

# Bethe–Salpeter Bound-State Solutions: Examining Semirelativistic Approaches

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**Abstract.** Within the formalism of relativistic quantum field theory an adequate framework for the description of two-particle bound states, such as, for instance, all conventional (i.e., non-exotic) mesons, is provided by the Poincaré-covariant homogeneous Bethe–Salpeter equation. In applications, however, this approach usually proves to be rather involved, whence it is not always quite easy to extract the predictions sought. In view of this, a coarse idea of the bound-state spectrum to be expected might be gained by adhering to some simplifying approximations – which constitutes an entirely legitimate first step. The reliability of the insights inferred from the arising simpler bound-state equation may be straightforwardly examined by taking into account a couple of rigorous constraints on the obtained discrete spectrum. Application of these tools is illustrated for popular potentials.

## 1 Bound States of Spinless Constituents: Semirelativistic Approach

A main issue in both relativistic quantum physics and quantum field theories, such as quantum electrodynamics and quantum chromodynamics, is to devise appropriate (and, if manageable, convenient) approaches to the bound states expected within specific settings. In this context, a very useful tool is provided by the spinless Salpeter equation, i.e., the eigenvalue equation of a Hamiltonian operator composed of the relativistic kinetic energies of the involved bound-state constituents and some static interaction potential  $V$  assumed to reflect the bound-state relevant aspects of any underlying quantum (field) theory. If  $V$  meets these expectations, this approach should return a rough idea of the bound-state spectrum. For two bound-state constituents with relative coordinates  $\mathbf{x}$  and  $\mathbf{p}$  in configuration and momentum space, this Hamiltonian becomes

$$H = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(\mathbf{x}) \quad (1)$$

(where  $m_1$  and  $m_2$  denote both interacting particles' masses) and, for  $m_1 = m_2 = m$ , reduces to

$$\tilde{H} = 2\sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{x}). \quad (2)$$

Conceptually, the spinless Salpeter equation can be formulated along two opposite directions:

- On the one hand, the spinless Salpeter equation can be understood as an improvement of the nonrelativistic Schrödinger equation that is achieved upon allowing, in the Hamiltonian, the kinetic part to retain its proper relativistic form instead of sticking to its nonrelativistic limit.

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- On the other hand, upon starting from the Poincaré-covariant *Bethe–Salpeter formalism* [1], representing an appropriate framework for the description of bound states within the realms of relativistic quantum field theories, the spinless Salpeter equation can be easily derived by applying several simplifying approximations to the homogeneous Bethe–Salpeter equation. The detailed recipe reads: discard all timelike coordinates (which leads to the *instantaneous* Bethe–Salpeter formalism [2]), require free propagation of bound-state constituents (which produces the Salpeter equation [3]), drop all negative-energy contributions (which results in what is called the *reduced* Salpeter equation) and, finally, ignore the spin degree of freedom of all bound-state constituents in order to eventually arrive at the *spinless* Salpeter equation.

## 2 Analysis of Semirelativistic Hamiltonian Operators: Some Insights

The nonlocality of the Hamiltonian (1), induced by the square-root operators of the relativistic kinetic energies, impedes an entirely analytic treatment of any emerging bound-state problem. Nonetheless, by utilization of instruments supplied by functional analysis a qualitative or even semiquantitative picture of expectable solutions of the spinless Salpeter equation may be drawn. In particular, establishing the mere existence of discrete bound states by proving boundedness from below of the spectrum of Hamiltonian operators is, beyond doubt, of utmost importance.

### 2.1 Spectrum of Semirelativistic Hamiltonian Operator: Boundedness from Below

Quite generally, well-definedness of a Hamiltonian requires, among others, this operator to be bounded from below. Noting the manifest positivity of the relativistic kinetic-energy operator, for semirelativistic Hamiltonians – such as those presented in Eqs. (1) or (2) – the crucial issue is the *possibly singular* behaviour of the interaction potentials showing up in the Hamiltonian. In such a context, the (in some sense) worst-case scenario originates in the Coulomb potential:

$$V_C(\mathbf{x}) = V_C(|\mathbf{x}|) \equiv -\frac{\alpha}{|\mathbf{x}|}, \quad \alpha > 0. \quad (3)$$

For the spinless relativistic Coulomb problem, in turn, the spectral theory has been thoroughly worked out [4, 5]: A semirelativistic Hamiltonian (2) with a Coulomb interaction potential (3) is bounded from below if and only if the involved coupling parameter  $\alpha$  satisfies the constraint

$$\alpha < \frac{4}{\pi} = 1.273239 \dots \quad (4)$$

Only for these strengths  $\alpha$ , the lower bound to the spectrum  $\sigma(\tilde{H})$  of the Hamiltonian (2) reads

$$\sigma(\tilde{H}) \geq 2m \sqrt{1 - \left(\frac{\pi\alpha}{4}\right)^2}. \quad (5)$$

For a more refined range,  $\alpha \leq 1$ , a somewhat increased lower bound on  $\sigma(\tilde{H})$  can be found [6]:

$$\sigma(\tilde{H}) \geq 2m \sqrt{\frac{1 + \sqrt{1 - \alpha^2}}{2}}. \quad (6)$$

### 2.2 Spinless-Salpeter Hamiltonian Operators: Number of Discrete Eigenstates [7]

Rigorous upper limits on the total number of bound states described by some spinless Salpeter equation may be formulated [7] in terms of (Lebesgue) spaces of measurable functions,  $L^p$ , on the Euclidean space  $\mathbb{R}^3$ , for  $p = 3/2, 3$ . For a Hamiltonian operator acting on the Hilbert space  $L^2(\mathbb{R}^3)$  and a potential that is a negative smooth function of compact support, hence, subject to

$$V(\mathbf{x}) \in C_0^\infty(\mathbb{R}^3), \quad V(\mathbf{x}) \leq 0, \quad (7)$$

in equal-mass two-particle problems (2) the total number of bound states,  $N$ , is constrained by

$$N \leq \frac{C}{48\pi^2} \int d^3x [|V(\mathbf{x})| (|V(\mathbf{x})| + 4m)]^{3/2}, \quad C = \begin{cases} 14.107590867 & \text{if } m > 0, \\ 6.074898097 & \text{if } m = 0, \end{cases} \quad (8)$$

with numerical results for  $C$  taken from Ref. [8]. Finiteness forces the potential  $V(\mathbf{x})$  to satisfy

$$V(\mathbf{x}) \in L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3). \quad (9)$$

### 2.3 Semirelativistic Hamiltonian Operators: Upper Limits on Discrete Eigenvalues

Upon having succeeded to establish – in the delicate instance of singular interaction potentials by rather shameless abuse of the findings gained for the spinless relativistic Coulomb problem [4, 5] (reviewed in Subsect. 2.1) – the rigorous boundedness from below of the semirelativistic Hamiltonian in one’s focus of interest, that is, of that operator’s spectrum, most likely one will be tempted to try, as a next step, to acquire (at least, limited) information on the actual location of its lowest-lying discrete eigenvalues. Such goal can be achieved by, for instance, narrowing down the conceivable range of any of these eigenvalues by finding an upper bound to its range. The theoretical foundation of any attempt of this kind is the minimum–maximum theorem [9]:

- Consider some self-adjoint operator,  $H$ , bounded from below, with its (ordered) eigenvalues

$$E_0 \leq E_1 \leq E_2 \leq \dots \quad (10)$$

- Define the operator  $\widehat{H}$  by restricting  $H$  to some  $d$ -dimensional subspace of the domain of  $H$ .
- Let  $\widehat{E}_0, \widehat{E}_1, \dots, \widehat{E}_{d-1}$  be the  $d$  eigenvalues of the restriction  $\widehat{H}$ , likewise ordered according to

$$\widehat{E}_0 \leq \widehat{E}_1 \leq \widehat{E}_2 \leq \dots \leq \widehat{E}_{d-1}; \quad (11)$$

- these form upper bounds to the first  $d$  eigenvalues of  $H$  below its essential spectrum’s onset:

$$E_k \leq \widehat{E}_k \quad \forall \quad k = 0, 1, 2, \dots, d-1. \quad (12)$$

Below, in Sect. 3, we will show slight predilection for interactions governed by spherically symmetric potentials. If dealing with central potentials, mere convenience dictates to span the  $d$ -dimensional subspace required by the minimum–maximum theorem by a basis the elements of which are functions,  $\psi_{k\ell}(\mathbf{x})$ , that are products of a spherical harmonic  $\mathcal{Y}_\ell$  for orbital angular momentum  $\ell$  and a radial factor involving the generalized-Laguerre polynomials  $L_k^{(\gamma)}$  [10, 11]:

$$\psi_{k\ell}(\mathbf{x}) \propto |\mathbf{x}|^{\ell+\beta-1} \exp(-\mu|\mathbf{x}|) L_k^{(2\ell+2\beta)}(2\mu|\mathbf{x}|) \mathcal{Y}_\ell(\Omega_{\mathbf{x}}), \quad k \in \mathbb{N}_0, \quad \ell \in \mathbb{N}_0, \quad (13)$$

$$L_k^{(\gamma)}(x) \equiv \sum_{t=0}^k \binom{k+\gamma}{k-t} \frac{(-x)^t}{t!}, \quad \mu > 0, \quad \beta > -\frac{1}{2};$$

herein,  $\Omega_{\mathbf{x}}$  indicates the  $\mathbf{x}$ -space solid angle, and  $\mu$  and  $\beta$  represent two variational parameters.

### 2.4 Spinless-Salpeter Hamiltonian Operator: Quality Assurance of its Eigenstates

Irrespective of the actual origin of approximate eigenstates  $|\chi\rangle$  of a Hamiltonian operator, their accuracy [12, 13] can be judged (or even quantified) by their degree of fulfilment of the *master* or *relativistic* virial theorem [14, 15] relating the expectation values of the radial derivatives of all kinetic terms and of the potential. This relation comprises, of course, the well-known virial theorem of nonrelativistic quantum theory as a special case. For the operators (1), it reduces to

$$\left\langle \chi \left| \frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + m_1^2}} + \frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + m_2^2}} \right| \chi \right\rangle = \left\langle \chi \left| \mathbf{x} \cdot \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \right| \chi \right\rangle. \quad (14)$$

### 2.5 Trivial Upper Bounds: Nonrelativistic Limit of Spinless-Salpeter Hamiltonians

The (undeniable) concavity of the square-root operator of the proper relativistic expression for the kinetic energy regarded as a function of  $\mathbf{p}^2$  implies that its nonrelativistic limit is tangent to its relativistic precursor at the point of tangency  $\mathbf{p}^2 = 0$ . Thus, the nonrelativistic Hamiltonian

$$H_{\text{NR}} = m_1 + m_2 + \frac{\mathbf{p}^2}{2\hat{m}} + V(\mathbf{x}), \quad \hat{m} \equiv \frac{m_1 m_2}{m_1 + m_2}, \quad (15)$$

forms an upper bound to its originator (1), with expectable impact on their sets of eigenvalues:

$$H \leq H_{\text{NR}} \quad \implies \quad E_k \leq E_{\text{NR},k}, \quad k \in \mathbb{N}_0. \quad (16)$$

With respect to the total number  $N$  of bound states of the spinless Salpeter equation, the above observation implies that this number can never be less than its nonrelativistic counterpart [16].

## 3 Informative Application to Two Spinless Semirelativistic Problems

In the past, the *effortlessness* of applying all the findings collected in Sect. 2 to semirelativistic bound-state problems has been demonstrated [8, 17–22] for a number of frequently employed, rather popular interaction potentials. Therefore, let's take a brief look at two of these analyses, where simplicity motivated us to confine ourselves to spherically symmetric central potentials

$$V(\mathbf{x}) = V(r), \quad r \equiv |\mathbf{x}|. \quad (17)$$

### 3.1 Short-Range Singular Example: Spinless Relativistic Hulthén Problem [18, 19]

The Hulthén potential  $V_S(r)$ , originally introduced in nuclear physics but since then in physics widely used, is defined by two parameters, the coupling strength  $\eta$  and an exponential range  $b$ :

$$V_S(r) \equiv -\frac{\eta}{\exp(br) - 1}, \quad b > 0, \quad \eta \geq 0. \quad (18)$$

As shown in Fig. 1, the potential  $V_S(r)$  approaches, for short distances, from above a Coulomb potential  $V_C(r)$  with strength  $\eta/b$  (i.e., exhibits Coulombic behaviour) and, for large distances, from below a negative exponential potential  $V_E(r)$ , or increases exponentially damped to zero:

$$V_S(r) \xrightarrow{r \rightarrow 0} V_C(r) \equiv -\frac{\eta}{br}, \quad V_S(r) \xrightarrow{r \rightarrow \infty} V_E(r) \equiv -\eta \exp(-br). \quad (19)$$

Conversely, the Hulthén potential (18) is bounded from below by the Coulomb potential (3) iff

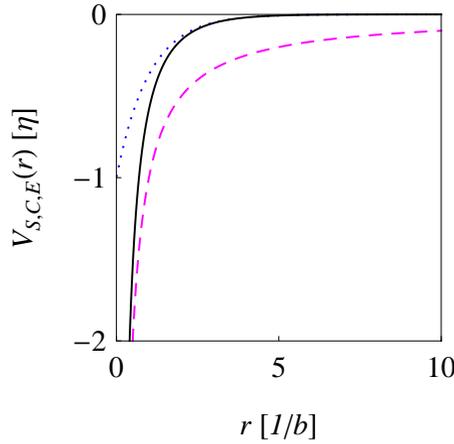
$$\alpha \geq \frac{\eta}{b}. \quad (20)$$

Consequently, the Hulthén potential (18) inevitably exhibits a Coulomb-like singularity at spatial origin  $r = 0$ . Nevertheless, according to Subsect. 2.1 the assigned Hamiltonian (1) and bound-state spectrum are bounded from below if the two involved potential parameters satisfy

$$\frac{\eta}{b} < \frac{4}{\pi}. \quad (21)$$

For the *nonrelativistic* limit (15) of the Hamiltonian (1) with Hulthén potential (18) and all states of orbital angular momentum  $\ell = 0$ , the eigenvalues  $E_{\text{NR}}$  can be found analytically [23]:

$$E_{\text{NR},n} = m_1 + m_2 - \frac{(2\hat{m}\eta - n^2 b^2)^2}{8\hat{m}n^2 b^2}, \quad \hat{m} \equiv \frac{m_1 m_2}{m_1 + m_2}, \quad n \in \mathbb{N}, \quad (22)$$



**Figure 1.** Hulthén potential  $V_S(r)$  (solid black line), Eq. (18), and asymptotes (19): negative exponential potential  $V_E(r)$  (dotted blue line) and Coulomb potential  $V_C(r)$  (dashed magenta line), for strength  $\eta = 1$ .

where the *radial* quantum number  $n$  is constrained by the two potential characteristics  $\eta$  and  $b$ ,

$$n \leq \frac{\sqrt{2\hat{m}\eta}}{b}. \quad (23)$$

In the spirit of Subject. 2.5, the expressions (22) may serve to provide, at least, a vague idea of the actual location of the discrete eigenstates of any spinless semirelativistic Hulthén problem.

### 3.2 Category of Spinless Semirelativistic Generalized-Hellmann Problems [21, 22]

Generalized Hellmann potentials [21, 22] establish a class of central interaction potentials that encompasses the shape introduced, some time ago [24, 25], for applications in atomic-physics theory. Each member of this set is defined to be a linear combination of an attractive Coulomb contribution  $V_C(r)$  and an optionally *either attractive or repulsive* Yukawa contribution  $V_Y(r)$ ,

$$V_H(r) \equiv V_C(r) + V_Y(r) = -\frac{\kappa}{r} - \nu \frac{\exp(-br)}{r}, \quad (24)$$

Coulomb coupling  $\kappa$ , Yukawa coupling  $\nu$ , and exponential-range parameter  $b$  hence satisfying

$$\kappa \geq 0, \quad \nu \gtrless 0, \quad b > 0. \quad (25)$$

(Note that, in the notation (24), the interaction potential originally proposed by Hellmann [24] corresponds to assuming for the Yukawa coupling parameter  $\nu$  a strictly negative value  $\nu < 0$ .) At large distances  $r$ , all generalized Hellmann potentials approach their Coulomb component,

$$V_H(r) \xrightarrow{r \rightarrow \infty} V_C(r). \quad (26)$$

This guarantees that any discrete eigenvalue  $E_k$  of  $H$  is nonpositive, thus bounded from above:

$$E_k \leq 0 \quad \forall \quad k \in \mathbb{N}_0. \quad (27)$$

Due to the asymptotic approach (26) to Coulomb-type behaviour, every generalized Hellmann potential (24) fails to meet all requirements imposed in Subsect. 2.2, particularly, because [21]

$$V_H(r) \notin L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3), \quad (28)$$

whence no finite upper bound (8) on the total number of its discrete bound states can be found.

*Nonsingular* generalized Hellmann potentials, in the classification of Table 1 identified by  $\kappa + \nu \leq 0$ , are bounded from below, as likewise (because of  $\sqrt{\mathbf{p}^2 + m^2} \geq 0$ ) their Hamiltonians:

$$H \geq V_H(r) \geq \min_{0 \leq r < \infty} V_H(r) > -\infty \quad \text{for} \quad \nu < -\kappa, \quad (29)$$

$$H \geq V_H(r) \geq V_H(0) = \nu b \quad \text{for} \quad \nu = -\kappa. \quad (30)$$

*Singular* generalized Hellmann potentials, in their systematics of Table 1 characterized by  $\kappa + \nu > 0$ , develop *negative singularities* at  $r = 0$  bounded from below by a Coulomb potential:

$$V_H(r) \geq -\frac{\alpha}{r} \quad \text{with} \quad \begin{cases} \alpha = \kappa + \nu & \text{for} \quad \nu > 0, \\ \alpha = \kappa & \text{for} \quad \nu \leq 0. \end{cases} \quad (31)$$

As consequence of this, *mutatis mutandis* the relevant insights collected in Subsect. 2.1 apply. Most importantly, spinless relativistic generalized-Hellmann problems are well defined [21] if

$$\kappa + \nu \leq \frac{4}{\pi}. \quad (32)$$

**Table 1.** Characterization of the seven types of generalized Hellmann potentials  $V_H(r)$  distinguished by comparing the Yukawa coupling  $\nu$  with the (by assumption nonnegative) Coulomb coupling  $\kappa \geq 0$  [21].

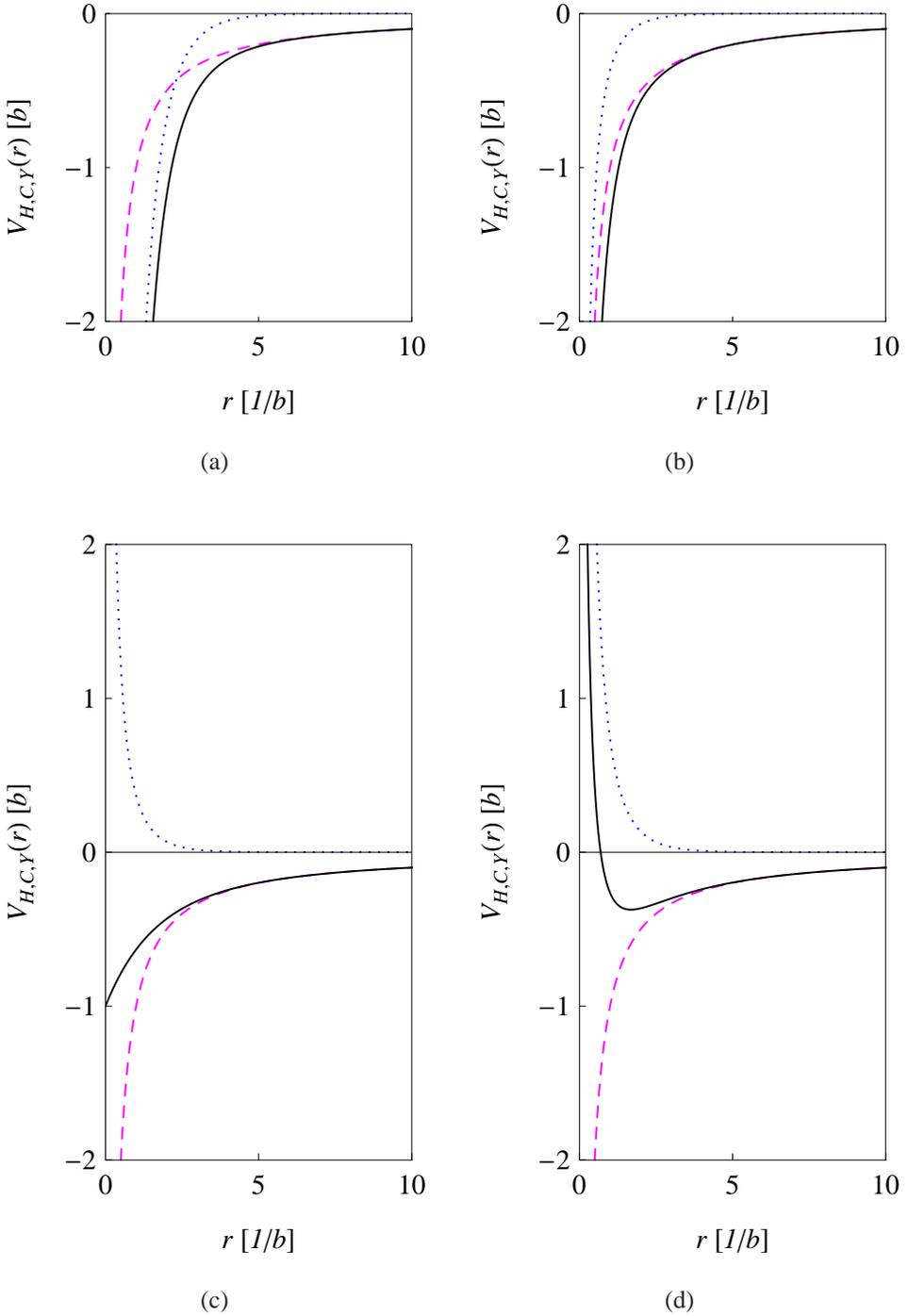
Boundedness from below	Characteristic of potential	Behaviour near the origin $r = 0$	Sign of sum of couplings	Relation between couplings $\nu$ and $\kappa$
unbounded	“singular”  pure Coulomb	$V_H(r) \xrightarrow[r \rightarrow 0]{} -\infty$	$\kappa + \nu > 0$	$\nu > \kappa$
				$\nu = \kappa$
				$0 < \nu < \kappa$
				$\nu = 0$
				$-\kappa < \nu < 0$
bounded	finite at origin	$V_H(r) \xrightarrow[r \rightarrow 0]{} \nu b$	$\kappa + \nu = 0$	$\nu = -\kappa$
	repulsive core	$V_H(r) \xrightarrow[r \rightarrow 0]{} +\infty$	$\kappa + \nu < 0$	$\nu < -\kappa$

Merely for illustrative purposes, Fig. 2 depicts a typical representative of four of our seven classes of generalized Hellmann potentials (24) identified in Table 1. There is no need to show the trivial case of the pure Coulomb potential arising for vanishing Yukawa coupling strength.

In order to exemplify the calculation of upper bounds on energies along the lines sketched in Subsect. 2.3, Table 2 provides, for few low-lying eigenstates of three generalized-Hellmann problems and a fixed choice of the numerical values of the parameters  $\mu$  and  $\beta$  in the basis (13) of some  $d$ -dimensional subspace, upper bounds on the associated binding energies, defined by

$$B_k \equiv E_k - 2m, \quad k \in \mathbb{N}_0. \quad (33)$$

The upper bounds in Table 2 can straightforwardly be optimized variationally, in two respects: definitely by increasing the dimension  $d$  of the adopted subspace, and potentially by trying out different values for (i.e., varying) the parameters  $\mu$  and/or  $\beta$  defining the subspace’s basis (13).



**Figure 2.** Generic representatives (solid black lines) of four of the only seven distinguishable categories in our systematic classification (Table 1) of generalized Hellmann potentials  $V_H(r)$  defined in Eq. (24) by linear combinations of Coulombic (dashed magenta lines) and Yukawa (dotted blue lines) contributions, choosing  $\kappa = 1$  and (a)  $\nu = 10$ , (b)  $\nu = 1$ , (c)  $\nu = -1$ , or (d)  $\nu = -2$  for the involved coupling strengths.

**Table 2.** Bounds on binding energies, for states of radial excitation  $n_r$  and orbital angular momentum  $\ell$ , for exemplary spinless relativistic generalized-Hellmann problems [21, 22] ( $b = \mu = m, \beta = 1, d \gtrsim 29$ ).

Bound state		Upper bounds on $B_{n_r\ell}$ [m]		
		$\kappa = \nu = \frac{1}{2}$ [case Fig. 2(b)]	$\kappa = 1, \nu = -1$ [case Fig. 2(c)]	$\kappa = 1, \nu = -2$ [case Fig. 2(d)]
$n_r$	$\ell$			
0	0	-0.11673	-0.17951	-0.14410
0	1	-0.01579	-0.06294	-0.06157
0	2	-0.00616	-0.02813	-0.02812
1	0	-0.02107	-0.05464	-0.04786
1	1	-0.00509	-0.02810	-0.02762
2	0	-0.00688	-0.02566	-0.02338
Lower bound on $B_{n_r\ell}$ [m]		-0.58578 ...	-1	-0.37336 ...

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