formal continuum description here. The Cardy-Rabinovici model consists of $U(1)$ gauge field, charge-N Higgs matter, and charge-1 magnetic matter. The partition function of this theory can be symbolically written as follows[1],

$$Z_{\mathrm{CR}}^{\tau} = \int \mathcal{D}a \exp\left(-S_{U(1)}^{\tau}[\mathrm{d}a]\right) \sum_{C,C':\,\mathrm{loops}} W^N(C)H(C'), \tag{1}$$

[1] In terms of the worldline representation, there should be some weight $e^{-S_{mat}[C,C']}$ with the desirable properties. See [9] for details. For simplicity, we set $S_{mat}[C,C'] = 0$, corresponding to a naive low-energy limit of charged matters.

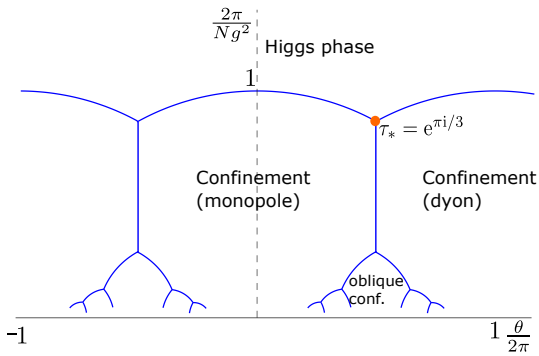

Figure 1. Conjectured phase diagram of the Cardy-Rabinovici model. The electric particle condenses in the weak coupling region (Higgs phase), and the magnetic particle condenses in the strong coupling region (confinement phase). Because of the Witten effect, the dyon tends to condense by increasing θ. Taken from [8].

where we have introduced the Maxwell action

$$S^{\tau}_{U(1)}[\mathrm{d}a] := \frac{1}{2g^2}\int \mathrm{d}a \wedge *\mathrm{d}a - \frac{\mathrm{i}N\theta}{8\pi^2}\int \mathrm{d}a \wedge \mathrm{d}a, \tag{2}$$

and the complex coupling $\tau := \frac{\theta}{2\pi} + \mathrm{i}\frac{2\pi}{Ng^2}$. The integration over the matter fields is represented as the sum of all possible worldlines of the charge-N electric particle and charge-1 magnetic particle. Note that the Cardy-Rabinovici model has the $\mathbb{Z}_N$ 1-form symmetry, which we denote by $\mathbb{Z}_N^{[1]}$, since the electric matter is charge-N.

Here, we have introduced the unusual N factor in the theta term. Since magnetic lines are explicitly present in this model, there is no conventional 2π periodicity here because a magnetic line acquires an electric line by the Witten effect, namely $\langle H(C')\rangle_{S^{\tau+1/N}_{U(1)}} = \langle H(C')W(C')\rangle_{S^{\tau}_{U(1)}}$. Still, since the extra charge-N electric line can be absorbed by the integration over the Higgs field, we have the 2π periodicity of θ angle with this convention: $Z^{\tau+1}_{\mathrm{CR}} = Z^{\tau}_{\mathrm{CR}}$.

1.2 Phase diagram and $SL(2,\mathbb{Z})$ duality

Cardy and Rabinovici estimated which particle may condensate by a rough free-energy argument and obtained the conjectured phase diagram [2, 3]. For the case where there is always some condensation[2], the phase diagram is shown in Fig. 1 In the weak-coupling region, the electric matter tends to condense, and the Higgs phase appears. In the strong-coupling region, monopole condensation wins, and the theory is in the confinement phase. As θ increases, the dyon will condense as inferred from the Witten effect. For stronger coupling, more exotic dyon condensates, the oblique confinement phases, appear. At the fixed strong coupling, the θ-dependence of this phase diagram is qualitatively the same as that expected in the Yang-Mills theory: monopole condensation at $\theta = 0$, dyon condensation at $\theta = 2\pi$, and spontaneous breakdown of CP symmetry (or possible nontrivial phase) at $\theta = \pi$. Therefore, this model gives an interesting playground for studying θ angle dependence[3]

[2]Coulomb phase, where no charged particle condenses, may appear.

[3]Moreover, the mixed anomaly structure of $\mathbb{Z}_N^{[1]} \times CP$ symmetry of this model is identical to that of the $SU(N)$ Yang-Mills theory, and the oblique confinement phase nontrivially satisfies the matching condition of this anomaly [9].

A notable feature we will use is that this phase diagram has the apparent $SL(2,\mathbb{Z})$ invariance generated by the S (electromagnetic) and T ($\theta \to \theta + 2\pi$) transformations. Let $\begin{pmatrix} n \\ m \end{pmatrix}$ denote a particle or line operator with electric charge Nn and magnetic charge m. Then, the phase diagram has the invariance under the following S and T transformation.

$$S : \tau \mapsto -\frac{1}{\tau}, \quad \begin{pmatrix} n \\ m \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} -m \\ n \end{pmatrix}, \tag{3}$$

$$T : \tau \mapsto \tau + 1, \quad \begin{pmatrix} n \\ m \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} n-m \\ m \end{pmatrix}. \tag{4}$$

From this invariance, it is tempting to speculate that the Cardy-Rabinovici model has the electromagnetic $SL(2,\mathbb{Z})$ self-duality. However, the standard electromagnetic S-transformation is *not* the duality of the Cardy-Rabinovici model. Indeed, the Cardy-Rabinovici model has electric charge-N and magnetic charge-1 matters, but the S-transformed model has electric charge-1 and magnetic charge-N matters. Note also that the S-transformed model has magnetic $\mathbb{Z}_N^{[1]}$ symmetry instead of the electric $\mathbb{Z}_N^{[1]}$ symmetry.

Then, what kind of duality does the $SL(2,\mathbb{Z})$ invariance of the phase diagram suggest? In Sec. 3.1, we see that the Cardy-Rabinovici model has $SL(2,\mathbb{Z})$ duality accompanied by the $\mathbb{Z}_N^{[1]}$ gauging.

2 Non-invertible duality symmetries

As mentioned above, symmetries are characterized by symmetry generators, or topological defects. Based on this view, we can consider various generalizations of symmetries. One direction, which many studies has focused on in the last few years, is to allow non-group-like fusion rules for the topological defects, so called non-invertible symmetry. This kind of topological defects has been known in 2d CFTs for long years, but not in higher-dimensional QFTs until very recently. In this section, a basic idea for a systematic construction of one class ("duality symmetries") of non-invertible symmetries is briefly reviewed [5–7]. The class of non-invertible symmetries we consider is a generalization of the Kramers-Wannier duality defect in 2d Ising model, which is one of the famous non-invertible defects.

It is well-known that the 2d Ising model has the Kramers-Wannier duality which relates the high-temperature Ising model and low-temperature one. On a general manifold, the duality operation is accompanied by the gauging of the spin-flipping $\mathbb{Z}_2$ symmetry. Therefore, at the critical temperature, the self-duality can be written as

$$\mathcal{T}_{\text{Ising}}/\mathbb{Z}_2 \simeq \mathcal{T}_{\text{Ising}}, \tag{5}$$

where the critical Ising model is denoted by $\mathcal{T}_{\text{Ising}}$. The Ising model at the self-dual point is invariant under $\mathbb{Z}_2$ gauging.

Using this kind of duality, we can construct the co-dimension-1 non-invertible defect by half-space gauging. Let us decompose the spacetime manifold X into two parts $X = X^+ \cup X^-$ with identical boundary $M = \partial X^+ = -\partial X^-$. Then, gauging $\mathbb{Z}_2$ symmetry in the half-space X^+ with the Dirichlet boundary condition for the $\mathbb{Z}_2$ gauge field on M defines a co-dimension-1 topological defect (Fig. 2).

The invariance under gauging (5) assures that the half-space gauged theory is equivalent to the original theory $\mathcal{T}_{\text{Ising}}$ with a co-dimension-1 defect $\mathcal{D}(M)$. In addition, since the gauge field for a discrete symmetry is topological, we can continuously deform the boundary M on which the Dirichlet boundary condition for the discrete gauge field is imposed. The defect we

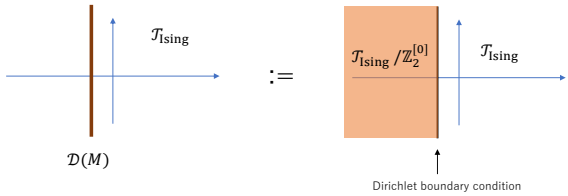

Figure 2. Schematic picture of the definition of the Kramers-Wannier duality defect in the 2d critical Ising model. Because of the invariance under the gauging (5), the interface between $\mathcal{T}_{\text{Ising}}/\mathbb{Z}_2$ and $\mathcal{T}_{\text{Ising}}$ gives the $\mathcal{T}_{\text{Ising}}$ with a co-dimension-1 defect.

defined here is co-dimension-1 and topological, so this defines a symmetry. However, this is not an ordinary symmetry, but a non-invertible symmetry with the following nontrivial fusion rules (with $\mathbb{Z}_2$ symmetry defect $\eta(M)$)

$$\mathcal{D}(M) \times \mathcal{D}(M) = 1 + \eta(M), \quad \mathcal{D}(M) \times \eta(M) = \eta(M) \times \mathcal{D}(M) = \mathcal{D}(M) \tag{6}$$

Since gauging $\mathbb{Z}_2$ twice is a trivial identity operation, one might naively expect that $\mathcal{D}(M)$ would be an inverse of $\mathcal{D}(M)$. However, due to the half-space gauging construction, the fusion product $\mathcal{D}(M) \times \mathcal{D}(M)$ leaves the $\mathbb{Z}_2$ gauging operation on M: $1 + \eta(M)$ (known as a "condensation operator" of 1-gauging [10]).

In short, the above discussion indicates that, when the theory is invariant under the gauging of a discrete symmetry, there exists a non-invertible duality defect constructed by the half-space gauging. In this viewpoint, the generalization to higher dimensional cases is straightforward: an invariance under gauging a discrete symmetry leads to a non-invertible symmetry [6].

3 Non-invertible symmetries in the Cardy-Rabinovici model

In Sec. 1.2, we have seen that the phase diagram respects some sort of $SL(2,\mathbb{Z})$ duality, but the Cardy-Rabinovici model itself does not have the standard electromagnetic S duality. First, in Sec. 3.1, we shall see that a correct description of the "$SL(2,\mathbb{Z})$ duality" is an invariance under gauging $\mathbb{Z}_N$ 1-form symmetry. As presented in the previous section, this invariance leads to a non-invertible symmetry by half-space gauging. Then, in Sec. 3.2, we obtain nontrivial constraints on the phase diagram when such a noninvertible symmetry exists.

3.1 Invariance under gauging, duality defects, and fusion rules

To state our claim, we introduce a few notations: We define the partition function with a $\mathbb{Z}_N^{[1]}$ background field $B \in H^2(X;\mathbb{Z}_N)$ as,

$$Z^{\tau}_{\text{CR}}[B] = \int \mathcal{D}a \exp(-S^{\tau}_{U(1)}[\mathrm{d}a + B]) \sum_{C,C':\,\text{loops}} W^N_{\mathrm{d}a+B}(C) H_{\mathrm{d}a+B}(C'). \tag{7}$$

and that of the gauged model with a dual $\mathbb{Z}_N^{[1]}$ background field B as[4],

$$Z^{\tau}_{\text{CR}/(\mathbb{Z}_N^{[1]})_p}[B] = \int \mathcal{D}b\ \mathcal{Z}^{\tau}_{\text{CR}}[b] \exp\left(\frac{\mathrm{i}Np}{4\pi}\int_X b\wedge b + \frac{\mathrm{i}N}{2\pi}\int_X b\wedge B\right). \tag{8}$$

[4]We set the normalization of $\int \mathcal{D}b$ as $\int \mathcal{D}b = \frac{|H^0(X;\mathbb{Z}_N)|}{|H^1(X;\mathbb{Z}_N)|}\sum_{b\in H^2(X;\mathbb{Z}_N)}$.

where we write $CR/(\mathbb{Z}_N^{[1]})_p$ for $\mathbb{Z}_N^{[1]}$-gauged the Cardy-Rabinovici model with the discrete θ term $\frac{\mathrm{i}Np}{4\pi}\int_X b\wedge b$, $(p=0,\cdots,N-1)$.

One of our main claims is that there is a duality between the Cardy-Rabinovici model and its gauged one:

$$\mathrm{CR}^\tau/(\mathbb{Z}_N^{[1]})_p \simeq \mathrm{CR}^{ST^p(\tau)} \tag{9}$$

In terms of partition functions, we have

$$\begin{aligned} Z^\tau_{\mathrm{CR}/(\mathbb{Z}_N^{[1]})_p}[B] &= N^{\frac{\chi(X)}{2}}(ST^p(\tau))^{\frac{\chi(X)+\sigma(X)}{4}}(ST^p(\overline{\tau}))^{\frac{\chi(X)-\sigma(X)}{4}}\, Z^{ST^p(\tau)}_{\mathrm{CR}}[B] \\ &= N^{\frac{\chi(X)}{2}}(\tau+p)^{-\frac{\chi(X)+\sigma(X)}{4}}(\overline{\tau}+p)^{-\frac{\chi(X)-\sigma(X)}{4}}\, Z^{-(\tau+p)^{-1}}_{\mathrm{CR}}[B], \end{aligned} \tag{10}$$

where $\chi(X)$ is the Euler number and $\sigma(X)$ is the signature of the 4d spacetime manifold X. In particular, at the "self-dual" coupling of some ST^p operation, $\tau = ST^p(\tau)$, this relation means the invariance under gauging a discrete symmetry. Therefore, by a parallel discussion of the Kramers-Wannier duality defect, we can construct non-invertible symmetries at such couplings.

Let us briefly describe an intuitive understanding of the claim (9).

We firstly focus on the $p=0$ case, the S duality: $\mathrm{CR}^\tau/\mathbb{Z}_N^{[1]} \simeq \mathrm{CR}^{S(\tau)}$. The integration of a $U(1)$ gauge field consists of (a) sum over bundle structure and (b) local fluctuation of the connection. Correspondingly, the field strength is decomposed as $\mathrm{d}a = m + \mathrm{d}(\delta a)$, where the Chern class part $m \in H^2(X;\mathbb{Z})$ characterizes the topological sector, and δa is a globally-defined 1-form for local fluctuations. The $\mathbb{Z}_N^{[1]}$ gauging procedure replaces $\mathrm{d}a$ by $\mathrm{d}a + b$ with the $\mathbb{Z}_N^{[1]}$ gauge field $b \in H^2(X;\mathbb{Z}_N)$. This replacement $\mathrm{d}a + b = \mathrm{d}(\delta a) + (m+b)$ effectively fractionalizes the Chern-class part by $1/N$, $(m+b) \in \frac{1}{N}H^2(X;\mathbb{Z})$[5]. Therefore, with a rescaling of the local fluctuation part δa, the $\mathbb{Z}_N^{[1]}$ gauging results in the $1/N$ rescaling of the $U(1)$ gauge field: $\mathrm{d}a + b = \mathrm{d}a'/N$ with a new $U(1)$ gauge field a'.

How do electric and magnetic lines are affected by this rescaling? The effect for the Wilson loop is simple: $W_{\mathrm{d}a+b}(C,\Sigma)$ becomes $W^{1/N}_{\mathrm{d}a'}(C,\Sigma)$, where $W_{\mathrm{d}a+b}(C,\Sigma) := e^{i\int_\Sigma \mathrm{d}a+b}$ and Σ is the surface whose boundary is C. In particular, the charge-N Wilson loop, $W^N_{\mathrm{d}a+b}(C) = e^{iN\int_C a}$, which is a genuine line operator in the $\mathbb{Z}_N^{[1]}$-gauged theory, becomes the unit charge Wilson loop for a', $W_{\mathrm{d}a'}(C)$. For the magnetic line, it is convenient to recall the (continuum) definition of the 't Hooft loop. The 't Hooft loop for a, $H_{\mathrm{d}a}(C)$, introduces a defect on C with the following boundary condition: for small two-spheres S^2 linking to the loop C, we impose $\int_{S^2}\mathrm{d}a = 2\pi$, which becomes $\int_{S^2}\mathrm{d}a' = 2\pi N$ in terms of a'. Thus, the charge-1 't Hooft line $H_{\mathrm{d}a+b}(C)$ becomes the charge-N line $H^N_{\mathrm{d}a'}(C)$.

Therefore, since the Cardy-Rabinovici model is a $U(1)$ gauge theory with electric charge-N and magnetic charge-1 matters, the $\mathbb{Z}_N^{[1]}$-gauged model is equivalent to a $U(1)$ gauge theory with electric charge-1 and magnetic charge-N matters. Then, the S duality of the $U(1)$ gauge theory transforms this model with electric charge-1 and magnetic charge-N matters into the original Cardy-Rabinovici model. For the coupling τ, the rescaling $a \to a/N$ makes $\tau \to \tau/N^2$, and the S transformation[6] makes $\tau/N^2 \to -1/\tau = S(\tau)$. To sum up, we obtain the S-duality: $\mathrm{CR}^\tau/\mathbb{Z}_N^{[1]} \simeq \mathrm{CR}^{S(\tau)}$. The prefactor of the right-hand side of (10) arises from the rescaling of the continuum part $\delta a'$ and the S-transformation as the Witten's computation [11]. For details, see [8].

[5]Here, for simplicity, we assume that the homology of the spacetime manifold $H^*(X;\mathbb{Z})$ has no torsion.

[6]Note that the complex coupling $\tau = \frac{\theta}{2\pi} + \mathrm{i}\frac{2\pi}{Ng^2}$ is different from the usual complex coupling of the Maxwell theory by $1/N$ factor.

For nonzero p, the discrete theta term $e^{\frac{\mathrm{i}Np}{4\pi}\int_X b\wedge b}$ can be written as $e^{\frac{\mathrm{i}p}{4\pi N}\int_X da'\wedge da'}$. On the other hand, the theta term for a becomes $e^{\frac{\mathrm{i}N\theta}{4\pi}\int_X (\mathrm{d}a+b)\wedge(\mathrm{d}a+b)} = e^{\frac{\mathrm{i}\theta}{4\pi N}\int_X da'\wedge da'}$. Hence, stacking the discrete theta term $e^{\frac{\mathrm{i}Np}{4\pi}\int_X b\wedge b}$ corresponds to the shift of θ: $\theta \to \theta + 2\pi p$ in the $\mathbb{Z}_N^{[1]}$-gauged theory. This implies the desired duality (9).

Following the construction of the Kramers-Wannier duality defect, we can construct a topological defect in the Cardy-Rabinovici model at a self-dual coupling by the half-space gauging.

As the simplest example, let us consider S defect[7], which can be constructed at $\tau = i$, where the invariance under $\mathbb{Z}_N^{[1]}$-gauging holds: $\mathrm{CR}^{\tau=i}/\mathbb{Z}_N^{[1]} \simeq \mathrm{CR}^{\tau=i}$. This defect $\mathcal{D}(M)$ satisfies the following Kramers-Wannier-like fusion rules

$$\mathcal{D}(M)\times\mathcal{D}(M) = C(M)\frac{1}{N}\sum_{\Sigma\in H_2(M;\mathbb{Z}_N)}\eta(\Sigma),\quad \mathcal{D}(M)\times\eta(\Sigma) = \eta(\Sigma)\times\mathcal{D}(M) = \mathcal{D}(M), \tag{11}$$

where $C(M)$ is the charge conjugation defect, $\eta(\Sigma)$ is the co-dimension-2 defect of the $\mathbb{Z}_N^{[1]}$ symmetry, and Σ is two-cycle on M. The charge conjugation defect appears because of $S^2 = C$. In addition to the charge conjugation, the fusion $\mathcal{D}(M)\times\mathcal{D}(M)$ leaves $\mathbb{Z}_N^{[1]}$ gauging on M, which is the condensation defect $\frac{1}{N}\sum_{\Sigma\in H_2(M;\mathbb{Z}_N)}\eta(\Sigma)$. Due to the half-space gauging construction, the duality defect absorbs the $\mathbb{Z}_N^{[1]}$ generator $\eta(\Sigma)$. These fusion rules are not group-like, so the duality defect represents a non-invertible symmetry.

As a more interesting example, we consider the ST^{-1} defect at $\tau = \tau_* = e^{\frac{\mathrm{i}\pi}{3}}$: $ST^{-1}(\tau_*) = \tau_*$. Reflecting the triality $(ST^{-1})^3 = C$, the fusion rules are[8],

$$\mathcal{D}(M)\times\mathcal{D}(M)\times\mathcal{D}(M) \propto C(M)\sum_{\Sigma\in H_2(M;\mathbb{Z}_N)}\eta(\Sigma),\quad \mathcal{D}(M)\times\eta(\Sigma) = \eta(\Sigma)\times\mathcal{D}(M) = \mathcal{D}(M), \tag{12}$$

We can construct various noninvertible defects from the $SL(2,\mathbb{Z})$ "duality" by using the (9) repeatedly.

3.2 Mixed gravitational anomaly and constraint on dynamics

For the ST^{-1} defect, the underlying self-duality relation reads, from (10),

$$Z^{\tau_*}_{\mathrm{CR}/(\mathbb{Z}_N^{[1]})_{-1}}[B] = N^{\frac{\chi(X)}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}\sigma(X)}Z^{\tau_*}_{\mathrm{CR}}[B]. \tag{13}$$

This relation, especially the nontrivial phase $\mathrm{e}^{-\frac{\pi\mathrm{i}}{3}\sigma(X)}$, excludes the trivially gapped vacuum. This phase can be seen as a mixed gravitational anomaly of the non-invertible symmetry.

If the vacuum is trivially gapped, the response to a background field should be described by an SPT phase. It is known that a $\mathbb{Z}_N^{[1]}$ SPT phase is classified by $Z_k[B] = \exp\left(\mathrm{i}\frac{Nk}{4\pi}\int_X B\wedge B\right)$, $k = 0,\cdots,N-1$. By a direct computation on a 4d manifold with a nontrivial signature, say a K3 surface, we can verify that none of the $\mathbb{Z}_N^{[1]}$ SPT phases can satisfy the self-duality (13) [8].

Lastly, we note how the phase diagram (Fig. 1) can satisfy this constraint. Natural guesses for low-energy theories of Higgs, monopole-condensed, and dyon-condensed phases are the level-N BF theory $Z_{\mathrm{Higgs}}[B] = \int\mathcal{D}a\mathcal{D}b\exp\left(\frac{\mathrm{i}N}{2\pi}\int b\wedge(\mathrm{d}a+B)\right)$ and two SPT phases

[7] For Maxwell theory, this type of defects is studied in [6].

[8] Here, for simplicity, we assume that $H^*(M;\mathbb{Z})$ is torsion-free. For a general case, there is an interesting subtlety to these fusion rules. See Ref. [10].

$Z_{\text{mon}}[B] = 1$, $Z_{\text{dyon}}[B] = \exp\left(\frac{\mathrm{i}N}{4\pi}\int B\wedge B\right)$. Since the point $\tau = \tau_*$ in Fig. 1 is the triple point of the Higgs, monopole-condensed, and dyon-condensed phases, we can speculate that a combination of them can match the anomaly (13). Indeed, the following combination

$$Z_{\text{CR}}^{\tau_*}[B] = Z_{\text{mon}}[B] + \mathrm{e}^{\frac{\pi\mathrm{i}}{3}\sigma(X)} Z_{\text{dyon}}[B] + N^{-\frac{\chi(X)}{2}} \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}\sigma(X)} Z_{\text{Higgs}}[B], \tag{14}$$

satisfies (13). Therefore, the obtained constraint is compatible with the conjectured phase diagram (Fig. 1).

Similar anomaly constraints and consistencies can be observed at other points, e.g., $\tau = \frac{\sqrt{3}+\mathrm{i}}{2\sqrt{3}}$ which is a fixed point of the $ST^{-1}ST^2S$ transformation and is a triple point where the monopole $(n,m) = (0,1)$ condensed phase, the dyon $(n,m) = (-1,1)$ condensed phase, and the oblique confinement phase with exotic dyon $(n,m) = (-1,2)$ condensation meet in Fig. 1.

4 Summary

As a new tool for studying QFTs, the generalization of symmetries has attracted more attention in recent years. We have considered a new type of symmetries with non-group-like algebra in the Cardy-Rabinovici model, a toy model for dynamics related to the θ angle.

Based on a self-duality of the Cardy-Rabinovici model, we have constructed non-invertible defects and found its mixed gravitational anomaly for some cases, which gives a constraint on the phase diagram.

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