The effective complex heavy-quark potential in an anisotropic quark-gluon plasma

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Abstract. We introduce a method for reducing anisotropic heavy-quark potentials to isotropic potentials by using an effective screening mass that depends on the quantum numbers $l$ and $m$ of a given state. We demonstrate that, using the resulting 1D effective potential model, one can solve a 1D Schrödinger equation and reproduce the full 3D results for the energies and binding energies of low-lying heavy-quarkonium bound states to relatively high accuracy. This includes the splitting of different p-wave polarizations. The resulting 1D effective model provides a way to include momentum anisotropy effects in open quantum system simulations of heavy-quarkonium dynamics in the quark-gluon plasma.

1 Introduction

The survival probability of the heavy quarkonium states, such as $J/\Psi$ and $\Upsilon$ has been widely used as a sensitive probe to study the quark-gluon plasma (QGP) formed in relativistic heavy-ion experiments at RHIC and LHC \cite{1,2}. Due to non-relativistic nature of heavy quarkonium states, one can obtain their in-medium properties, such as masses and decay rates by solving a Schrödinger equation with a complex heavy-quark (HQ) potential. The real part of the HQ potential provides the binding energy, whereas the imaginary part provides information about the decay of a quarkonium state via wave function decoherence \cite{3,4,5,6,7,8}. One can obtain the HQ potential at short distances by making use of hard-thermal-loop (HTL) resummed perturbation theory in the weak-coupling limit. Recently, several attempts have been made to develop complex-valued potential models to understand the in-medium properties of quarkonia quantitatively \cite{10,11}. During the last decade, many prior works have treated the QGP as an anisotropic medium by incorporating momentum-space anisotropies generated by longitudinal expansion into the underlying parton distribution functions. To make a phenomenological study of this effect in heavy-ion collisions, we consider the following spheroidal distribution function ansatz in the local rest frame (LRF) of the QGP \cite{20}

\begin{equation}
\label{eq:1}
f_{\text{aniso}}^{\text{LRF}} (k) \equiv f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{k^2 + \xi (k \cdot n)^2} \right).
\end{equation}

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This form takes into account the rapid longitudinal expansion of the QGP at early times and allows for explicit pressure anisotropies in the LRF \cite{21,22}. Here, \( f_{\text{iso}} \) is an arbitrary isotropic distribution function, \( \lambda \) is a temperature-like scale, which becomes the temperature \( T \) of the system in the thermal equilibrium limit. The degree of momentum-space anisotropy (\( \xi \)) in the range \(-1 < \xi < \infty \) is given by

\[
\xi = \frac{1}{2} \frac{(k^2)}{\langle k^2 \rangle} - 1,
\]

where \( k \equiv \mathbf{k} \cdot \mathbf{n} \) and \( k_{\perp} \equiv \mathbf{k} - \mathbf{n} (\mathbf{k} \cdot \mathbf{n}) \) correspond to the particle momenta along and perpendicular to the direction of anisotropy (\( \mathbf{n} \)), respectively. Many prior works have studied heavy quarkonium physics by considering the momentum-space anisotropy inside the QGP \cite{10,13,16,18,20,21,23}. Here we focus on how to efficiently take momentum-space anisotropy into account in a one-dimensional effective theory and compare the one- and three-dimensional results for static and dynamical quantities numerically.

In this proceedings contribution, we summarize our previous works where the real part of a 3D anisotropic HQ potential has been reduced to 1D effective potential \cite{24,25}. This work is organized as follow: In sec. 2 we describe the isotropic complex HQ potential model, in sec. 3 we obtain the anisotropic complex HQ potential model, in sec. 4 we obtain our effective complex HQ potential model, in sec. 5 we present our static results, and in sec. 6 we present our dynamic results.

## 2 Isotropic Potential Model

The Fourier transform of the real time gluon propagator in the static limit gives the complex HQ potential in an isotropic QGP \cite{26}.

\[
V(\lambda, r) = -g^2 C_F \int \frac{d^3 \mathbf{p}}{(2\pi)^3} (e^{\mathbf{p} \cdot \mathbf{r}} - 1) D^{00}(p_0 = 0, \mathbf{p}, \lambda) .
\]

### 2.1 Perturbative Contribution

The perturbative contribution to the complex HQ potential can be obtained from HTL resummed perturbation theory. The real and imaginary parts of this perturbative contribution are given by

\[
\text{Re} \; V_{\text{pt}}(\lambda, r) = -g^2 C_F \int \frac{d^3 \mathbf{p}}{(2\pi)^3} (e^{\mathbf{p} \cdot \mathbf{r}} - 1) \left( \frac{1}{p^2 + m_D^2} - \frac{1}{p^2} \right) \equiv \alpha m_D (I_1(\hat{r}) - 1) ,
\]

\[
\text{Im} \; V_{\text{pt}}(\lambda, r) = -g^2 C_F \int \frac{d^3 \mathbf{p}}{(2\pi)^3} (e^{\mathbf{p} \cdot \mathbf{r}} - 1) \frac{-\pi \lambda m_D^2}{p(p^2 + m_D^2)^2} \equiv \alpha \lambda (I_2(\hat{r}) - 1) ,
\]

where the integrals \( I_1(\hat{r}) \) and \( I_2(\hat{r}) \) are

\[
I_1(\hat{r}) = 4\pi \int \frac{d^3 \hat{p}}{(2\pi)^3} \frac{1}{\hat{p}^2 (\hat{p}^2 + 1)} = \frac{1 - e^{-\rho}}{\rho} ,
\]

\[
I_2(\hat{r}) = 4\pi^2 \int \frac{d^3 \hat{p}}{(2\pi)^3} \frac{1}{\hat{p}^2 (\hat{p}^2 + 1)} = \phi_2(\hat{r}) ,
\]

with

\[
\phi_\mu(\hat{r}) = 2 \int_0^\infty \frac{dz}{z^2} \frac{\sin(z\hat{r})}{z^2 + 1} z .
\]

Here, \( \hat{p} \equiv p/m_D, \hat{r} \equiv r/m_D \), and the strong coupling constant \( \alpha = g^2 C_F/(4\pi) \). We also subtracted a term \( 1/p^2 \) in eq. \[4\] to make the \( r \)-independent part finite.
2.2 Non-perturbative Contribution

The gluon propagator also contains a non-perturbative string contribution which arises from a dimension two gluon condensate. Its Fourier transform gives us the non-perturbative contributions [25].

\[
\begin{align*}
\text{Re } V_{\text{np}}(\lambda, r) &= -g^2 C_F m_G^2 \int \frac{d^3 p}{(2\pi)^3} \left( \frac{p^2 + 5 m_D^2}{(p^2 + m_D^2)^2} \right) = -\frac{2\sigma}{m_D} (I_3(\hat{r}) - 1), \quad (8) \\
\text{Im } V_{\text{np}}(\lambda, r) &= -g^2 C_F m_G^2 \int \frac{d^3 p}{(2\pi)^3} \left( \frac{4\pi \lambda m_D^2 (p^2 - 2 m_D^2)}{p^2 + m_D^2} \right) \frac{1}{p^2 + m_D^2} \approx -\frac{4\sigma \lambda}{m_D^2} (I_4(\hat{r}) - 1), \quad (9)
\end{align*}
\]

where \( \sigma = a m_G^2 / 2 \) and \( m_G^2 \) is a dimensionful constant. The integrals appearing above are

\[
\begin{align*}
I_3(\hat{r}) &= 4\pi \int \frac{d^3 \hat{p}}{(2\pi)^3} e^{i \hat{p} \cdot \hat{r}} \hat{p}^2 + 5 = (1 + \hat{r}/2) e^{-\hat{r}}, \\
I_4(\hat{r}) &= 8\pi^2 \int \frac{d^3 \hat{p}}{(2\pi)^3} e^{i \hat{p} \cdot \hat{r}} \frac{2 - \hat{p}^2}{\hat{p}^2 (\hat{p}^2 + 1)^4} = -2\phi_3(\hat{r}) + 6\phi_4(\hat{r}). \quad (10)
\end{align*}
\]

2.3 Total Isotropic potential

The sum of the perturbative and non-perturbative contributions give us the total complex isotropic HQ potential

\[
\begin{align*}
\text{Re } V_{\text{iso}}(r) &= \text{Re } V_{\text{pr}}(\lambda, r) + \text{Re } V_{\text{np}}(\lambda, r) \\
&= \lambda m_D \left( 1 - e^{-r/m_D} \right) - \alpha m_D - \frac{\sigma}{m_D} \left( 2 + r/m_D \right) e^{-r/m_D} + \frac{2\sigma}{m_D} \frac{\alpha}{r}, \quad (11) \\
\text{Im } V_{\text{iso}}(r) &= \text{Im } V_{\text{pr}}(\lambda, r) + \text{Im } V_{\text{np}}(\lambda, r) \\
&= \alpha \lambda \phi_2 (r/m_D) - \alpha \lambda - \frac{8\sigma \lambda}{m_D^2} \phi_3 (r/m_D) + \frac{24\sigma \lambda}{m_D^2} \phi_4 (r/m_D) - \frac{4\sigma \lambda}{m_D^2}. \quad (12)
\end{align*}
\]

We include a relativistic correction, \(-0.8\sigma / (m_{b/c} / r)\), in the potential model while solving the Schrödinger equation for charmonia and bottomonia [10], where the masses of the charm and bottom quarks are taken to be \( m_c = 1.3 \text{ GeV} \) and \( m_b = 4.7 \text{ GeV} \), respectively.

3 3D Anisotropic Potential Model

The real and imaginary part of the 3D anisotropic potential model as derived in our previous work [25] are

\[
\begin{align*}
\text{Re } V_{\text{Aniso}}(r, \theta, \xi) &= \alpha \lambda^2 \phi_2 \left( m_D^2 \right) - \alpha \lambda^2 \left( m_D^2 / 6 \right) e^{-r/m_D^2} + \frac{2\sigma}{m_D^2} \frac{\alpha}{r}, \quad (13) \\
\text{Im } V_{\text{Aniso}}(r, \theta, \xi) &= \alpha \lambda^2 \phi_2 \left( m_D^2 \right) - \alpha \lambda^2 \left( m_D^2 / 6 \right) e^{-r/m_D^2} + \frac{24\sigma \lambda^4}{m_D^4} \phi_4 \left( m_D^2 \right) - \frac{4\sigma \lambda^4}{m_D^4}, \quad (14)
\end{align*}
\]

where,

\[
m_D^2 = m_D \left( 1 - \frac{\xi}{6} \right), \quad \lambda^2 = \lambda \left( 1 - \frac{\xi}{6} \right), \quad (15)
\]
and

\[ m^R_D = m_D \left[ 1 + \xi \left(0.108 \cos 2\theta - 0.131\right)\right], \quad m^I_D = m_D \left[ 1 + \xi \left(0.026 \cos 2\theta - 0.158\right)\right]. \] (16)

Eq. (15) assures a correct asymptotic behavior of the potential. Eq. (16) was obtained by matching effective and exact result at \( \hat{r} = 1 \) as described in [25].

### 4 1D Effective Potential Model

Due to the angular dependence in the 3D anisotropic potential model, solving a 3D Schrödinger equation to find various in-medium properties of the quarkonium states is rather time consuming and much more complicated. One possible solution to this problem is to introduce an angle-averaged effective screening mass \( M_{im}(\lambda, \xi) \) [24]

\[
M_{im}^{R,I}(\lambda, \xi) = \langle Y_{lm}(\theta, \phi)|m^R_I D(\lambda, \xi, \theta)|Y_{lm}(\theta, \phi)\rangle, \quad \int_{-1}^{1} d\cos \theta \int_{0}^{2\pi} d\phi Y_{lm}(\theta, \phi) m^R_I D(\lambda, \xi, \theta) Y^*_{lm}(\theta, \phi), \tag{17}
\]

and where \( Y_{lm}(\theta, \phi) \) refers to the spherical harmonics with azimuthal quantum number \( l \) and magnetic quantum number \( m \). The main advantage of using an angle-averaged effective screening mass \( M_{im}(\lambda, \xi) \) is to utilize the spherical symmetry in the potential model which significantly simplifies the numerics.

The real and imaginary part of the 1D effective potential model as derived in our previous work [25] are

\[
\text{Re } V_{\text{eff}}(r, \xi) = \alpha m^A_D \left(1 - e^{-r M^A_{im}} \right) - \alpha m^A_D - \frac{\sigma}{m^A_D} \left(2 + r M^A_{im}\right) e^{-r M^A_{im}} + \frac{2\sigma}{m^A_D} - \frac{\alpha}{r}, \tag{18}
\]

\[
\text{Im } V_{\text{eff}}(r, \xi) = \alpha \lambda^4 \phi_2 \left(r M^I_{im}\right) - \alpha \lambda^4 - \frac{8\sigma \lambda^4}{(m^A_D)^2} \phi_3 \left(r M^I_{im}\right) + \frac{24\sigma \lambda^4}{(m^A_D)^2} \phi_4 \left(r M^I_{im}\right) - \frac{4\sigma \lambda^4}{(m^A_D)^2}, \tag{19}
\]

where,

\[
K_{lm} = \frac{2(l+1) - 2m^2 - 1}{4l(l+1) - 3}, \tag{20}
\]

and

\[
M^R_{im} = m_D \left[ 1 + \xi \left(0.216 K_{im} - 0.239\right)\right], \quad M^I_{im} = m_D \left[ 1 + \xi \left(0.052 K_{im} - 0.184\right)\right]. \tag{21}
\]

The \( l \) and \( m \) values of various quarkonium states are given in Table [1]

### 5 Static Results

For the static solutions, we used a previously developed 3D eigensolver called quantumFDTD [27, 28]. Using this code, we compared results obtained with the 1D effective potential and the full 3D anisotropic potential. In Table [2] we list the exact results of the eigenenergies (Re \( E \)), decay widths (Im \( E \)) and the binding energies (\( E_{\text{bind}} \)) with the anisotropy parameter \( \xi = 1 \) for \( \Upsilon(1S) \) and \( J/\Psi \).

In the numerical evaluations, we took \( \alpha = 0.272 \) and \( \sigma = 0.215 \text{ GeV}^2 \). For the \( \Upsilon(1S) \) state, we used a lattice size of \( N^3 = 512^3 \) with a lattice spacing of \( a = 0.020 \text{ GeV}^{-1} \approx 0.004 \text{ fm} \) giving a lattice size of \( L = Na \approx 2.05 \text{ fm} \). For the \( J/\Psi \), we used a lattice size of \( N^3 = 256^3 \) with a lattice spacing of \( a = 0.085 \text{ GeV}^{-1} \approx 0.017 \text{ fm} \) giving a lattice size of \( L = Na \approx 4.35 \text{ fm} \).
Table 1. The exact 3D results of the complex eigenenergies ($E$) and binding energies ($E_{\text{bind}}$) for different quarkonium states at various temperatures with $\xi = 1$. $\delta E$ are the differences in results obtained using 1D effective and 3D anisotropic potentials. Here $T_0$ is 192 MeV and all results are in MeV [25].

<table>
<thead>
<tr>
<th>$T(1S)$</th>
<th>$\text{Re}E$</th>
<th>$\delta\text{Re}E$</th>
<th>$E_{\text{bind}}$</th>
<th>$\text{Im}E$</th>
<th>$\delta\text{Im}E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>182.869</td>
<td>0.611</td>
<td>$-662.669$</td>
<td>11.838</td>
<td>0.027</td>
</tr>
<tr>
<td>$1.1T_0$</td>
<td>174.957</td>
<td>0.593</td>
<td>$-570.612$</td>
<td>14.830</td>
<td>0.031</td>
</tr>
<tr>
<td>$1.2T_0$</td>
<td>166.556</td>
<td>0.573</td>
<td>$-493.689$</td>
<td>18.190</td>
<td>0.034</td>
</tr>
<tr>
<td>$1.4T_0$</td>
<td>148.439</td>
<td>0.531</td>
<td>$-372.540$</td>
<td>26.004</td>
<td>0.039</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$J/\Psi$</th>
<th>$\text{Re}E$</th>
<th>$\delta\text{Re}E$</th>
<th>$E_{\text{bind}}$</th>
<th>$\text{Im}E$</th>
<th>$\delta\text{Im}E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>439.336</td>
<td>1.230</td>
<td>$-406.202$</td>
<td>41.980</td>
<td>0.107</td>
</tr>
<tr>
<td>$1.1T_0$</td>
<td>422.207</td>
<td>1.163</td>
<td>$-323.362$</td>
<td>51.467</td>
<td>0.105</td>
</tr>
<tr>
<td>$1.2T_0$</td>
<td>404.597</td>
<td>1.095</td>
<td>$-255.648$</td>
<td>61.698</td>
<td>0.098</td>
</tr>
<tr>
<td>$1.3T_0$</td>
<td>386.604</td>
<td>1.028</td>
<td>$-199.583$</td>
<td>72.564</td>
<td>0.086</td>
</tr>
<tr>
<td>$1.4T_0$</td>
<td>368.301</td>
<td>0.963</td>
<td>$-152.678$</td>
<td>83.958</td>
<td>0.070</td>
</tr>
</tbody>
</table>

6 Dynamical Results

In order to solve the 3D Schrödinger equation in real time, we used a split-step pseudospectral method [29] with temporal step size $\Delta t = 0.001$ fm/c. Once again we compare results obtained with the full 3D anisotropic potential to those obtained with the 1D effective potential. We evolve the wave function from $\tau = 0$ fm/c to $\tau = 0.25$ fm/c in the vacuum ($T = 0$). Starting at $\tau = \tau_0 = 0.25$ fm/c, we consider a fixed anisotropy parameter $\xi = 1$ and boost-invariant Bjorken evolution for the hard scale

$$\lambda(\tau) = \lambda_0 \left( \frac{T_0}{\tau} \right)^{1/3}. \tag{22}$$

Here we take the initial hard scale to be $\lambda_0 = 630$ MeV. Further details of the numerical method can be found in [25].
Figure 1. The top row shows the overlaps of $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ resulting from real-time solution of the Schrödinger equation. Here we initialized the wave function as pure $\Upsilon(1S)$ eigenstate. The bottom row shows the time evolution of the bottomonium p-wave overlaps resulting from initialization with different p-wave polarizations \[25\].

6.1 Bottomonium

For bottomonium states we take the box size to be $L = 2.56$ fm, $m_b = 4.7$ GeV, and use $N = 128$ lattice points in each direction. The top row of fig. 1 shows the time evolution of overlaps of the $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ using a pure $\Upsilon(1S)$ eigenstate as the initial condition. Whereas the bottom row shows the time evolution of the bottomonium p-wave overlaps resulting from initialization with different p-wave polarizations. Results with pure $\Upsilon(2S)$ and $\Upsilon(3S)$ eigenstate and a Gaussian as the initial condition can be found in Ref. [25].

6.2 Charmonium

For charmonium states we take $L = 5.12$ fm, $m_c = 1.3$ GeV, and use $N = 128$ lattice points in each direction. The top row of the fig. 2 shows the time evolution of overlaps of the $J/\psi$, $\psi(2S)$, and $\psi(3S)$ by using pure $J/\psi$ eigenstate as the initial condition. Whereas the bottom row shows the time evolution of the charmonium p-wave overlaps resulting from initialization with different p-wave polarizations. The results with pure $\psi(2S)$ and $\psi(3S)$ eigenstate and Gaussian as the initial condition can be found in Ref. [25].
Figure 2. The top row shows the overlaps of $J/\psi$, $\psi(2S)$, and $\psi(3S)$ resulting from real-time solution of the Schrödinger equation. Here we initialized the wave function as pure $J/\psi$ eigenstate. The bottom row shows the time evolution of the charmonium p-wave overlaps resulting from initialization with different p-wave polarizations [25].

7 Conclusions

We have reduced anisotropic heavy-quark potentials to isotropic ones by introducing an effective screening mass that depends on the quantum numbers $l$ and $m$ of a given state. We demonstrated that, using the resulting 1D effective potential model, one can reproduce the full 3D results for the energies and binding energies of low-lying heavy-quarkonium bound states to relatively high accuracy. This finding is important because it can be used to incorporate anisotropy effects into one-dimensional real-time Schrödinger equations which underpin phenomenological calculations of bottomonium suppression in open quantum systems approaches.

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