Determination of α_s value from tau decays with a renormalon-motivated approach

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Abstract. We apply Borel-Laplace sum rules to the data of the semihadronic tau decay rate. For the higher order terms of the Adler function in the leading-twist ($D = 0$) contribution we use a renormalon-motivated model, where the correct leading anomalous dimensions are taken into account in the IR $u = 3$ (and $u = 2$) renormalon contributions. In the evaluation of $D = 0$ contribution of the sum rules we apply two methods: (a) fixed order perturbation theory (FO) and (b) Borel resummation of the singular part with the Principal Value prescription (PV). We use as data the ALEPH data for the (V+A)-channel, and a combined set of data for the V-channel. In the $D = 6$ OPE term of the Adler function we account for the leading nonzero (and noninteger) anomalous dimension. In the OPE for the Adler function we include terms with dimension up to $D = 10$ for the (V+A)-channel, and up to $D = 14$ for the V-channel. In such cases, the extracted values of the coupling α_s and of the condensates show a reasonably good convergence under the increase of OPE terms. In order to suppress the quark-hadron duality violations, our sum rules are doubly-pinched in the Minkowskian point. We obtain the averaged extracted values of the coupling $\alpha_s(m_\tau^2) = 0.3169^{+0.0070}_{-0.0096}$, corresponding to $\alpha_s(M_Z^2) = 0.1183^{+0.0009}_{-0.0012}$.

1 Introduction

In QCD, one of the main problems is the determination of the value of the (perturbative) coupling parameter $\alpha_s(M_Z^2)$. However, a somewhat related problem is the behaviour of the running coupling $\alpha_s(Q^2)$ at lower momenta $|Q^2| \lesssim 1\ \mathrm{GeV}^2$. The latter behaviour is of particular interest because we know that perturbative QCD (pQCD) approach starts failing once the values of $|Q^2|$ move into this regime, principally because of the vicinity of the Landau singularities of the pQCD coupling. On the other hand, there are well-measured data of the semihadronic τ-lepton, by OPAL [1, 2] and ALEPH [3–6] Collaborations; these are QCD data for momenta of $|Q^2| \lesssim m_\tau^2$ ($\approx 3\ \mathrm{GeV}^2$), i.e., in the mentioned low-momentum regime. Therefore, they represent, to a certain degree, a challenge for the (p)QCD theory.

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The theoretical quantity for the strangeless semihadronic τ-decays is the u-d quark current correlator function $\Pi(Q^2)$ and the related Adler function $\mathcal{D}(Q^2) \propto d\Pi(Q^2)/d\ln Q^2$. The theoretical framework used for the description of the Adler function in this intermediate regime $|Q^2| \sim 1\ \mathrm{GeV}^2$ is usually the Operator Product Expansion (OPE). This theoretical approach was developed and applied to the mentioned τ-decays in the works [7–12]. On the other hand, the perturbation expansion of the leading-twist (dimension $D = 0$) contribution to the Adler function, $d(Q^2)_{D=0}$, has been calculated up to $\mathcal{O}(\alpha_s^4)$ [13] (and references therein).

In our approach, we apply Borel-Laplace sum rules to the mentioned decay data. We use for the $\mathcal{O}(\alpha_s^5)$ term in $d(Q^2)_{D=0}$ an estimated value, and estimate (generate) the higher order terms by a renormalon-motivated model. In the OPE we take into account up to $D = 10$ terms in the (V+A)-channel, and up to $D = 14$ terms in the V-channel. We use ALEPH data [3–6] for the (V+A)-channel, and a combined set of data [14] for the V-channel. The weight functions for the sum rules are all double-pinched in the Minkowskian point $Q^2 = -\sigma_{\max}$ ($\equiv -\sigma_{\mathrm{m}}$), in order to suppress the quark-hadron duality violations. In comparison with the version presented at the conference, this is an updated version based on results of [15]. In Sec. 2 we make a brief description of the appplied sum rules, and in Sec. 3 of the renormalon-motivated model. In Sec. 4 we describe the numerical analysis and present the results of the analysis.

2 Sum rules

The u-d quark current correlator function $\Pi(Q^2)$, e.g., for the (V+A)-channel, is

$$\Pi(Q^2)_{\mathrm{V+A}} = \Pi_{\mathrm{V}}^{(1)}(Q^2) + \Pi_{\mathrm{A}}^{(1)}(Q^2) + \Pi_{\mathrm{A}}^{(0)}(Q^2), \tag{1}$$

where we neglect the term $\Pi_{\mathrm{V}}^{(0)}$ because of the small masses m_u, m_d. The above terms enter the correlator

$$\Pi_{\mathrm{J},\mu\nu}(q) = i \int d^4x\, e^{iq\cdot x} \langle T J_\mu(x) J_\nu(0)^\dagger \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi_{\mathrm{J}}^{(1)}(Q^2) + q_\mu q_\nu \Pi_{\mathrm{J}}^{(0)}(Q^2), \tag{2}$$

where $J_\mu = \bar{u}\gamma_\mu d$ and $\bar{u}\gamma_\mu\gamma_5 d$ for $J = V, A$, respectively. We note that $Q^2 \equiv -q^2$ where q^2 is the square of the momentum transfer [$q^2 = (q^0)^2 - \vec{q}^2$]. The Adler function $\mathcal{D}(Q^2)$ is the derivative of the correlator function

$$\mathcal{D}(Q^2)_{\mathrm{V+A}} \equiv -2\pi^2 \frac{d\Pi(Q^2)_{\mathrm{V+A}}}{d\ln Q^2}. \tag{3}$$

The (truncated) OPE for this quantity is

$$\mathcal{D}_{\mathrm{th}}(Q^2)_{\mathrm{V+A}} = d(Q^2)_{D=0} + 1 + 4\pi^2 \frac{\langle O_4 \rangle_{\mathrm{V+A}}}{(Q^2)^2} + 6\pi^2 \frac{a(Q^2)^{k^{(1)}} \langle O_6^{(j)} \rangle_{\mathrm{V+A}}}{(Q^2)^3} + 2\pi^2 \sum_{p=4}^{p_{\max}} \frac{p \langle O_{2p} \rangle_{\mathrm{V+A}}}{(Q^2)^p}, \tag{4}$$

where $D = 4$ term is known to have zero anomalous dimension, $D = 6$ term has approximate anomalous dimension $k^{(1)} \equiv \gamma^{(1)}(O_6^{(1)})/\beta_0 \approx 0.222$ [16] [and we use the notation $a(Q^2) \equiv \alpha_s(Q^2)/\pi$], and for $D > 6$ terms we assume (approximate) zero anomalous dimensions.

The sum rules are obtained by applying the Cauchy theorem in the complex Q^2-plane to an integral containing $\Pi(Q^2)$ and a holomorphic weight function $g(Q^2)$. In this approach, it is implicitly taken into account that $\Pi(Q^2)$ has no Landau singularities, i.e., it has only singularities along the negative Q^2-semiaxis [in pQCD this is explicitly violated because pQCD coupling $a(Q^2)$ has Landau singularities]. The Cauchy theorem gives

$$\int_0^{\sigma_{\mathrm{m}}} d\sigma g(-\sigma) \omega_{\mathrm{exp}}(\sigma) = -i\pi \oint_{|Q^2|=\sigma_{\mathrm{m}}} dQ^2 g(Q^2) \Pi_{\mathrm{th}}(Q^2), \tag{5}$$

where on the right-hand side we have integration along a circle of radius $\sigma_{\rm max} \equiv \sigma_{\rm m}$, and on the left-hand side appears the (experimentally measured) discontinuity (spectral) function of the (V+A)-channel polarisation function

$$\omega(\sigma) \equiv 2\pi \ {\rm Im} \ \Pi(Q^2 = -\sigma - i\epsilon) \ . \tag{6}$$

The right-hand side expression in Eq. (5) can be reexpressed by integration by parts, giving us the contour integral involving the (theoretical) Adler function

$$\int_0^{\sigma_{\rm m}} d\sigma g(-\sigma)\omega_{\rm exp}(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \ \mathcal{D}_{\rm th}(\sigma_{\rm m} e^{i\phi}) G(\sigma_{\rm m} e^{i\phi}), \tag{7}$$

where $G(Q^2)$ is such that $dG(Q^2)/dQ^2 = g(Q^2)$ and $G(-\sigma_{\rm m}) = 0$. On the right-hand side of the sum rule Eq. (7) we use the truncated OPE (4). For the Borel-Laplace sum rules, the double-pinched weight function is

$$g_{M^2}(Q^2) = \left(1 + \frac{Q^2}{\sigma_{\rm m}}\right)^2 \frac{1}{M^2} \exp\left(\frac{Q^2}{M^2}\right). \tag{8}$$

3 Renormalon-motivated construction of $D = 0$ Adler function

Now we briefly explain the construction of the leading-twist contribution $d(Q^2)_{D=0}$ appearing in the OPE (4), which is based on the renormalon-motivated approach of [17]. We have the perturbation expansion in $a(Q^2) \equiv \alpha_s(Q^2)/\pi$

$$d(Q^2)_{D=0,{\rm pt}} = d_0 \ a(Q^2) + d_1 \ a(Q^2)^2 + \ldots + d_n \ a(Q^2)^{n+1} + \ldots, \qquad (d_0 = 1), \tag{9}$$

where the exact values of the first four expansion coefficients ($d_0 = 1; d_1; d_2; d_3$) are known [13] (and references therein). For the next coefficient d_4 we have an estimate [13, 18–20]

$$d_4 = 275. \pm 63. \tag{10}$$

The expansion (9) in powers a^n can be reorganised in logarithmic derivatives $\widetilde{a}_n$

$$d(Q^2)_{D=0,{\rm lpt}} = \widetilde{d}_0 \ a(Q^2) + \widetilde{d}_1 \ \widetilde{a}_2(Q^2) + \ldots + \widetilde{d}_n \ \widetilde{a}_{n+1}(Q^2) + \ldots, \tag{11}$$

where

$$\widetilde{a}_n(Q^2) \equiv \frac{(-1)^{n-1}}{(n-1)! \beta_0^{n-1}} \left(\frac{d}{d \ln Q^2}\right)^{n-1} a(Q^2) \qquad (n = 1, 2, \ldots). \tag{12}$$

We note that $\widetilde{a}_n = a^n$ only in the one-loop approximation. We can now construct an auxiliary Adler quantity $\widetilde{d}(Q^2)$ in which we replace $\widetilde{a}_n \mapsto a^n$

$$\widetilde{d}(Q^2)_{\rm ser} \equiv a(Q^2) + \widetilde{d}_1 \ a(\kappa Q^2)^2 + \ldots \widetilde{d}_n \ a(Q^2)^{n+1} + \ldots, \tag{13}$$

The Borel transforms of $d(Q^2)_{D=0}$ and $\widetilde{d}(Q^2)$ have the series

$$\mathcal{B}[d](u;\kappa)_{\rm ser.} \quad \equiv \quad d_0 + \frac{d_1(\kappa)}{1! \beta_0} u + \ldots + \frac{d_n(\kappa)}{n! \beta_0^n} u^n + \ldots \ , \ \Rightarrow \tag{14}$$

$$\mathcal{B}[\widetilde{d}](u;\kappa)_{\rm ser.} \quad \equiv \quad \widetilde{d}_0 + \frac{\widetilde{d}_1(\kappa)}{1! \beta_0} u + \ldots + \frac{\widetilde{d}_n(\kappa)}{n! \beta_0^n} u^n + \ldots, \tag{15}$$

where $\beta_0 = (11 - 2n_f/3)/4 = 9/4$ is the leading (i.e., one-loop) beta-coefficient (we take the number of quark flavors $n_f = 3$). The true (resummed) Borel transforms have singularities in the complex u-plane. It can be shown that we have the following relation between the two Borel transforms [17, 21]:

$$\mathcal{B}[\widetilde{d}](u)_{p,\kappa_p} = \frac{\pi \widetilde{d}_p^{\rm IR}}{(p-u)^{\kappa_p}} \;\Rightarrow\; \mathcal{B}[d](u)_{p,\kappa_p} = \frac{\pi d_p^{\rm IR}}{(p-u)^{\kappa_p + p\beta_1/\beta_0^2}} \Big\{ 1 + \mathcal{O}((p-u)^1) \Big\}, \qquad (16)$$

where $\beta_1 = (1/16)(102 - 38 N_f/3)$ is the two-loop beta-function coefficient. Furthermore, the ambiguity $\delta d(Q^2)_{p,\kappa_p}$ of the Borel resummation of the above contribution has a specific Q^2-dependence

$$\delta d(Q^2)_{p,\kappa_p} \sim \frac{1}{\beta_0} {\rm Im} \int_{+i\varepsilon}^{+\infty+i\varepsilon} du \exp\left(-\frac{u}{\beta_0 a(Q^2)} \right) \frac{1}{(p-u)^{\kappa_p + p\beta_1/\beta_0^2}} \sim \frac{1}{(Q^2)^p} a(Q^2)^{1-\kappa_p} \left[1 + \mathcal{O}(a) \right]. \qquad (17)$$

In order for the OPE term with dimension $D = 2p$ to cancel this ambiguity, it must have the same Q^2 dependence as above. In the case of $D = 6$ term ($p = 3$) this implies that $\kappa_3 = 1 - k = 1 - \gamma^{(1)}(O_6)/\beta_0$, where $k = \gamma^{(1)}(O_6)/\beta_0$ is the corresponding anomalous dimension in the $D = 2p$ term of the OPE. According to the work [16], the first two leading anomalous dimensions for $D = 6$ operators ($p = 3$) are $k^{(1)} \approx 0.222$ and $k^{(2)} \approx 0.625$, thus $\kappa^{(1)} = 0.778$ and $\kappa^{(2)} = 0.375$. On the other hand, as mentioned, in the OPE (4), we take in $D = 6$ term only the smallest anomalous dimension, i.e., the term $\sim a(Q^2)^{k^{(1)}} = a(Q^2)^{0.222}$.

We regard as the basis for the Adler leading-twist contribution the Borel transform $\mathcal{B}[\widetilde{d}](u)$, which is the generator of the (Adler function) coefficients $\widetilde{d}_n$. By taking into account the above aspects, we make for the Borel transform $\mathcal{B}[\widetilde{d}](u)$ (with $\kappa = 1$) in the $\overline{\rm MS}$ scheme the ansatz which contains the mentioned discussed singularities at $u = 2, 3$ (IR renormalons, $p = 2, 3$), and at $u = -1$ (UV renormalon, $-p = -1$)

$$\mathcal{B}[\widetilde{d}](u) = \exp\left(\widetilde{K} u\right) \pi \Big\{ \widetilde{d}_{2,1}^{\rm IR} \left[\frac{1}{(2-u)} + \widetilde{\alpha}(-1) \ln\left(1 - \frac{u}{2}\right) \right] + \frac{\widetilde{d}_{3,2}^{\rm IR}}{(3-u)^{\kappa^{(2)}}} + \frac{\widetilde{d}_{3,1}^{\rm IR}}{(3-u)^{\kappa^{(1)}}} + \frac{\widetilde{d}_{1,2}^{\rm UV}}{(1+u)^2} \Big\}, \qquad (18)$$

where we use the mentioned indices $\kappa^{(1)} = 0.778$ and $\kappa^{(2)} = 0.375$. Here, $\widetilde{\alpha} = -0.255$ is determined [17] from the knowledge of a subleading term related with the Wilson coefficient of $D = 4$ OPE term. The other five parameters in (18) are determined from the knowledge of the five first coefficients $\widetilde{d}_n$ ($n = 0, \ldots, 4$) of the expansion (11) [which is obtained from the first five coefficients d_n of the expansion (9)], where for d_4 we use the estimate (10).

We point out that the obtained expression Eq. (18) generates all the terms $\widetilde{d}_n$ (and thus d_n) of the expansions (9)-(11) of $D = 0$ Adler function $d(Q^2)_{D=0}$. Further, the expression Eq. (18) then leads, as mentioned in Eq. (16), to the following resummed form of the Borel transform $\mathcal{B}[d](u)$ of $D = 0$ Adler function:

$$\begin{aligned} \frac{1}{\pi} \mathcal{B}[d](u) \;\; = \;\; & \frac{2.565390}{(2-u)^{209/81}} \left[1 + \mathcal{O}(2-u) \right] + \frac{9.310304}{(3-u)^{3.14837}} \left[1 + \mathcal{O}(3-u) \right] \\ & + \frac{(-3.203771)}{(3-u)^{2.74537}} \left[1 + \mathcal{O}(3-u) \right] + \frac{(-0.0111528)}{(1+u)^{98/81}} \left[1 + \mathcal{O}(1+u) \right], \qquad (19) \end{aligned}$$

where the coefficients at $\mathcal{O}(p-u)$ ($p = 2, 3$) and $\mathcal{O}(1+u)$ can be reasonably well determined up to quadratic order ($\sim (p-u)^2, (1+u)^2$).

4 Evaluations and results

We take two approaches for the evaluation of $D = 0$ part of the sum rules: (a) the fixed order (FO) perturbation theory, where we Taylor-expand the powers $a(\sigma_{\rm m} e^{i\phi})^n$, appearing in $d(\sigma_{\rm m}e^{i\phi})_{D=0}$ of the contour integral in the sum rules, in powers of $a(\sigma_{\rm m})$; (b) we Borel integrate, taking the Principal Value (PV), the singular part of the Borel transform Eq. (19), where we account for the relative corrections up to $\sim (2-u)^2$, $(3-u)^2$, $(1+u)^2$. In the latter method (PV), we must add a (truncated) power series to the Borel integral to correct for the missing terms. Both methods thus involve in practice a truncation at $a(\sigma_{\rm m})^{N_t}$ and $a(\sigma_{\rm m}e^{i\phi})^{N_t}$, respectively, where N_t is a chosen truncation index.

The fitting of the Borel-Laplace sum rules (7) is performed in practice for the real part of the Borel-Laplace sum rules $B(M^2;\sigma_{\rm m})$

$$\mathrm{Re}B_{\rm exp}(M^2;\sigma_{\rm m}) = \mathrm{Re}B_{\rm th}(M^2;\sigma_{\rm m}), \tag{20}$$

where $B(M^2;\sigma_{\rm m})$ is the sum rule for the weight function $g_{M^2}(Q^2)$ given in Eq. (8). We perform the fit by minimising the following quantity:

$$\chi^2 = \sum_{\alpha=1}^{n} \left(\frac{\mathrm{Re}B_{\rm th}(M^2_\alpha;\sigma_{\rm m}) - \mathrm{Re}B_{\rm exp}(M^2_\alpha;\sigma_{\rm m})}{\delta_B(M^2_\alpha)} \right)^2, \tag{21}$$

where M^2_α are Borel-Laplace scales along three rays in the complex M^2-plane: $M^2 = |M^2|\exp(i\psi)$, where $\psi = 0,\ \pi/6,\ \pi/4$. For the lengths of the rays we choose $0.9\ \mathrm{GeV}^2 \le |M^2| \le 1.5\ \mathrm{GeV}^2$. In practice, we use 3 equidistant points along each ray (covering the endpoints of the rays), i.e., in total 9 points ($n = 9$).

4.1 (V+A)-channel

For the experimental data $\omega_{\rm V+A}(\sigma)$ we use the high precision data of the ALEPH Collaboration [3–6], with the last two bins excluded for the central value case, due to high uncertainty (thus: $\sigma_{\rm max} \equiv \sigma_{\rm m} = 2.8\ \mathrm{GeV}^2$). The extracted values for the parameters, i.e., the values of $a(\sigma_{\rm m})$ and of the condensates, depend on the $D = 0$ truncation index N_t, and on the truncation of the OPE. It turns out that reasonably good behaviour of these values is obtained when the OPE is truncated at the dimension $D = 10$. Then, for each chosen N_t we extract the values of the parameters which we then use to evaluate the sum rule (double-pinched) momenta $a^{(2,n)}(\sigma_{\rm m})$ ($n = 0, 1, 2$), i.e., these are the sum rules for weight functions

$$g^{(2,n)}(Q^2) = \left(\frac{n+3}{n+1}\right)\frac{1}{\sigma_m}\left(1+\frac{Q^2}{\sigma_m}\right)^2 \sum_{k=0}^{n}(k+1)(-1)^k\left(\frac{Q^2}{\sigma_m}\right)^k. \tag{22}$$

We then determine the optimal N_t as (the lowest) such value where the resulting momenta $a^{(2,n)}(\sigma_{\rm m})$ vary minimally under the change $(N_t - 1) \mapsto N_t$. This gives us $N_t = 8$ in the FO approach, and $N_t = 7$ in the PV approach. The extracted values of the coupling are then [15]

$$\begin{aligned}
\alpha_s(m_\tau^2)^{\rm (FO)} &= 0.3202 \pm 0.0028(\mathrm{exp})^{-0.0014}_{+0.0087}(\kappa)^{-0.0087}_{+0.0052}(d_4)^{+0.0028}_{-0.0038}(N_t) \\
&\quad \pm 0.0043(O_{14}) \pm 0.0032(N_{\rm bin}) && (23)\\
&= 0.3202^{+0.0122}_{-0.0113} \approx 0.320^{+0.012}_{-0.011} \qquad \left(N_t = 8^{+2}_{-3};\ \mathrm{V+A}\right), && (24)
\end{aligned}$$

$$\begin{aligned}
\alpha_s(m_\tau^2)^{\rm (PV)} &= 0.3216 \pm 0.0027(\mathrm{exp})^{-0.0003}_{+0.0002}(\kappa)^{-0.0095}_{+0.0029}(d_4)^{-0.0001}_{+0.0002}(N_t) \\
&\quad \pm 0.0030(O_{14}) \pm 0.0032(N_{\rm bin}) \mp 0.0001(\mathrm{amb}) && (25)\\
&= 0.3216^{+0.0059}_{-0.0107} \approx 0.322^{+0.006}_{-0.011}, \qquad (N_t = 7 \pm 2;\ \mathrm{V+A}). && (26)
\end{aligned}$$

The uncertainties at '(κ)' are those due to the variation of the renormalisation scale parameter $\kappa \equiv \mu^2/Q^2$ ($\kappa = 1^{+1}_{-1/2}$); those at '(d_4)' are due to the variations of d_4 Eq. (10); those at '(N_t)'

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Table 1. The extracted values of $\alpha_s(m_\tau^2)$, for FO approach, for different truncations of the OPE: $D_{\max}$ is the maximal dimension $D = D_{\max}$ at which the OPE is truncated. The values are given for the (V+A)-channel and the V-channel. The truncation index of the $D = 0$ sum rule contribution is $N_t = 8$.

channel	$D_{\max} = 6$	$D_{\max} = 8$	$D_{\max} = 10$	$D_{\max} = 12$	$D_{\max} = 14$	$D_{\max} = 16$
V+A	0.3079	0.3148	0.3202	0.3226	0.3245	0.3261
V	0.3493	0.3459	0.3274	0.3181	0.3128	0.3124

are due to variation of N_t, in the ranges indicated above; those at '(O_{14})' are due to OPE truncation, i.e., the variation when we add to OPE $D = 12$ and $D = 14$ terms; those at '$(N_{\rm bin})$' are variation when we add the additional two data bins (which have high uncertainties) to the analysis. In the PV approach, the uncertainty at '(amb)' is coming from the ambiguity of the Borel integration due to the IR renormalon singularities.

The uncertainties at '(exp)' are coming from experimental uncertainties, and were determined by an analysis using the experimental correlation matrix, the analysis following the formalism outlined in Appendix of Ref. [22]. We can see from Eqs. (23) and (25) that the theoretical uncertainties, especially those due to d_4, dominate over the experimental uncertainties.

4.2 *V*-channel

The fitting for the V-channel is performed in the same way as described in Sec. 4.1, but using the data of [14, 23] which combine ALEPH and OPAL V-channel data and electroproduction data. The last two experimental bins in this case do not have increased uncertainties, therefore we include them, i.e., in this case we have $\sigma_{\rm m} = 3.0574\ {\rm GeV}^2$. The truncation of OPE is made at $D = 14$, because the extracted values are reasonably stable when we increase the number of OPE terms beyond $D = 14$. Also in the case of V-channel, the optimal $D = 0$ truncation index N_t is $N_t = 8$ for FO and $N_t = 7$ for PV. The extracted values for the coupling from this analysis of the V-channel are [15]

$$\begin{aligned}
\alpha_s(m_\tau^2)^{\rm (FO)} &= 0.3128 \pm 0.0053({\rm exp})^{-0.0015}_{+0.0038}(\kappa)^{-0.0071}_{+0.0040}(d_4)^{+0.0012}_{-0.0042}(N_t) \mp 0.0004(O_{16}) & (27)\\
&= 0.3128^{+0.0078}_{-0.0099} \approx 0.313^{+0.008}_{-0.010} \qquad \left(N_t = 8^{+2}_{-3};\ {\rm V-channel}\right), & (28)
\end{aligned}$$

$$\begin{aligned}
\alpha_s(m_\tau^2)^{\rm (PV)} &= 0.3131 \pm 0.0053({\rm exp}) \mp 0.0003(\kappa)^{-0.0068}_{+0.0023}(d_4)^{-0.0000}_{+0.0002}(N_t) \mp 0.0007(O_{16}) \\
&\quad \mp 0.0001({\rm amb}) & (29)\\
&= 0.3131^{+0.0058}_{-0.0087} \approx 0.313^{+0.006}_{-0.009}, \qquad (N_t = 7 \pm 2;\ {\rm V-channel})\,. & (30)
\end{aligned}$$

The (small) uncertainty at '(O_{16})' is coming when we include in the OPE the additional $D = 16$ term. We can see that for V-channel, in contrast to the (V+A)-channel, the experimental uncertainties are comparable to the theoretical ones. In Table 1 we show the extracted values of $\alpha_s(m_\tau^2)$ for the (V+A) and V-channels, for different truncations of the OPE. It is seen that for the V-channel the values stabilise starting at $D_{\max} = 14$; for the (V+A)-channel, we get a slow-down of the growth starting at $D_{\max} = 10$.

In Figs. 1(a), (b), we present the extracted values of $\alpha_s(m_\tau^2)$ as a function of the $D = 0$ truncation index N_t, for the (V+A)-channel and the V-channel. We see that especially the PV method gives results which are quite stable under the variation of this theoretical parameter.

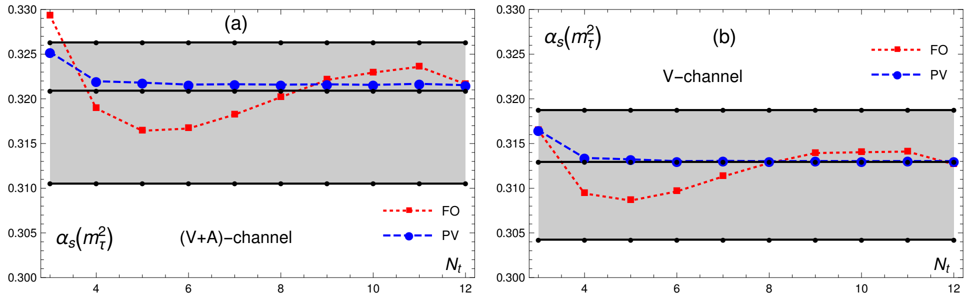

Figure 1. (coloured online) (a) The values of $\alpha_s(m_\tau^2)$ extracted from the Borel-Laplace sum rule of the (V+A)-channel, as a function of the truncation index N_t, for each of the two methods (FO, PV). The light grey (blue) band represents the band of the final averaged values for the (V+A)-channel [cf. Eq. (31)]. (b) The same, but extracted from the V-channel data.

4.3 Final results

The average of the FO and PV results, for the (V+A)-channel and V-channel separately, is

$$\alpha_s(m_\tau^2)_{\rm V+A} = 0.3209^{+0.0059}_{-0.0107}; \qquad \alpha_s(m_\tau^2)_{\rm V} = 0.3129^{+0.0058}_{-0.0087}. \tag{31}$$

Averaging the above results also over the (V+A) and V-cases gives us our final averaged result

$$\alpha_s(m_\tau^2) = 0.3169^{+0.0070}_{-0.0096} \quad \Rightarrow \quad \alpha_s(M_Z^2) = 0.1183^{+0.0009}_{-0.0012}, \tag{32}$$

where we presented also the corresponding value at M_Z^2 (obtained with 5-loop RGE evolution in $\overline{\rm MS}$ and 4-loop quark thresholds).

The obtained results for $\alpha_s(m_\tau^2)$ are somewhat lower than the results of the sum rule analyses in [25, 26] (0.320 ± 0.012) and higher than those of [24] (0.296 ± 0.010) and [14] (0.3077 ± 0.0065). The latter three results were obtained from specific sum rule analyses using the FO approach. In [24, 25] ALEPH data were used, while in [14] the mentioned combined V-channel data were used. In [14, 24] a Duality Violation model was used with the OPE approach.

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